The contraction principle in extended context

Mihály Bessenyei

University of Debrecen

Analysis Research Seminar

Department of Analysis (Debrecen), April 22, 2015

http://www.math.unideb.hu/bessenyei-mihaly
Theorem (Banach, 1922; Cacciopoli, 1930)
Any contraction of a complete metric space has exactly one fixed point.

Proof (Palais, 2007)
Let $x \in X$ be fixed and consider the sequence $x_{n+1} = Tx_n$, where $x_1 = x$. We shall prove that $(x_n)$ is Cauchy sequence. For all $u, v \in X$, we have

$$d(u, v) \leq d(u, Tu) + d(Tu, Tv) + d(Tv, v);$$

hence

$$d(u, v) \leq \frac{d(u, Tu) + d(Tv, v)}{1 - q}.$$

Therefore,

$$d(x_n, x_m) \leq \frac{d(x_n, x_{n+1}) + d(x_{m+1}, x_m)}{1 - q} \leq \cdots \leq \frac{(q^n + q^m)}{1 - q}d(x_1, x_2),$$

yielding the desired Cauchy property.
The orbit and the double orbit induced by $T$ are the sets given by

$$O(x) := \{ T^n x \mid n \in \mathbb{N} \cup \{0\} \}; \quad O(x, y) := O(x) \cup O(y).$$

Under a comparison function we mean an increasing, upper semicontinuous function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\varphi(0) = 0$ and $\varphi(t) < t$ for $t > 0$. 
Definition
Let $(X, d)$ be a metric space and $\varphi$ be a comparison function. A mapping $T : X \to X$ is called a weak $\varphi$-quasicontraction if

$$d(Tx, Ty) \leq \varphi(\text{diam} \ O(x, y)) \quad (x, y \in X)$$

holds. A strong $\varphi$-quasicontraction is a mapping $T : X \to X$ with the property

$$d(Tx, Ty) \leq \varphi(\text{diam} \{x, y, Tx, Ty\}) \quad (x, y \in X).$$

Lemma
The sequence of composite iterates of a comparison function tends to zero pointwise. If $T$ is a weak $\varphi$-quasicontraction, then $T^n$ is a weak $\varphi^n$-quasicontraction.
The Main Result

Theorem

If a weak quasicontraction of a complete metric space induces bounded orbits, then it has a unique fixed point.

Proof, step 1

First we show that \((T^n x)\) is Cauchy. For \(\varepsilon > 0\), there exists \(n_0 \in \mathbb{N}\), such that

\[ \varphi^{n_0} (\text{diam } \mathcal{O}(x)) < \varepsilon / 2. \]

For \(n > n_0\),

\[ d(T^{n_0} x, T^n x) \leq \varphi^{n_0} (\text{diam } \mathcal{O}(x, T^{n-n_0} x)) = \varphi^{n_0} (\text{diam } \mathcal{O}(x)) < \varepsilon / 2. \]

Hence the triangle inequality leads to \(d(x_n, x_m) < \varepsilon\) whenever \(n, m > n_0\).
The Main Result

Proof, step 2
By completeness, $T^n x \to x_0$. Claim: $T^n x_0 \to x_0$ also holds. Indeed,

\[ d(x_0, T^n x_0) \leq d(x_0, T^n x) + d(T^n x, T^n x_0) \]
\[ \leq d(x_0, T^n x) + \varphi^n(\text{diam } O(x, x_0)) \to 0 \quad (n \to \infty). \]

Proof, step 3
Aim: diam $O(x_0) = 0$. If this is not the case, then

\[ d(T^n x_0, T^{n+k} x_0) \leq \varphi^n(\text{diam } O(x_0, T^k x_0)) \leq \varphi(\text{diam } O(x_0)). \]

Therefore,

\[ \sup_{n, m \in \mathbb{N}} d(T^n x_0, T^m x_0) \leq \varphi(\text{diam } O(x_0)) < \text{diam } O(x_0); \]

\[ \text{diam } O(x_0) = \sup_{n \in \mathbb{N}} d(x_0, T^n x_0) = \max\{d(x_0, T^k x_0) \mid k = 1, \ldots, n_0\}. \]
The Main Result

