MULTIPlicative Loops
of Topological Quasifields

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Abstract. Locally compact connected topological non-Desarguesian translation planes have been a popular subject for research in geometry since the seventies of the last century. These planes are coordinatized by locally compact quasifields $(Q, +, \cdot)$ such that the kernel of $Q$ is either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. In recent papers we determined the algebraic structure of the multiplicative loops $Q^* = (Q \setminus \{0\}, \cdot)$ of quasifields $Q$ such that $Q$ has dimension 2 over its kernel. Now we compare these cases and give a unified treatment of our results. In particular, we deal with multiplicative loops which either have a one-dimensional normal subloop or contain a compact subgroup.

1. Introduction. The first impulse to study non-associative structures came from the investigation of coordinate systems of non-Desarguesian planes. The translation planes are affine planes with a transitive group of translations. The translation planes are coordinatized by planar quasifields. The finite translation planes and the finite semifields are thoroughly studied in [D], [J], [GN1], [GN2]. The locally compact connected topological non-Desarguesian translation planes and the locally compact quasifields were fruitfully investigated in [B3], [B1–B6], [C], [H1–H3], [K], [O], [PS], [S] including classifications.

In [F1] and [F2] we gave loop theoretical characterizations of the algebraic structure of the multiplicative loops $Q^* = (Q \setminus \{0\}, \cdot)$ of locally compact quasifields $Q$ having dimension 2 over its kernel $K_r$. If $K_r = \mathbb{R}$, then $(Q, +)$ is the vector group $\mathbb{R}^2$ and the topological loop $Q^*$ is homeomorphic to $\mathbb{R} \times S^1$. If $K_r = \mathbb{C}$, then $(Q, +)$ is the vector group $\mathbb{C}^2$ and the loop $Q^*$ is homeomorphic to $\mathbb{R} \times S^3$. In this paper we compare these cases and give a unified treatment of our results.

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In Proposition 3.1 we determine the Lie groups $G$ topologically generated by the left translations of the multiplicative loops $Q^*$ of $Q$ and the continuous sharply transitive sections $\sigma$ belonging to the loops $Q^*$. If $Q^*$ is a two-dimensional proper loop, then P. T. Nagy and K. Strambach showed that the group $G$ is the connected component of $GL_2(\mathbb{R})$ (cf. [NS1], Section 29, p. 345). In this case we give a new characterization for the functions parametrizing the continuous section $\sigma$. For $\dim Q = 4$ and $K_r = \mathbb{C}$ the group $G$ is one of the following groups: $\text{Spin}_3(\mathbb{R}) \times \mathbb{R}$, $\text{Spin}_3(\mathbb{R}) \times \mathbb{C}$, $\text{SL}_2(\mathbb{C}) \times \mathbb{R}$, $GL_2(\mathbb{C})$. We give a new detailed proof of this assertion.

In Section 3 we mostly deal with those properties of the two- and four-dimensional multiplicative loops which emphasize their common features. To do this we give the precise description of the connections between the special algebraic properties of these loops $Q^*$ and the functions parametrizing the continuous section $\sigma$ corresponding to $Q^*$ (cf. Propositions 3.3, 3.6 and Theorem 3.4). We use here a slight modification of the notion of the decomposable multiplicative loop, the normal subloop decomposition of a multiplicative loop which is a central extension of a one-dimensional normal subgroup by a compact loop and the corresponding proofs comparing with those given in [F1] and [F2].

The last section is devoted to the differences between the two- and four-dimensional multiplicative loops $Q^*$. Although each locally compact two-dimensional nearfield is the field of complex numbers ([G], XI.12.2 Proposition, p. 348) there are proper Kalscheuer’s nearfields of dimension four. Any locally compact two-dimensional semifield is the field of complex numbers (cf. [PS]). In contrast to this we describe a class of four-dimensional proper semifields having the field $\mathbb{C}$ as their kernel. We show that the group generated by all left and right translations of the multiplicative loops $Q^*$ of this class is the group $\text{SL}_4(\mathbb{R}) \times \mathbb{R}$. This fact differs from the two-dimensional proper multiplicative loops since they have infinite-dimensional group as the group generated by all translations (cf. [NS1], Theorem 29.1, p. 345).

2. Preliminaries. For basic facts on loops we refer to [NS1], Section 1. Here we collect the often used notions.

**Definition 2.1.** A set $L$ with a binary operation $(x, y) \mapsto x \cdot y$ is called a loop if there exists an element $1 \in L$ such that $x = 1 \cdot x = x \cdot 1$ holds for all $x \in L$ and for any given $a, b \in L$ the equations $a \cdot y = b$ and $x \cdot a = b$ have unique solutions which are denoted by $y = a \backslash b$ and $x = b/a$. A loop $L$ is proper if it is not a group. The kernel of a homomorphism $\alpha : L_1 \rightarrow L_2$ from a loop $L_1$ to a loop $L_2$ is a normal subloop $N$ of $L_1$, i.e. a subloop of $L_1$ such that

$$x \cdot N = N \cdot x, \quad (x \cdot N) \cdot y = x \cdot (N \cdot y), \quad (N \cdot x) \cdot y = N \cdot (x \cdot y)$$

(1)

hold for all $x, y \in L_1$. The centre $Z(L)$ of a loop $L$ consists of all elements $z$ which satisfy the equations $zx \cdot y = z \cdot xy, xy \cdot z = x \cdot yz, xz \cdot y = x \cdot yz, xz = zx$ for all $x, y \in L$. If $L$ has a central subgroup $L_1$ and the factor loop $L/L_1$ is isomorphic to the loop $L_2$ then $L$ is called the central extension of $L_1$ by $L_2$. A loop $L$ is called topological, if it is a topological space and the binary operations $(a, b) \mapsto a \cdot b,$ $(a, b) \mapsto b/a,$ $(a, b) \mapsto a \backslash b : L \times L \rightarrow L$ are continuous.
For all \( a \in L \), the left translations \( \lambda_a : L \to L, x \mapsto a \cdot x \), as well as the right translations \( \rho_a : L \to L, x \mapsto x \cdot a \), are bijections of \( L \). For topological loops the left and right translations of \( L \) are homeomorphisms of \( L \). Every topological connected loop \( L \) having a Lie group \( G \) as the group topologically generated by the left translations of \( L \) corresponds to a sharply transitive continuous section \( \sigma : G/H \to G \), where \( G/H = \{ xH \mid x \in G \} \) consists of the left cosets of the stabilizer \( H \) of \( 1 \in L \) such that \( \sigma(H) = 1_G \) and \( \sigma(G/H) \) generates \( G \). The section \( \sigma \) is sharply transitive if the image \( \sigma(G/H) \) acts sharply transitively on \( G/H \), which means that to any \( xH, yH \) there exists precisely one \( z \in \sigma(G/H) \) with \( zxH = yH \).

