MAPS ON THE POSITIVE DEFINITE CONE OF A C^*-ALGEBRA PRESERVING CERTAIN QUASI-ENTROPIES

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Abstract. We describe the structure of those bijective maps on the cone of all positive invertible elements of a C^*-algebra with a normalized faithful trace which preserve certain kinds of quasi-entropy. It is shown that essentially any such map is equal to a Jordan *-isomorphism of the underlying algebra multiplied by a central positive invertible element.

1. Introduction

Relative entropy is a numerical quantity which is of fundamental importance in information sciences. It is used to measure dissimilarity between probability distributions in classical information theory, or between quantum states (represented by density operators) in quantum information theory. The original concept of relative entropy was generalized in many different ways in the past decades. In the classical theory, probably the most extensively studied such generalization is the so-called 'f-divergence' introduced by I. Csiszár. As for quantum information science, a natural quantum analogue appeared under the name 'quantum f-divergence', but in fact an even more general concept called 'quasi-entropy' was introduced by D. Petz and then studied in details. For its precise definition in the setting of finite quantum systems we refer the reader to the paper [15] and to the monograph [13] (Part II, Section 7).

Motivated by the classical result of Wigner concerning the structure of quantum mechanical symmetry transformations (i.e., maps on pure states preserving transition probability), in the paper [6] we described the bijective maps of the set of all density operators (positive semidefinite operators with trace 1) on a finite dimensional Hilbert space which preserve the usual quantum relative entropy, i.e., the one due to Umegaki. After this, first in [11] we were able to drop the condition of bijectivity in the former result and then in [9] we could present the complete description of all transformations on the set of density operators which preserve any quantum f-divergence, f being any strictly convex function. (For some more preservers on quantum structures we refer to Chapter 2 in the volume [5].)

For several reasons, in many cases (even in physically motivated ones) transformations defined not only on the set of density operators but on the whole cone of positive (definite or semidefinite) operators are also studied. Especially, when differential geometrical considerations are made and corresponding tools are used, it is very natural to consider the cone of all positive invertible (i.e., positive definite) operators.

As recent literature on investigations in this direction we mention our paper [10] where bijective maps on the cone of positive definite matrices preserving rather general Bregman divergences or Jensen divergences were described and also the paper [17] by Virosztek who managed to determine the structure of all bijective maps on the cone of all positive semidefinite matrices which preserve a general quantum f-divergence.

Our aim here is to make steps towards the descriptions of maps preserving quasi-entropies which concept, as mentioned above, is more general than the one called quantum f-divergence.

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A closer look at the clever proof given in [17] clearly shows and can convince anybody that the description of general quasi-entropy preservers on cones is most probably a very difficult problem, if it is doable at all. Therefore, in what follows we consider the physically most important quasi-entropies and determine the structures of their preservers.

For these reasons, that is since here we deal with some particular quasi-entropies, we do not give their general definition but refer the reader to the sources [15], [13]. Let us merely recall that Petz's general quasi-entropies form a large class of numerical quantities of two variables varying over the positive definite cone (i.e., the set of all positive invertible elements) of a finite dimensional C^*-algebra $\mathcal{A}$ which are parametrized by two objects: on the one hand, by a continuous numerical function on the positive real line and, on the other hand, by an element of $\mathcal{A}$. It is written on page 113 in [13] that most of the numerical functions are in fact physically irrelevant, the important ones are just the following: $x \mapsto x \log x$, $x \mapsto -\log x$ and $x \mapsto x^\alpha$ with some exponent $\alpha \in \mathbb{R}$.

As mentioned above the bijective quantum f-divergence preservers on the cone of all positive semidefinite operators on a finite dimensional Hilbert space have been determined in [17]. As to quasi-entropies, this means the cases where the parametrizing real functions of quasi-entropies under considerations are general but the other parameter, namely the element of the C^*-algebra behind, is the identity. We have already pointed out above that the fully general cases (general real functions and general C^*-algebra elements as parameters) seem at the moment hopeless to handle, therefore we deal with the physically most important concrete cases regarding the function parameter and, on the other hand, we consider the operator parameter rather general (any invertible element). We also mention that although in [15] finite-dimensional C^*-algebras were treated, in this paper we do not make restriction on dimension and hence obtain, from the mathematical point of view, general results.

