COMMENT FOR THE "FROM THE EDITOR-IN-CHIEF" COLUMN IN LAA

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Let H be a Hilbert space with $1 < \dim H < \infty$ and denote by $\mathcal{S}(H)$ the convex set of all positive semidefinite operators on H with unit trace. The elements of $\mathcal{S}(H)$ are called density operators and they represent quantum states of a finite quantum system. The von Neumann entropy $S(\rho)$ of a density operator $\rho \in \mathcal{S}(H)$ is defined by $S(\rho) = -\operatorname{Tr}(\rho \log_2 \rho)$.

The main result Theorem 2.1 in the recent paper

K. He, Q. Yuan and J. Hou, *Entropy-preserving maps on quantum states,* Linear Algebra Appl. **467** (2015), 243–253.
reads as follows.

Theorem 2.1. *A surjective map $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ satisfies*

$$S(t\rho + (1-t)\sigma) = S(t\phi(\rho) + (1-t)\phi(\sigma))$$

for all $t \in [0,1]$ and $\rho, \sigma \in \mathcal{S}(H)$ if and only if it is implemented by a unitary or antiunitary operator U on H meaning that ϕ is of the form $\phi(\rho) = U\rho U^*$, $\rho \in \mathcal{S}(H)$.

The sufficiency part of the result is trivial, so the real content relies in the necessity part. In the references of the paper one can find our work

L. Molnár and G. Nagy, *Transformations on density operators that leave the Holevo bound invariant,* Int. J. Theor. Phys. **53** (2014), 3273–3278.

We formulate its main result, Theorem 1.

Theorem 1. *Let $t_1, \ldots, t_n$ be a given collection of positive real numbers with sum 1. Assume that $\phi : \mathcal{M} \to \mathcal{S}(H)$ is a transformation defined on a dense subset $\mathcal{M}$ of $\mathcal{S}(H)$ with the property that*

$$S(\sum_{k=1}^n t_k \phi(\rho_k)) - \sum_{k=1}^n t_k S(\phi(\rho_k)) = S(\sum_{k=1}^n t_k \rho_k) - \sum_{k=1}^n t_k S(\rho_k)$$

holds for all collections $\rho_1, \dots, \rho_n$ of density operators in $\mathcal{M}$. Then there is a unitary or antiunitary operator U on H such that

$$\phi(\rho) = U\rho U^*, \quad \rho \in \mathcal{M}.$$

The following is just a trivial consequence of Theorem 1.

Corollary. *Let $t_1, \dots, t_n$ be a given collection of positive real numbers with sum 1. Assume that $\phi : \mathcal{M} \to \mathcal{S}(H)$ is a transformation defined on a dense subset $\mathcal{M}$ of $\mathcal{S}(H)$ with the property that*

$$S(\sum_{k=1}^{n} t_k \phi(\rho_k)) = S(\sum_{k=1}^{n} t_k \rho_k)$$

holds for all collections $\rho_1, \dots, \rho_n$ of density operators in $\mathcal{M}$. Then there is a unitary or antiunitary operator U on H such that

$$\phi(\rho) = U\rho U^*, \quad \rho \in \mathcal{M}.$$

Indeed, assuming $\phi : \mathcal{M} \to \mathcal{S}(H)$ is a transformation with the property in the last displayed formula, plugging $\rho_1 = \rho_2 = \ldots = \rho_n$ into it we immediately obtain that ϕ preserves the von Neumann entropy meaning that $S(\phi(\rho)) = S(\rho)$, $\rho \in \mathcal{S}(H)$. Theorem 1 clearly applies and we have the desired conclusion.

Apparently, Corollary is a considerably stronger result than Theorem 2.1 for the following reasons:

- In Corollary the preservation of the von Neumann entropy of convex combinations is assumed only for one particular nontrivial convex combination and not for all,
- there the transformation ϕ needs not to be assumed surjective and
- the domain of ϕ needs not to be the whole set $\mathcal{S}(H)$, only a dense subset of it (e.g., the set of nonsingular states which is an important subset in considerations where differential geometrical tools are applied concerning the state space).

Beside all these, one may find that the proof of Theorem 1 is shorter and simpler than that of Theorem 2.1.

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