**Definition 2.2.** A set \( Q \) with two binary operations \(+, \cdot : Q \times Q \to Q\) is called a (left) quasifield if \((Q, +)\) is an abelian group with neutral element 0, \((Q \setminus \{0\}, \cdot)\) is a loop, \(0 \cdot x = x \cdot 0 = 0\), and between these operations the (left) distributive law \( x \cdot (y+z) = x \cdot y + x \cdot z\) holds. A quasifield is proper if it is not a (skew) field. The kernel \( K_r \) of a (left) quasifield \( Q \) is a skew field defined by \((x+y) \cdot k = x \cdot k + y \cdot k\) and \((x \cdot y) \cdot k = x \cdot (y \cdot k)\) for all \( x, y, k \in Q \), \( k \in K_r \). The centre \( Z \) of \( Q \) is the set \( \{ z \in K_r \mid z \cdot x = x \cdot z \text{ for all } x \in Q \} \). A left quasifield \( Q \) is a semifield, if in \( Q \) also the right distributive law holds. A nearfield is a quasifield with associative multiplication. A locally compact connected topological quasifield is a locally compact connected topological space \( Q \) such that \((Q, +)\) is a topological group, \((Q \setminus \{0\}, \cdot)\) is a topological loop, the multiplication \( \cdot : Q \times Q \to Q \) is continuous and the mappings \( \lambda_a : x \mapsto a \cdot x \) and \( \rho_a : x \mapsto x \cdot a \) with \( 0 \neq a \in Q \) are homeomorphisms of \( Q \).

The (left) quasifield \( Q \) is a right vector space over \( K_r \) and for all \( a \in Q \) the map \( \lambda_a : Q \to Q, x \mapsto a \cdot x \), is \( K_r \)-linear. Every locally compact connected nearfield is isomorphic to \( \mathbb{R}, \mathbb{C} \), or to the nearfield \( \mathbb{H}_r = (\mathbb{H}, +, \circ) \) obtained from the skew field \((\mathbb{H}, +, \cdot)\) of quaternions by using the new multiplication \( x \circ y = x \cdot \varphi(x)^{-1} \cdot y \cdot \varphi(x) \), where \( \varphi(x) = \exp(\imath r \log |x|) \), for some \( r \in \mathbb{R} \). The nearfields \( \mathbb{H}_r, r \neq 0 \), are called proper Kalscheuer’s nearfields (cf. [S], 64.19, 64.20, p. 363).

### 3. Sections corresponding to the multiplicative loops \( Q^* \) of \( Q \)

Here a locally compact connected topological quasifield \( Q \) having dimension 2 over its kernel \( K_r \) is treated. Assume that \( B = \{ e_1, e_2 \} \) is a fixed basis of \( Q \) as right vector space over \( K_r \) with the scalar multiplication induced by \( K^*_r \), where \( e_1 \in K_r \) is the identity element of the multiplicative loop \( Q^* \) of \( Q \). Hence \( Q \) is the vector space of pairs \((x, y)^t \in K^2_r, K_r \) is the subspace of pairs \((x, 0)^t \) and \((1, 0)^t \) is the identity element of \( Q^* \). Since the set \( \Lambda_Q \) of all left translations of \( Q \) is a spread set of the vector space \( Q \) (cf. Proposition 1.14 in [K], p. 12) the set \( \Lambda_Q \) is a set of \((2 \times 2)\)-matrices such that for any \((\alpha, \gamma)^t \in K^2_r \) there exists a unique matrix of \( \Lambda_Q \) having \((\alpha, \gamma)^t \) as its first column (cf. [FT], p. 2595).

**Proposition 3.1.** Let \( Q^* \) be the multiplicative loop for a locally compact topological proper quasifield \( Q \) having dimension two over its kernel \( K_r \). If \( K_r = \mathbb{R} \), then the group \( G \) topologically generated by the left translations of \( Q^* \) is the connected component \( \text{GL}_2^+(\mathbb{R}) \) of the group \( \text{GL}_2(\mathbb{R}) \). If \( K_r = \mathbb{C} \), then the group \( G \) is one of the following Lie groups: \( \text{Spin}_3(\mathbb{R}) \times \mathbb{R} \), \( \text{Spin}_3(\mathbb{R}) \times \mathbb{C} \), \( \text{SL}_2(\mathbb{C}) \times \mathbb{R} \), \( \text{GL}_2(\mathbb{C}) \). If \( G = \text{Spin}_3(\mathbb{R}) \times \mathbb{R} \), then \( Q \) is a proper Kalscheuer’s nearfield. If \( Q^* \) is proper, then it corresponds to a continuous
sharply transitive section of the form \( \sigma : G/H_{(k,l,s)} \to G \):

\[
\begin{pmatrix}
ux & -uy \\
uy & u\bar{x}
\end{pmatrix} H \mapsto \begin{pmatrix}
ux & -uy \\
uy & u\bar{x}
\end{pmatrix} \begin{pmatrix}
a(u, x, y) & b(u, x, y) \\
0 & a^{-1}(u, x, y)e^{ic(u, x, y)}
\end{pmatrix} = M_{(u, x, y)} \tag{2}
\]

such that \( u > 0 \), \((x, y) \in \mathbb{C}^2\), \( x\bar{x} + y\bar{y} = 1 \) and \( a(u, x, y), b(u, x, y), c(u, x, y) \) are continuous functions with positive, complex, real values, respectively. If \( G \) is \( GL_2(\mathbb{C}) \), then \( a(1, 1, 0) = 1, b(1, 1, 0) = 0 = c(1, 1, 0) \). If \( G \) is \( SL_2(\mathbb{C}) \times \mathbb{R} \), then \( a(1, 1, 0) = 1, b(1, 1, 0) = 0, \) and \( c(u, x, y) \) is the constant function \( 0 \). If \( G \) is \( Spin_3(\mathbb{R}) \times \mathbb{C} \), then \( a(u, x, y) \) is the constant function \( 1 \), \( b(u, x, y) \) is the constant function \( 0 \) and \( c(1, 1, 0) = 0 \). If \( G \) is \( GL_2^+(\mathbb{R}) \), then \( x = \cos t, y = -\sin t, t \in [0, 2\pi) \), \( a(u, t) > 0, b(u, t) \in \mathbb{R}, c(u, t) \) is the constant function \( 0 \) with \( a(1, 0) = 1, b(1, 0) = 0 \).