Let us now fix the notation and give some necessary definitions. In what follows let $\mathcal{A}$ be an arbitrary (unital) C^*-algebra with unit 1. Denote by $\mathcal{A}_s$ the space of all self-adjoint elements of $\mathcal{A}$, let $\mathcal{A}_+$ stand for the set of all positive elements of $\mathcal{A}$, and denote by $\mathcal{A}_+^{-1}$ the set of all invertible elements of $\mathcal{A}_+$. We call $\mathcal{A}_+^{-1}$ the positive definite cone of $\mathcal{A}$.

By a normalized trace on $\mathcal{A}$ we mean a positive linear functional Tr which satisfies $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$, $x, y \in \mathcal{A}$, and $\mathrm{Tr}\, 1 = 1$. It is called faithful if for any $a \in \mathcal{A}_+$, the equality $\mathrm{Tr}\, a = 0$ implies $a = 0$.

In what follows we consider a fixed normalized trace Tr on $\mathcal{A}$ (this implicitly means that we assume that such a trace does exist). We define the quasi-entropies in what we are interested in this paper as follows. For an element $w \in \mathcal{A}$ we set

$$(1) \qquad \begin{aligned} S_r^w(a,b) &= \mathrm{Tr}(b(\log b)ww^* - bw(\log a)w^*) \\ &= \mathrm{Tr}(ww^*b\log b - w(\log a)w^*b), \end{aligned}$$

$$(2) \qquad \begin{aligned} S_s^w(a,b) &= \mathrm{Tr}(a^{1/2}w^*wa^{1/2}\log a - waw^*\log b) \\ &= \mathrm{Tr}(w^*wa\log a - w^*(\log b)wa), \end{aligned}$$

$$(3) \qquad S_\alpha^w(a,b) = \mathrm{Tr}(a^{1-\alpha}w^*b^\alpha w)$$

for all $a, b \in \mathcal{A}_+^{-1}$. These are the quasi-entropies corresponding to the numerical functions $x \mapsto x \log x$, $x \mapsto -\log x$, and $x \mapsto x^\alpha$, respectively, see [13], page 113. For example, if $\mathcal{A}$ is the algebra of all $n \times n$ complex matrices and w is the identity matrix, then the second quantity is just the usual Umegaki relative entropy.

Observe that the quantities S_r^w, S_s^w are closely related. Namely, we clearly have

$$(4) \qquad S_s^w(a,b) = S_r^{w^*}(b,a), \quad a, b \in \mathcal{A}_+^{-1}.$$

In what follows we determine the structure of all bijective maps of the positive definite cone of a C^*-algebra which are symmetries with respect to the above defined quasi-entropies meaning that they are bijective maps which preserve any of those quantities.

2. Results

To formulate our results we need the concept of Jordan *-isomorphisms. A Jordan *-isomorphism between $*$-algebras is a bijective linear map J which respects the square and the * operations, i.e., which satisfies $J(x^2) = J(x)^2$ (or, equivalently, $J(xy + yx) = J(x)J(y) + J(y)J(x)$) and $J(x^*) = J(x)^*$ for all x (and y) from its domain.

Recall moreover that a factor von Neumann algebra is a von Neumann algebra with trivial center. It is well-known that on any finite von Neumann factor there exists a unique faithful normalized trace (cf., [7, Proposition 2]).

Our first result reads as follows.

Theorem 1. *Assume $\mathcal{A}$ is a C^*-algebra,* Tr *is a faithful normalized trace on $\mathcal{A}$, and pick an invertible element $w \in \mathcal{A}$. Let $\phi : \mathcal{A}_+^{-1} \to \mathcal{A}_+^{-1}$ be a bijective map. Then ϕ preserves the quasi-entropy S_r^w, i.e., ϕ satisfies*

$$S_r^w(\phi(a), \phi(b)) = S_r^w(a, b), \quad a, b \in \mathcal{A}_+^{-1} \tag{5}$$

*if and only if we have a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{A}$ and a central element $c \in \mathcal{A}_+^{-1}$ such that*

$$\phi(a) = cJ(a), \quad a \in \mathcal{A}_+^{-1} \tag{6}$$

and c, J fulfill

$$\mathrm{Tr}(cwJ(x)w^*J(y)) = \mathrm{Tr}(wxw^*y) \tag{7}$$

for every $x, y \in \mathcal{A}$.