Proof, step 4

Let \( k \in \{1, \ldots, n_0\} \) be the index via the diameter is represented. Then,

\[
d(x_0, T^k x_0) \leq d(x_0, T^{n+k} x_0) + d(T^{n+k} x_0, T^k x_0) \\
\leq d(x_0, T^{n+k} x_0) + \varphi^k (\text{diam } O(T^n x_0, x_0)) \\
= d(x_0, T^{n+k} x_0) + \varphi^k (\text{diam } O(x_0)) \\
\leq d(x_0, T^{n+k} x_0) + \varphi (\text{diam } O(x_0)).
\]

Passing the limit,

\[
\text{diam } O(x_0) = d(x_0, T^k x_0) \leq \varphi (\text{diam } O(x_0)) < \text{diam } O(x_0).
\]

This contradiction completes the proof.
<table>
<thead>
<tr>
<th>Theorem (Hegedűs–Szilágyi, 1980; Walter, 1981)</th>
</tr>
</thead>
<tbody>
<tr>
<td>If a weak quasicontraction of a complete metric space induces bounded orbits, then it has a unique fixed point.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem (Ćirić, 1974)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Any linear strong quasicontraction in a complete metric space has a unique fixed point.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem (Browder, 1968; Matkowski, 1975)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Any quasicontraction in a complete metric space has a unique fixed point.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ((X, d)) is a complete metric space, (\varphi) is comparison function satisfying (\varphi^n(t) \leq c_n t) with a convergent series (\sum c_n), then any strong (\varphi)-quasicontraction has a unique fixed point in (X).</td>
</tr>
</tbody>
</table>
It suffices to verify the boundedness of orbits. Fix $x \in X$ arbitrarily. Then,

\[
\begin{align*}
\text{diam } O_{n+1}(x) & \leq \text{diam } O_n(x) + d(T^n x, T^{n+1} x) \\
& \leq \text{diam } O_n(x) + \varphi(\text{diam } \{T^{n-1} x, T^n x, T^{n+1} x\}) \\
& \leq \text{diam } O_n(x) + \varphi^n(\text{diam } O_{n+1}(x)).
\end{align*}
\]

That is, $a_n = \text{diam } O_n(x)$ fulfills $a_{n+1} \leq a_n + \varphi^n(a_{n+1})$. Applying telescoping summation on the inequalities $a_{k+1} - a_k \leq \varphi^k(a_{k+1})$, we get

\[
a_{n+1} - a_{n_0} \leq \sum_{k=n_0}^{n+1} \varphi^k(a_{k+1}) \leq \sum_{k=n_0}^{\infty} \varphi^k(a_{n+1}) \leq \sum_{k=n_0}^{\infty} c_k a_{n+1}.
\]

Choose $n_0$ such that $\sum_{k=n_0}^{\infty} c_k$ be smaller then 1. Then the rearranged form of the inequality gives the boundedness of $(a_n)$. 

M. Bessenyei

The contraction principle in extended context
Applications

Remark

The assumptions of the main result and the previous application, in general, cannot be relaxed. Let \((X, d)\) be the complete metric space of the reals, and define \(\varphi: \mathbb{R}^+ \to \mathbb{R}^+\) by \(\varphi(t) = t - \arctan(t)\). Then, \(\varphi\) is a comparison function. Clearly, \(Tx = x + \pi/2\) is a fixed point free mapping. However,

\[
\varphi(\text{diam}\{x, y, Tx, Ty\}) = |x - y| + \frac{\pi}{2} - \arctan\left(|x - y| + \frac{\pi}{2}\right).
\]

In other words, \(T\) is a strong \(\varphi\)-quasicontraction with no fixed points.

M. Bessenyei, *The contraction principle in extended context*, (2015), manuscript.


One more application

Corollary (Archimedes)

Give me a place to stand on (that is: a fixed point), and I will move the Earth.

Thank You the kind attention!