**Proof.** The first assertion was proved in [NS1], Theorem 29.1, p. 345. If \( K_r = \mathbb{C} \) then the group \( G \) is a connected closed subgroup of \( GL_2(\mathbb{C}) \). As the loop \( Q^* \) is homeomorphic to \( S^3 \times \mathbb{R} \) the group \( G \) operates transitively on the sphere \( S^3 \) of oriented lines through 0 in \( \mathbb{R}^4 \) and it has a four-dimensional subgroup \( Spin_3(\mathbb{R}) \times \mathbb{R} \). According to [V], p. 24, the group \( G \) is the product \( ST \) such that \( S \) is conjugate to the group \( SL_2(\mathbb{C}) \) or to \( SU_2(\mathbb{C}) = Spin_3(\mathbb{R}) \) and \( T \) is a Lie subgroup of the centralizer of \( S \) in \( GL_4(\mathbb{R}) \). Since \( SU_2(\mathbb{C}) \) is a maximal compact subgroup of \( SL_2(\mathbb{C}) \) the group \( T \) is a connected subgroup of the group \( \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C} \setminus \{0\} \} \) and the group \( G \) is one of the groups listed in the assertion. In these cases \( dim Q^* = 4 \). If \( G \) is the group \( Spin_3(\mathbb{R}) \times \mathbb{R} \), then \( Q^* \) is isomorphic to \( G \) and hence \( Q \) is a proper Kalscheuer’s nearfield. If \( Q^* \) is proper, then we may assume that the stabilizer \( H \) of the identity of \( Q^* \) is the subgroup

\[
H_{(k,l,s)} = \left\{ \begin{pmatrix} k & l \\ 0 & k^{-1}e^{is} \end{pmatrix} \mid k > 0, \ l \in \mathbb{C}, \ s \in \mathbb{R} \right\} \tag{3}
\]

if \( G = GL_2(\mathbb{C}) \), the subgroup \( H_{(k,1,0)} \) if \( G = SL_2(\mathbb{C}) \times \mathbb{R} \), the subgroup \( H_{(1,0,s)} \) if \( G = Spin_3(\mathbb{R}) \times \mathbb{C} \). The elements \( g \) of \( G \) have a unique decomposition as the product

\[
g = \begin{pmatrix} ux & -uy \\ uy & u\bar{x} \end{pmatrix} h \quad \text{with} \quad x, y \in \mathbb{C}, \quad x\bar{x} + y\bar{y} = 1, \quad u > 0, \quad h \in H_{(k,l,s)}. \tag{4}
\]

In particular if \( G \) is the group \( GL_2^+(\mathbb{R}) \), then \( dim Q^* = 2 \) and the stabilizer \( H \) of the identity of \( Q^* \) may be chosen as the subgroup \([3]\) with \( k > 0, \ l \in \mathbb{R}, \ s = 0 \). The elements \( g \) of \( G = GL_2^+(\mathbb{R}) \) can be written uniquely as the product \([4]\) such that \( x = \cos t, \ y = -\sin t, \ t \in [0, 2\pi) \). Hence a continuous section corresponding to a loop \( Q^* \) has form \([2]\) satisfying the properties as in the assertion.■

Description for the functions \( a, b \) of the section \( \sigma \) for a multiplicative loop \( Q^* \) with \( dim Q^* = 2 \): Using the sharply transitivity property we characterized in [F2] the continuous functions \( a, b, c \) defining the section \( \sigma \) belonging to a four-dimensional multiplicative loop \( Q^* \). Now we give a similar description for a two-dimensional multiplicative loop \( Q^* \). The section \( \sigma \) corresponding to \( Q^* \) is sharply transitive precisely if for all pairs \((u_1, t_1),\)
Comparing in both sides of matrix equation (6) the elements in the first column we have

\[
\begin{pmatrix}
  u \cos t & u \sin t \\
  -u \sin t & u \cos t
\end{pmatrix}
\begin{pmatrix}
  a(u, t) & b(u, t) \\
  0 & a^{-1}(u, t)
\end{pmatrix}
\begin{pmatrix}
  u_1 \cos t_1 & u_1 \sin t_1 \\
  -u_1 \sin t_1 & u_1 \cos t_1
\end{pmatrix}
= \begin{pmatrix}
  u_2 \cos t_2 & u_2 \sin t_2 \\
  -u_2 \sin t_2 & u_2 \cos t_2
\end{pmatrix}
\begin{pmatrix}
k & l \\
0 & k^{-1}
\end{pmatrix} \cdot (5)
\]

The determinants of the matrices on both sides of (5) are equal. Hence \( u = u_1^{-1}u_2 \) and matrix equation (5) reduces to

\[
\begin{pmatrix}
  \cos t & \sin t \\
  -\sin t & \cos t
\end{pmatrix}
\begin{pmatrix}
  a(u_1^{-1}u_2, t) & b(u_1^{-1}u_2, t) \\
  0 & a^{-1}(u_1^{-1}u_2, t)
\end{pmatrix}
= \begin{pmatrix}
  \cos t_2 & \sin t_2 \\
  -\sin t_2 & \cos t_2
\end{pmatrix}
\begin{pmatrix}
k & l \\
0 & k^{-1}
\end{pmatrix}
\begin{pmatrix}
\cos t_1 & -\sin t_1 \\
\sin t_1 & \cos t_1
\end{pmatrix} \cdot (6)
\]

Comparing in both sides of matrix equation (6) the elements in the first column we have

\[
\cos ta(u_1^{-1}u_2, t) = \cos t_2 \cos t_1 k + \cos t_2 \sin t_1 l + \sin t_2 \sin t_1 k^{-1}, \quad (7)
\]

\[- \sin ta(u_1^{-1}u_2, t) = - \sin t_2 \cos t_1 k - \sin t_2 \sin t_1 l + \cos t_2 \sin t_1 k^{-1}. \quad (8)
\]