If $\mathcal{A}$ is a finite von Neumann factor with unique normalized trace Tr*, then any bijective map ϕ on $\mathcal{A}_+^{-1}$ fulfills* (5) *if and only if it extends to a map Φ on $\mathcal{A}$ which is either an algebra *-isomorphism or an algebra *-antiisomorphism satisfying $\Phi(w) = \lambda w$ in the former case and satisfying $\Phi(w) = \lambda w^*$ in the latter one, where λ is some complex number of modulus 1.*

Remark that by the equality (4) we immediately have the structure of bijective maps on $\mathcal{A}_+^{-1}$ which preserve the quasi-entropy S_s^w. This is so straightforward that we do not formulate the corresponding result. Instead, let us see what the above result says in the case where $\mathcal{A}$ is the algebra $\mathbb{M}_n$ of all $n \times n$ complex matrices. Denote by $\mathbb{P}_n$ the positive definite cone of $\mathbb{M}_n$. Below ^{t} stands for the transpose of matrices.

Corollary 2. *Assume $w \in \mathbb{M}_n$ is a non-singular matrix. Let $\phi : \mathbb{P}_n \to \mathbb{P}_n$ be a bijective map. It satisfies*

$$S_r^w(\phi(a), \phi(b)) = S_r^w(a, b), \quad a, b \in \mathbb{P}_n$$

if and only if either we have a unitary matrix $u \in \mathbb{M}_n$ and a complex number λ of modulus 1 with $uw = \lambda wu$ such that

$$\phi(a) = uau^*, \quad a \in \mathbb{P}_n,$$

or we have a unitary matrix $u \in \mathbb{M}_n$ and a complex number λ of modulus 1 with $uw = \lambda w^* u$ such that

$$\phi(a) = ua^t u^*, \quad a \in \mathbb{P}_n.$$

Our second main result gives the precise structure of bijective maps on positive definite cones which preserve the quasi-entropy S_α^w. It reads as follows.

Theorem 3. *Assume $\mathcal{A}$ is a C^*-algebra and* Tr *is a faithful normalized trace on $\mathcal{A}$. Let α be a real number, $\alpha \neq 0, 1/2, 1$, and pick an invertible element $w \in \mathcal{A}$. The bijective map $\phi : \mathcal{A}_+^{-1} \to \mathcal{A}_+^{-1}$ preserves S_α^w, i.e., ϕ satisfies*

$$S_\alpha^w(\phi(a), \phi(b)) = S_\alpha^w(a, b), \quad a, b \in \mathcal{A}_+^{-1} \tag{8}$$

*if and only if we have a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{A}$ and a central element $c \in \mathcal{A}_+^{-1}$ such that*

$$\phi(a) = cJ(a), \quad a \in \mathcal{A}_+^{-1} \tag{9}$$

and c, J fulfill

$$\mathrm{Tr}(cwJ(x)w^*J(y)) = \mathrm{Tr}(wxw^*y) \tag{10}$$

for every $x, y \in \mathcal{A}$.

If $\mathcal{A}$ is a finite von Neumann factor with unique normalized trace Tr*, then the bijective map ϕ on $\mathcal{A}_+^{-1}$ fulfills* (8) *if and only if it extends to a map Φ which is either an algebra *-isomorphism or an algebra *-antiisomorphism of $\mathcal{A}$ satisfying $\Phi(w) = \lambda w$ in the former case and satisfying $\Phi(w) = \lambda w^*$ in the latter one with some complex number λ of modulus 1.*

Still assuming that $\mathcal{A}$ is a finite von Neumann factor, in the case where $\alpha = 1/2$, the bijective map $\phi : \mathcal{A}_+^{-1} \to \mathcal{A}_+^{-1}$ satisfies (8) *if and only if there is a map $\Phi : \mathcal{A} \to \mathcal{A}$ such that*

$$\phi(a) = (\phi(1)^{1/4}\Phi(a^{1/2})\phi(1)^{1/4})^2, \quad a \in \mathcal{A}_+^{-1}$$

*and Φ is either an algebra *-isomorphism or an algebra *-antiisomorphism satisfying*

$$\Phi(w) = \lambda\phi(1)^{1/4}w\phi(1)^{1/4}$$

in the former case and satisfying

$$\Phi(w) = \lambda\phi(1)^{1/4}w^*\phi(1)^{1/4}$$

in the latter one where λ is some complex number of modulus 1.