Taking the square of both sides of equations (7), (8) and adding these equations we obtain

\[a(u_1^{-1}u_2, t) = \sqrt{k^2 \cos^2 t_1 + (l^2 + k^{-2}) \sin^2 t_1 + 2kl \cos t_1 \sin t_1}, \quad (9)\]

\[\cos t = \frac{\cos t_2 \cos t_1 k + \cos t_2 \sin t_1 l + \sin t_2 \sin t_1 k^{-1}}{\sqrt{k^2 \cos^2 t_1 + (l^2 + k^{-2}) \sin^2 t_1 + 2kl \cos t_1 \sin t_1}}, \quad (10)\]

\[\sin t = \frac{\sin t_2 \cos t_1 k + \sin t_1 \sin t_2 l - \sin t_1 \cos t_2 \sin t_1 k^{-1}}{\sqrt{k^2 \cos^2 t_1 + (l^2 + k^{-2}) \sin^2 t_1 + 2kl \cos t_1 \sin t_1}}. \quad (11)\]

The elements in the second column in both sides of matrix equation (6) are

\[
\cos tb(u_1^{-1}u_2, t) + \sin ta^{-1}(u_1^{-1}u_2, t) = - \cos t_2 \sin t_1 k + \cos t_2 \cos t_1 l + \sin t_2 \cos t_1 k^{-1}, \quad (12)
\]

\[- \sin tb(u_1^{-1}u_2, t) + \cos ta^{-1}(u_1^{-1}u_2, t) = \sin t_2 \sin t_1 k - \sin t_2 \cos t_1 l + \cos t_2 \cos t_1 k^{-1}. \quad (13)\]

Multiplying equation (12) by \( \cos t \) and (13) by \( -\sin t \) and adding the obtained equations we have

\[b(u_1^{-1}u_2, t) = \frac{- \sin t_1 \cos t_1 k^2 + (\cos^2 t_1 - \sin^2 t_1)kl + (l^2 + k^{-2}) \cos t_1 \sin t_1}{\sqrt{k^2 \cos^2 t_1 + (l^2 + k^{-2}) \sin^2 t_1 + 2kl \cos t_1 \sin t_1}}, \quad (14)\]

If \( \dim Q^* = 2 \), then for a continuous sharply transitive section \( \sigma \) given by (5) \( u = u_1^{-1}u_2 > 0 \), \( t \) is given by (10), (11), and for all fixed \( u \) the functions \( a(u_1^{-1}u_2, t) \), \( b(u_1^{-1}u_2, t) \) are given by (9), (14). Any continuous sharply transitive section \( \sigma \) belonging to a two-dimensional multiplicative loop \( Q^* \) has this form with suitable \( k > 0 \), \( l \in \mathbb{R} \).

The left translation with an element \((s, z)\) of the multiplicative loop \( Q^* \) of a left quasifield \( Q \) is a linear transformation \( M_{(u,x,y)}(u,x,y) \in \sigma(G/H(k,l,s)) \) defined by

\[
\begin{pmatrix}
s & u \\
z & w
\end{pmatrix}
= M_{(u,x,y)}(u,x,y)
\begin{pmatrix}
x \cos t & -y \sin t \\
x \sin t & y \cos t
\end{pmatrix}
\begin{pmatrix}
0 & -b(u,x,y) \\
a^{-1}(u,x,y) & e^{ic(u,x,y)}
\end{pmatrix}
\begin{pmatrix}
v \\
w
\end{pmatrix} \cdot (15)
\]
where \( s = uxα(u, x, y), \ z = uyα(u, x, y) \). The elements of the kernel \( K_r \) of \( Q \) are \((0, 0)^t, (s, 0)^t\), \( s \in \mathbb{C} \setminus \{0\} \) if \( \dim Q = 4 \), and \( s \in \mathbb{R} \setminus \{0\} \) if \( \dim Q = 2 \). The matrix representation of the left translations with the elements of the one-dimensional connected subgroup \((r, 0)^t \mid r > 0\) of the kernel \( K_r \) of \( Q \) is

\[
M_{(u, 1, 0)} = \begin{cases} 
(ua(u, 1, 0) & ub(u, 1, 0) \\
0 & ua^{-1}(u, 1, 0)e^{ic(u, 1, 0)}
\end{cases} \quad u > 0
\]

(16)

with \( r = ua(u, 1, 0) \) and \( c(u, 1, 0) = 0 \) if \( \dim Q = 2 \). The subset

\[
\mathcal{T}_\mathbb{R} = \left\{ \begin{pmatrix} \cos t & \sin t \\
-\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1, t) & b(1, t) \\
a^{-1}(1, t) \end{pmatrix} \mid t \in [0, 2\pi) \right\}
\]

(17)

of the image of the section (2) consisting of elliptic elements acts sharply transitively on the oriented lines through \((0, 0)^t\) in \( \mathbb{R}^2 \). Therefore \( \mathcal{T}_\mathbb{R} \) is the set of all left translations of a one-dimensional compact loop. Similarly the subset

\[
\mathcal{T}_\mathbb{C} = \left\{ \begin{pmatrix} x & -\bar{y} \\
y & \bar{x} \end{pmatrix} \begin{pmatrix} a(1, x, y) & b(1, x, y) \\
a^{-1}(1, x, y)e^{ic(1,x,y)} \end{pmatrix} \mid x, y \in \mathbb{C}, \ x\bar{x} + y\bar{y} = 1 \right\}
\]

(18)

of the image of (2) seen as a set of \((4 \times 4)\)-real matrices acts sharply transitively on the oriented lines through \((0, 0, 0, 0)^t\) in \( \mathbb{R}^4 \). Hence \( \mathcal{T}_\mathbb{C} \) is the set of all left translations of a loop homeomorphic to \( S^3 \).

**Definition 3.2.** Let \( Q \) be a locally compact connected topological quasifield having dimension two over its kernel. The multiplicative loop \( Q^* \) of \( Q \) is called *decomposable*, if the set of all left translations of \( Q^* \) is a product \( \mathcal{T}_\mathcal{K} \), where \( \mathcal{T} \) is the set of all left translations of a compact loop given by (17), respectively (18), and \( \mathcal{K} \) is the set (16) of all left translations of \( Q^* \) belonging to the subgroup \((r, 0)^t \mid r > 0\) of the kernel \( K_r \) of \( Q \).

**Proposition 3.3.** Let \( Q \) be a locally compact connected topological quasifield having dimension two over its kernel. The multiplicative loop \( Q^* \) of \( Q \) is decomposable if and only if for all \( u > 0, \ x, y \in \mathbb{C}, \ x\bar{x} + y\bar{y} = 1 \) one has \( a(u, x, y) = a(1, x, y)a(u, 1, 0), \ c(u, x, y) = c(u, 1, 0) + c(1, x, y) \) and \( b(u, x, y) = a(1, x, y)b(u, 1, 0) + a^{-1}(u, 1, 0)e^{ic(u,1,0)}b(1, x, y) \) if \( \dim Q^* = 4 \) or for all \( u > 0, \ t \in [0, 2\pi) \), one has \( a(u, t) = a(1, t)a(u, 0), \ b(u, t) = a(1, t)b(u, 0) + a^{-1}(u, 0)b(1, t) \) if \( \dim Q^* = 2 \).

**Proof.** The set

\[
\{ M_{(u, x, y)} \mid u > 0, \ x, y \in \mathbb{C}, \ x\bar{x} + y\bar{y} = 1 \text{ or } x = \cos t, \ y = -\sin t, \ c(u, x, y) = 0, \ t \in [0, 2\pi) \}
\]

acts sharply transitively on \( Q^* \). Hence any point \((v, w)^t \setminus \{(0, 0)\}\) is the image of the point \((1, 0)^t\) under a unique linear mapping \( M_{(u, x, y)} \) given by (15). For all \( s > 0, \ m, n \in \mathbb{C}, \ m\bar{m} + n\bar{n} = 1 \) or \( m = \cos \varphi, \ n = -\sin \varphi, \ \varphi \in [0, 2\pi) \) the matrix equation

\[
\mathcal{T}_\mathbb{C} \left[ M_{(u, 1, 0)} \begin{pmatrix} \text{sma}(s, m, n) \\ \text{sna}(s, m, n) \end{pmatrix} \right] = \begin{pmatrix} ux & -uy \\
uy & ux \end{pmatrix} \begin{pmatrix} a(u, x, y) & b(u, x, y) \\
a^{-1}(u, x, y)e^{ic(u,x,y)} \end{pmatrix} \begin{pmatrix} \text{sma}(s, m, n) \\ \text{sna}(s, m, n) \end{pmatrix}
\]

(19)

holds if and only if the identities of the assertion are satisfied. □
The multiplicative loop $Q^*$ is homeomorphic either to $S^3 \times \mathbb{R}$ or to $S^1 \times \mathbb{R}$. We wish to study under which circumstances the loop $Q^*$ has a connected normal subloop $N^*$ such that the factor loop $Q^*/N^*$ is homeomorphic either to the sphere $S^3$ or to $S^1$.

**Theorem 3.4.** Let $Q$ be a locally compact connected topological quasifield having dimension two over its kernel. The multiplicative loop $Q^*$ of $Q$ has a one-dimensional connected normal subloop $N^*$ consisting of real elements such that the factor loop $Q^*/N^*$ is homeomorphic to $S^3$ or to $S^1$ if and only if $N^*$ is the group $\{(u,0)^t \mid u > 0\}$ isomorphic to $\mathbb{R}$ and $a(u,1,0) = 1$, $b(u,1,0) = 0 = c(u,1,0)$, $a(u,x,y) = a(1,x,y)$, $b(u,x,y) = b(1,x,y)$, $c(u,x,y) = c(1,x,y)$ for all $u > 0$, $x,y \in \mathbb{C}$, $x \bar{x} + y \bar{y} = 1$, or $a(u,0) = 1$, $b(u,0) = 0$, $a(u,t) = a(1,t)$, $b(u,t) = b(1,t)$, $c(u,t) = 0$ for all $t \in [0,2\pi)$. Then $Q^*$ is a central extension of the normal subgroup $N^*$ by a loop homeomorphic to $S^3$ or to $S^1$.

**Proof.** The left translations of a normal subloop $N^*$ of $Q^*$ generate a normal subgroup $N$ of the group $G$ topologically generated by all left translations of $Q^*$ (cf. Lemma 1.7 in [NS1], p. 19). Since $Q^*/N^*$ is homeomorphic to $S^3$ or to $S^1$ the subgroup $N^*$ is homeomorphic to $\mathbb{R}$. The group topologically generated by the left translations of a proper loop homeomorphic to $\mathbb{R}$ is the universal covering $PSL_2(\mathbb{R})$ of $PSL_2(\mathbb{R})$ (cf. [NS1], Section 18, p. 235). But $PSL_2(\mathbb{R})$ is not a subgroup of $G$ listed in Proposition 3.1. Hence $N^*$ is a group isomorphic to $\mathbb{R}$ and the set $\Lambda_{Q^*}$ of all left translations of $Q^*$ must contain the group $\{(u,0)^t \mid u > 0\}$ as a normal subgroup. $N^*$ has the form $\{(u,0)^t \mid u > 0\}$ which is a central subgroup of $Q^*$ such that the intersection of a compact subloop $\{1\}$ as well as of $\{1\}$ of $Q^*$ with $N^*$ is 1. Hence $Q^*$ is a central extension as in the assertion. According to (1) for all $u > 0$ one has $a(u,1,0) = 1$, $b(u,1,0) = 0$, $c(u,1,0) = 0$ if dim $Q^* = 4$ and $a(u,0) = 1$, $b(u,0) = 0$ if dim $Q^* = 2$. To obtain the necessary and sufficient conditions under which $N^*$ is normal in $Q^*$ we often use the fact that by (15) the element

$$
\begin{pmatrix}
ux \\
u y \\
u x \\
\end{pmatrix} \begin{pmatrix}
a(u,x,y) \\
0 \\
b(u,x,y) \\
\end{pmatrix} \begin{pmatrix}
a^{-1}(u,x,y)e^{ic(u,x,y)} \\
0 \\
1 \\
\end{pmatrix}
$$

belongs to the left translation of $(uxa(u,x,y),uya(u,x,y))^t$ with $u > 0$, $x,y \in \mathbb{C}$, $x \bar{x} + y \bar{y} = 1$ if dim $Q^* = 4$ and $x = \cos t$, $y = -\sin t$, the function $c$ is constant 0 if dim $Q^* = 2$. For all elements $q_1 := (x,y)^t$ of $S^3$ or $S^1$, $q_2 := (v,w)^t$ of $Q^*$ the condition $(N^* \cdot q_1) \cdot q_2 = N^* \cdot (q_1 \cdot q_2)$ of (1) holds if and only if