Observe that the cases where $\alpha = 0$ or $\alpha = 1$ are uninteresting, and also that one can easily obtain a result from the above theorem like Corollary 2 for the case of matrix algebras.

3. Proofs

This section is devoted to the proofs of our results. In what follows we shall use certain fundamental properties of Jordan *-isomorphisms that we collect below.

Let $\mathcal{A}$ be a C^*-algebra and $J : \mathcal{A} \to \mathcal{A}$ be a Jordan *-isomorphism. We have

$$J(aba) = J(a)J(b)J(a), \quad a, b \in \mathcal{A}, \tag{11}$$

and

$$J(a^n) = J(a)^n, \quad a \in \mathcal{A} \tag{12}$$

for every non-negative integer n, see [14, 6.3.2 Lemma]. In particular, J is unital meaning that $J(1) = 1$. Since J is positive (it sends positive elements to positive elements), it is bounded. In fact, it can be shown that J is an isometry, see, e.g., [4]. By [16, Proposition 1.3], J preserves invertibility, namely we have

$$J(a^{-1}) = J(a)^{-1} \tag{13}$$

for every invertible element $a \in \mathcal{A}$. It follows that J preserves the spectrum and, using continuous function calculus, from (12) we can infer that

$$J(f(a)) = f(J(a)) \tag{14}$$

holds for all $a \in \mathcal{A}_s$ and continuous real function f defined on the spectrum of a. According to [14, 9.9.16 Proposition], J sends central elements to central elements.

A fundamental theorem of Herstein says that if J is a Jordan isomorphism onto a prime algebra (an algebra $\mathcal{A}$ is called prime if, for any $a, b \in \mathcal{A}$, the equality $a\mathcal{A}b = \{0\}$ implies $a = 0$ or $b = 0$), then J is either a homomorphism (meaning that it is multiplicative) or an antihomomorphism (i.e., it is antimultiplicative). See, e.g., [14, 6.3.7 Theorem]. It follows rather easily from the comparison theorem for projections in von Neumann algebras that any von Neumann factor is a prime algebra. That means that Jordan *-isomorphisms onto such algebras are either algebra *-isomorphisms or algebra *-antiisomorphisms.

We shall need two characterizations of central elements in C^*-algebras, we first present those auxiliary results. For the proofs we recall the following. By Gelfand-Naimark-Segal theorem, any C^*-algebra $\mathcal{A}$ is isometrically (algebra) *-isomorphic to a C^*-subalgebra of the algebra $B(H)$ of all bounded linear operators on a Hilbert space H. Let us denote this algebra of operators by the same symbol $\mathcal{A}$. The weak (strong) closure $\mathcal{B}$ of $\mathcal{A}$ is a von Neumann algebra what we call von Neumann algebra envelop of $\mathcal{A}$. By the celebrated Kaplansky density theorem the positive part of the unit ball of $\mathcal{A}$ is weakly (strongly) closed in the positive part of the unit ball of $\mathcal{B}$.

Lemma 4. *Let $\mathcal{A}$ be a C^*-algebra, $c \in \mathcal{A}_s$ be a self-adjoint element such that the map*

$$\psi_c : a \mapsto \exp(\log a + c), \quad a \in \mathcal{A}_+^{-1}$$

is additive or, more generally, is a Jensen map meaning that it satisfies

$$\psi_c((a+b)/2) = (\psi_c(a) + \psi_c(b))/2, \quad a, b \in \mathcal{A}_+^{-1}.$$

Then c is a central element of $\mathcal{A}$.