$$
\begin{pmatrix}
u u' \\
x v' \\
y w' \\
\end{pmatrix} = \begin{pmatrix}
u u \\
x 0 \\
y 0 \\
\end{pmatrix} \cdot \begin{pmatrix}
u x \\
x 0 \\
y 0 \\
\end{pmatrix}
$$

for all $x,y \in \mathbb{C}$, $x \bar{x} + y \bar{y} = 1$, $(v,w) \in \mathbb{C}^2 \setminus \{(0,0)\}$, or for all $x = \cos t$, $y = -\sin t$, $(v,w) \in \mathbb{R}^2 \setminus \{(0,0)\}$ with suitable $u,u' > 0$. This is equivalent to

$$
\begin{pmatrix}
ra(r,m,n)mv + rb(r,m,n)mw - ra^{-1}(r,m,n)w\bar{e}^{ic(r,m,n)} \\
rar(r,m,n)nv + rb(r,m,n)nw + ra^{-1}(r,m,n)w\bar{e}^{ic(r,m,n)} \\
\end{pmatrix} = 
\begin{pmatrix}
u u'(a(1,x,y)xv + b(1,x,y)xw - a^{-1}(1,x,y)w\bar{e}^{ic(1,x,y)}) \\
u u'(a(1,x,y)yv + b(1,x,y)yw + a^{-1}(1,x,y)w\bar{e}^{ic(1,x,y)}) \\
\end{pmatrix}
$$
such that $ux = ra(r, m, n)m$, $uy = ra(r, m, n)n$ with $m + n\eta = 1$. As $u^2(x\bar{x} + y\bar{y}) = r^2a^2(r, m, n)(m\bar{m} + n\bar{n})$ we obtain $m = x$, $n = y$, $u = ra(r, x, y)$. Using this for all $(x, y)$ of $S^3$ or $S^1$ and $(v, w)$ of $C^2 \setminus \{(0, 0)\}$ or $R^2 \setminus \{(0, 0)\}$ we obtain

$$[(x(va(r, x, y) + wb(r, x, y)) - ywa^{-1}(r, x, y)e^{ic(r, x, y)}]
\times [y(va(1, x, y) + wb(1, x, y)) + \bar{x}wa^{-1}(1, x, y)e^{ic(1, x, y)}]
= [(y(va(r, x, y) + wb(r, x, y)) + \bar{x}wa^{-1}(r, x, y)e^{ic(r, x, y))}]
\times [x(va(1, x, y) + wb(1, x, y)) - ywa^{-1}(1, x, y)e^{ic(1, x, y)}].$$

The last equation holds if and only if for all $(v, w)$ in $C^2 \setminus \{(0, 0)\}$ or in $R^2 \setminus \{(0, 0)\}$

$$(a(r, x, y)a^{-1}(1, x, y)e^{ic(1, x, y)} - a^{-1}(r, x, y)a(1, x, y)e^{ic(r, x, y)})(x\bar{x} + y\bar{y})vw
+ (b(r, x, y)a^{-1}(1, x, y)e^{ic(1, x, y)} - a^{-1}(r, x, y)b(1, x, y)e^{ic(r, x, y)})(x\bar{x} + y\bar{y})w^2 = 0$$

and hence

$$a(r, x, y)a^{-1}(1, x, y)e^{ic(1, x, y)} - a^{-1}(r, x, y)a(1, x, y)e^{ic(r, x, y)} = 0,$$
$$b(r, x, y)a^{-1}(1, x, y)e^{ic(1, x, y)} - a^{-1}(r, x, y)b(1, x, y)e^{ic(r, x, y)} = 0.$$

Multiplying the last two equations by $e^{-ic(1, x, y)}$ one obtains

$$a(r, x, y)a^{-1}(1, x, y) - a^{-1}(r, x, y)a(1, x, y)e^{ic(r, x, y)} = 0,$$
$$b(r, x, y)a^{-1}(1, x, y) - a^{-1}(r, x, y)b(1, x, y)e^{ic(r, x, y)} = 0.$$

Since $a(r, x, y)$ is positive for all $r > 0$ we get $c(r, x, y) = c(1, x, y)$ and hence $a(r, x, y) = a(1, x, y)$, $b(r, x, y) = b(1, x, y)$ for all $r > 0$, $x, y \in \mathbb{C}$, $x\bar{x} + y\bar{y} = 1$ or for all $x = \cos t$, $y = -\sin t$. If we take into account the obtained restrictions for the functions $a(r, x, y)$, $b(r, x, y)$, $c(r, x, y)$, a straightforward computation shows that the condition $(q_1 \cdot N^*) \cdot q_2 = q_1 \cdot (N^* \cdot q_2)$ of $[\mathbb{I}]$ holds for all elements $q_1 := (x, y)t$ of $S^3$ or $S^1$, $q_2 := (v, w)t$ of $Q^\ast$. This proves the assertion.
Let \( R \) denotes the group \( \text{Spin}_3(\mathbb{R}) \) if \( \dim Q = 4 \) or the group \( \text{SO}_2(\mathbb{R}) \) if \( \dim Q = 2 \) precisely if \( \Lambda_{Q^*} \) has the form
\[
\begin{pmatrix}
x & -\bar{y} \\
y & \bar{x}
\end{pmatrix}
\begin{pmatrix}
ua(u, 1, 0) & ub(u, 1, 0) \\
0 & ua^{-1}(u, 1, 0)e^{i\psi(u, 1, 0)}
\end{pmatrix}
\bigg| u > 0, \ x, y \in \mathbb{C}, \ x\bar{x} + y\bar{y} = 1 \quad \text{or} \quad x = \cos t, \ y = -\sin t, \ c(u, 1, 0) = 0 \bigg\}
\]
with the continuous functions \( a(u, 1, 0) > 0, b(u, 1, 0) \in \mathbb{C} \text{ or } \mathbb{R}, c(u, 1, 0) \in \mathbb{R} \) such that \( ua(u, 1, 0) \) is strictly monotone. In this case \( Q^* \) is decomposable.

**Proof.** If \( \dim Q = 4 \), then the assertion is proved in Proposition 10 of [16], whereas if \( \dim Q = 2 \), then the proof of the assertion is given in Proposition 15 of [15].

4. **Applications.** Although the group topologically generated by the left translations of any two-dimensional proper multiplicative loop is \( \text{GL}_2^+(\mathbb{R}) \) (cf. [15], Section 29, p. 345), here we show that the groups \( \text{GL}_2(\mathbb{C}), \text{Spin}_3(\mathbb{R}) \times \mathbb{C} \) and \( \text{SL}_2(\mathbb{C}) \times \mathbb{R} \) are realized as the group generated by the left translations of a four-dimensional multiplicative loop. The given examples consist of loops which are multiplicative loops of semifields, central extensions of \( \mathbb{R} \) by a loop defined on \( S^3 \) or which contain the group \( \text{Spin}_3(\mathbb{R}) \) illustrating Theorem 3.4 and Proposition 3.6.

There are two classes of four-dimensional semifields \( Q \) having the field \( \mathbb{C} \) as their kernel (cf. [15], Section 6). In the first class are the Rees algebras which are fully characterized in [15], Section 29.2. The multiplicative loop \( Q^* \) of a semifield \( Q \) in the second class is given by
\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\star
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}
= \lambda_{(x_1, x_2)}
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}
= \begin{pmatrix}
x_1 & -cz_2 - x_2 \\
x_2 & x_1 + rz_2
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix},
\]
where \((x_1, x_2)^t, (y_1, y_2)^t \in \mathbb{C}^2 \setminus \{(0, 0)^t\}, \ r \geq 0 \text{ and } c = c_1 + ic_2 \in \mathbb{C}, c_2 \geq 0 \text{ are constants such that for all } v \in \mathbb{R} \text{ one has } 0 < P_{v,c}(v) = v^4 + (2Re c - r^2)v^2 - 2rv + |c|^2 - 1 \) (cf. [15], p. 83), \( \bar{z} \) is the complex conjugate of \( z \in \mathbb{C} \). The kernel \( K_r \) of the semifield \( Q_{(r,c)} \) defined by (21) is \( K_r = \{(k, 0)^t | k \in \mathbb{C}\} \) and the centre \( Z \) of \( Q_{(r,c)} \) is \( \{ (k, 0)^t | k \in \mathbb{R} \} \).

**Proposition 4.1.** The multiplicative loop \( Q^*_{(r,c)} \) is the direct product of the group \( \mathbb{R} \) and a loop \( L_{(r,c)} \) realized on \( S^3 \) and having the multiplication
\[
\begin{pmatrix}
x_1 & -cz_2 - x_2 \\
x_2 & x_1 + rz_2
\end{pmatrix}
\circ
\begin{pmatrix}
y_1 & -cy_2 - y_2 \\
y_2 & y_1 + ry_2
\end{pmatrix} = \begin{pmatrix}
z_1 & -cz_2 - z_2 \\
z_2 & z_1 + rz_2
\end{pmatrix},
\]
where \( z_1 = x_1y_1 - cxz_2y_2 - x_2y_2, z_2 = x_2y_1 + x_1y_2 + rz_2y_2, |\det(\lambda_{(x_1, x_2)})| = |\det(\lambda_{(y_1, y_2)})| = 1 = |\det(\lambda_{(z_1, z_2)})| \). The group generated by all left translations of \( Q^*_{(r,c)} \), respectively of \( L_{(r,c)} \), is the group \( \text{GL}_2(\mathbb{C}) \), respectively the group of complex \((2 \times 2)\)-matrices which have absolute value 1. The group generated by all translations of \( L_{(r,c)} \), respectively of \( Q^*_{(r,c)} \), is the group \( \text{SL}_4(\mathbb{R}) \), respectively the direct product \( \mathbb{R} \times \text{SL}_4(\mathbb{R}) \).

**Proof.** Let \( \lambda_{(x_1, x_2)} \) be a matrix in (21). If \( x_2 \neq 0 \) then
\[
\det(\lambda_{(x_1, x_2)}) \in \mathbb{C} \quad \text{and} \quad \lambda_{(x_1, x_2)}\lambda_{(x_1, x_2)^t} \notin \mathbb{R} \cdot I,
\]
where $I$ is the identity matrix. Hence the group $G_{Q_{(r,c)}^*}$ generated by the left translations of $Q_{(r,c)}^*$ is the group $GL_2(\mathbb{C})$. The loop $Q_{(r,c)}^*$ has a central subgroup $Z_0^* = \{(k,0)^t \mid k > 0\} \cong \mathbb{R}$. The set $S_{(r,c)}$ of matrices

$$
\lambda_{(x_1,x_2)} = \begin{pmatrix} x_1 - c\bar{x}_2 - x_2 \\ x_2 - \bar{x}_1 + rx_2 \end{pmatrix}, \quad |\det(\lambda_{(x_1,x_2)})| = 1,
$$

topologically generates the group $\Delta$ of complex matrices $A$ with $|\det(A)| = 1$ and the map $S_{(r,c)} \to S_{(r,c)}Z_0/Z_0$, where $Z_0$ is the group of the left translations by the elements of $Z_0^*$, is bijective. The product $\circ : S_{(r,c)} \times S_{(r,c)} \to S_{(r,c)}$ given by (22) in the assertion yields a loop $L_{(r,c)}$ diffeomorphic to $S^3$ because $L_{(r,c)}$ is a system of representatives with respect to the subgroup $\{(k_0 k^t e^t), k > 0, l \in \mathbb{C}, s \in \mathbb{R}\}$ in the group $\Delta$. Hence the multiplicative loop $Q_{(r,c)}^*$ of $Q_{(r,c)}$ is isomorphic to the direct product of $\mathbb{R}$ and $L_{(r,c)}$.