Proof. Observe first that the map $\psi_c : \mathcal{A}_+^{-1} \to \mathcal{A}_+^{-1}$ is bijective. Assume now that it is a Jensen map. The structure of such transformations was determined in [8, Proposition 1]. Namely, we have that there exists a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{A}$ such that

$$\psi_c(a) = \psi_c(1)^{1/2} J(a) \psi_c(1)^{1/2}, \quad a \in \mathcal{A}_+^{-1}.$$

From this equality it follows that

$$\exp(-c/2)\exp(\log a + c)\exp(-c/2) = J(a), \quad a \in \mathcal{A}_+^{-1}. \tag{15}$$

Now, on the one hand, we have $J(x^2) = J(x)^2$, $x \in \mathcal{A}$ which easily implies that

$$\exp(2\log a + c) = \exp(\log a + c)\exp(-c)\exp(\log a + c)$$

holds for all $a \in \mathcal{A}_+^{-1}$. This means that for every $x \in \mathcal{A}_s$ we have

$$\exp(2x + c) = \exp(x + c)\exp(-c)\exp(x + c).$$

Setting $y = x + c$ and $d = -c$ we obtain that

$$\exp(2y + d) = \exp(y)\exp(d)\exp(y) \tag{16}$$

is valid for all $y \in \mathcal{A}_s$. On the other hand, we have that J satisfies $J(z^{-1}) = J(z)^{-1}$ for every invertible element $z \in \mathcal{A}$ (see (13))). By (15) this implies

$$\exp(-c/2)\exp(-\log a + c)\exp(-c/2) = \exp(c/2)\exp(-\log a - c)\exp(c/2)$$

for all $a \in \mathcal{A}_+^{-1}$ and hence we have

$$\exp(d)\exp(y)\exp(d) = \exp(y + 2d) \tag{17}$$

for all $y \in \mathcal{A}_s$. Now, from (17) and (16) it follows that for the fixed element $f = \exp(d) = \exp(-c)$ and for arbitrary $a = \exp(2y) \in \mathcal{A}_+^{-1}$ we have

$$fa^2f = \exp(4y + 2d) = (\exp(2y + d))^2 = (a^{1/2}fa^{1/2})^2.$$

Equivalently, we obtain

$$fa^4f = afa^2fa \tag{18}$$

for every $a \in \mathcal{A}_+^{-1}$ and hence for every $a \in \mathcal{A}_+$, too. Plug $1 + \lambda a$ into this equality for a given $a \in \mathcal{A}_+^{-1}$, and let λ vary over the set of all positive real numbers. Computing the coefficients of the first power of λ on both sides of the equality

$$f(1 + \lambda a)^4 f = (1 + \lambda a)f(1 + \lambda a)^2 f(1 + \lambda a)$$

we infer

$$4faf = af^2 + 2faf + f^2a$$

and this implies that

$$2faf = af^2 + f^2a \tag{19}$$

holds for all $a \in \mathcal{A}_+^{-1}$ and then for all $a \in \mathcal{A}$, too (the linear hull of $\mathcal{A}_+^{-1}$ is $\mathcal{A}$). Consider the von Neumann algebra envelop $\mathcal{B}$ of the C^*-algebra $\mathcal{A}$. We obtain that (19) holds true also for every $a \in \mathcal{B}$. Moreover, by Kaplansky density theorem, we also have that (18) holds for the elements $a \in \mathcal{B}_+$ (multiplication is strongly continuous on bounded sets).

Now, on the one hand, using (19), for any projection $p \in \mathcal{B}$ we have

$$2fpf = pf^2 + f^2p.$$

Multiplying this equality by p from both sides, we obtain $2pfpfp = 2pf^2p$. On the other hand, with regard to (18) we also have $fpf = pfpfp$ and thus we deduce

$$fp^2f = pf^2p.$$

But for any two positive elements a, b of a C^*-algebra, $ab^2a = ba^2b$ holds if and only if a, b commute, see [1, Proposition 1]. Therefore, it follows that f commutes with all projections in $\mathcal{B}$. Since the norm closed linear span of the set of projections in $\mathcal{B}$ is just $\mathcal{B}$, we infer that f is a central element in $\mathcal{B}$ and hence in $\mathcal{A}$, too. The conclusion on the centrality of c is immediate hence completing the proof of the lemma. □

The other characterization of central elements in C^*-algebras what we need reads as follows.