The group generated by all translations of the compact loop $L_{(r,c)}$ is $SL_4(\mathbb{R})$ (cf. [F11], Proposition 11). Since $Q_{(r,c)}^*$ is the direct product of $\mathbb{R}$ and $L_{(r,c)}$, the group generated by all translations of $Q_{(r,c)}^*$ is the direct product $\mathbb{R} \times SL_4(\mathbb{R})$. This proves the assertion. $\blacksquare$

Let $\varphi : \mathbb{R} \to \text{Spin}_3(\mathbb{R})$, $\varphi(1) = 1$, be a continuous mapping and $\mathbb{H} = (\mathbb{R}^4, +, \cdot)$ be the skew field of quaternions. Then $\mathbb{H}_\varphi = (\mathbb{R}^4, +, \circ)$ with the multiplication $\circ$ given by $0 \circ x = 0$ and for $m \neq 0$ by $m \circ x = m \cdot x\varphi(|m|) = m \cdot \varphi(|m|)^{-1} \cdot x \cdot \varphi(|m|)$ is a four-dimensional topological quasifield. The kernel $K_\varphi$ of $\mathbb{H}_\varphi$ is isomorphic to the field $\mathbb{C}$ precisely if $\varphi(\mathbb{R}_{>0})$ lies in a subfield of $\mathbb{H}$ isomorphic to $\mathbb{C}$ (cf. [H1], pp. 234–238). The multiplicative loop $\mathbb{H}_\varphi^*$ of $\mathbb{H}_\varphi^*$ is defined by the multiplication

$$
m \circ x = M_{(m_1,m_2)}(x_1, x_2) = \begin{pmatrix} m_1 & -\bar{m}_2 \varphi(|m|)^2 \\ m_2 & \bar{m}_1 \varphi(|m|)^2 \end{pmatrix} (x_1, x_2),
$$

with $x = x_1 + jx_2$, $m = m_1 + jm_2$, $x_1, x_2, m_1, m_2 \in \mathbb{C}$.

**Proposition 4.2.** The set $\Lambda_{\mathbb{H}_\varphi^*}$ of all left translations of the loop $\mathbb{H}_\varphi^*$ contains the group $\text{Spin}_3(\mathbb{R})$ and the group topologically generated by $\Lambda_{\mathbb{H}_\varphi^*}$ is $\text{Spin}_3(\mathbb{R}) \times \mathbb{C}$.

**Proof.** For each matrix $M_{(m_1,m_2)}$ one has $M_{(m_1,m_2)}^t \cdot \bar{M}_{(m_1,m_2)} = (m_1 \bar{m}_1 + m_2 \bar{m}_2) \cdot I \in \mathbb{R}I$, where $I$ is the identity matrix, and $\det(M_{(m_1,m_2)}) = (m_1 \bar{m}_1 + m_2 \bar{m}_2)(\varphi(|m|)^2) \in \mathbb{C}^*$. Hence the group $G_{\mathbb{H}_\varphi^*}$ generated by the left translations of $\mathbb{H}_\varphi^*$ is the group $\text{Spin}_3(\mathbb{R}) \times \mathbb{C}$ and $\varphi(u, x, y) = 1, b(u, x, y) = 0$ (cf. Proposition 3.1). Since the matrix $M_{(u,x,y)}$ in (2) coincides with $M_{(m_1,m_2)}$ given by (23) and $\det(M_{(u,x,y)}) = u^2 e^{i c(u,x,y)}$, we obtain $u = \sqrt{|\det(M_{(m_1,m_2)})|^2} = |m|$ and $e^{i c(u,x,y)} = \frac{\det(M_{(m_1,m_2)})}{u^2} = (\varphi(|m|))^2$. Hence the function $c$ depends only on the variable $u = |m|$. According to Proposition 3.6 the set $\Lambda_{\mathbb{H}_\varphi^*}$ contains the group $\text{Spin}_3(\mathbb{R})$ and the assertion is proved. $\blacksquare$

Let $Q$ be the quasifield given by formula (2) in [H2], p. 87. For $a_2 \neq 0$ the multiplication of $Q$ is given by

$$
\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 x_1 - a_2 \bar{x}_2 + \frac{a_2 \varphi(a_2/|a_2|)}{\sqrt{1 + |\varphi(a_2/|a_2|)|^2}}(x_1 - x_1) \\ a_1 x_2 + a_2 \bar{x}_1 - \frac{a_2 \varphi(a_2/|a_2|)}{\sqrt{1 + |\varphi(a_2/|a_2|)|^2}}(x_2 - \bar{x}_2) \end{pmatrix},
$$

for $a_2 \neq 0$. The group $G_Q$ generated by the left translations of $Q^*$ is the group $GL_2(\mathbb{C})$. The loop $Q^*$ has a central subgroup $Z_0^* = \{(k,0)^t \mid k > 0\} \cong \mathbb{R}$.
where $a_i, x_i \in \mathbb{C}, i = 1, 2,$ and $\varrho : S^1 \to \{il | l \in \mathbb{R}\}$ is a continuous non-constant function having pure imaginary values, $(a_i^1) \circ (x_i^1) = (a_i x_i)$ for $a_2 = 0$. The kernel $K_1$ of the quasifield $Q_\varrho$ is $K_1 = \{(x) | x, y \in \mathbb{R}\}$ such that $(a_1^1) \circ (x) = (a_1 x - a_2 y)$. The coordinate change $T : \mathbb{C}^2 \to \mathbb{C}^2, (r + si, u + vi)^t \mapsto (r + ui, s + vi)^t$ transforms the kernel $K_1$ to $K_2 = \{(x + iy, 0)^t \mid x, y \in \mathbb{R}\} = \{(z, 0)^t \mid z \in \mathbb{C}\}$ and the multiplication of the loop $Q_\varrho^*$ is given by

\[
\begin{pmatrix}
    a_{11} + ia_{12} \\
    a_{21} + ia_{22} 
\end{pmatrix}
\circ
\begin{pmatrix}
    x_{11} + ix_{12} \\
    x_{21} + ix_{22} 
\end{pmatrix}

= T^{-1}
\begin{pmatrix}
    a_{11} + ia_{21} -a_{12} + ia_{22} + \frac{2a_{21} \Im(\varrho(a_2/|a_2|))}{\sqrt{1 + (\Im(\varrho(a_2/|a_2|))^2} \\
    a_{12} + ia_{22} a_{11} - ia_{21} + \frac{2a_{22} \Im(\varrho(a_2/|a_2|))}{\sqrt{1 + (\Im(\varrho(a_2/|a_2|))^2}} 
\end{pmatrix}
\begin{pmatrix}
    x_{11} + ix_{21} \\
    x_{12} + ix_{22} 
\end{pmatrix}.
\]

**Proposition 4.3.** The group topologically generated by the left translations of $Q_\varrho^*$ is the group $\text{SL}_2(\mathbb{C}) \times \mathbb{R}$. The loop $Q_\varrho^*$ is a central extension of $Z_0^* = \{(c, 0)^t \mid c > 0\} \cong \mathbb{R}$ by a loop homeomorphic to $S^3$.

**Proof.** The assertion is proved in Proposition 13 of [F2].

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**References**