Lemma 5. *Let $d \in \mathcal{A}_+^{-1}$ and t be a real number such that $t \notin \{-1, 0, 1\}$. Assume $(dyd)^t = d^ty^td^t$ holds for all $y \in \mathcal{A}_+^{-1}$. Then d is a central element of $\mathcal{A}$.*

Proof. Clearly, we may assume that $t > 0$. We assert that

$$d^{-1}(dyd)^t d^{-1} = y^{1/2}(y^{1/2}d^2y^{1/2})^{t-1}y^{1/2} \tag{20}$$

holds for all $y \in \mathcal{A}_+^{-1}$. Indeed, for every non-negative integer n we have

$$d^{-1}(dyd)^n d^{-1} = y^{1/2}(y^{1/2}d^2y^{1/2})^{-1}(y^{1/2}d^2y^{1/2})^n y^{1/2}. \tag{21}$$

For any $y \in \mathcal{A}_+^{-1}$, using uniform approximation by polynomials on the spectrum of dyd (that is equal to the spectrum of $y^{1/2}d^2y^{1/2}$), it follows that in (21) the nth power function can be replaced by any continuous real function defined on the positive half line, in particular, by the tth power function, too. It follows that we obtain (20) for any $y \in \mathcal{A}_+^{-1}$. Let $\mathcal{B}$ be von Neumann algebra envelope of $\mathcal{A}$. As in the proof of the previous lemma, we apply Kaplansky's density theorem as follows. Picking $y' \in \mathcal{B}_+^{-1}$, there is a positive real number k' such that $k'1 \leq y'$, i.e., $y' - k'1 \in \mathcal{B}_+$. Therefore, we have a bounded net (z_γ) in $\mathcal{A}_+$ such that $y_\gamma = z_\gamma + k'1$ converges strongly to y'. Since every bounded continuous function is strongly continuous (cf. [12, 4.3.2. Theorem]) meaning that it preserves strongly convergent nets of hermitian operators, and here the spectra of the elements y_γ lie in a compact subinterval of the positive half-line, we can take strong limit in (20) and obtain that

$$y^{1/2}(y^{1/2}d^2y^{1/2})^{t-1}y^{1/2} = d^{-1}(dyd)^t d^{-1} = d^{-1}(d^t y^t d^t)d^{-1} = d^{t-1}y^t d^{t-1} \tag{22}$$

holds for all $y \in \mathcal{B}_+^{-1}$. Assume $t > 1$. Then the above equality clearly holds for all $y \in \mathcal{B}_+$, too. Choosing any projection p in $\mathcal{B}$ and plugging $y = p$ into (22), we have

$$p(pd^2p)^{t-1}p = d^{t-1}pd^{t-1}$$

which implies

$$pd^{t-1}pd^{t-1} = d^{t-1}pd^{t-1}.$$

Since d is invertible, it follows that

$$pd^{t-1}p = d^{t-1}p.$$

But the element on the left hand side of this equality is self-adjoint, and hence so must be the one on the right hand side. This means that for every projection p in $\mathcal{B}$, the elements d^{t-1} and p commute which immediately gives us that d is a central element in $\mathcal{B}$ and hence also in $\mathcal{A}$.

Let us now consider the case where $0 < t < 1$. Let k be a positive real number such that $k1 \leq d^2$. We have $ky \leq y^{1/2}d^2y^{1/2}$. Since the power function with exponent $1-t$ is operator monotone, we have $(y^{1/2}d^2y^{1/2})^{t-1} \leq k^{t-1}y^{t-1}$, $y \in \mathcal{B}_+^{-1}$. Therefore, from (22) we infer that

$$d^{t-1}y^t d^{t-1} \leq k^{t-1}y^t, \quad y \in \mathcal{B}_+^{-1}.$$

Letting y converge in norm to any projection $p \in \mathcal{B}$, it follows that

$$d^{t-1}pd^{t-1} \leq k^{t-1}p.$$

From this we can deduce that $pd^{t-1}pd^{t-1} = d^{t-1}pd^{t-1}$ holds for any projection $p \in \mathcal{B}$. Indeed, from the last displayed inequality it follows that the range of $d^{t-1}pd^{t-1}$ as a Hilbert space operator is included in the range of p. We can finish the proof just as in the case $t > 1$ above. □

For the proofs of our main results we shall also need the following simple observation.

Lemma 6. *Assume $\mathcal{A}$ is a C^*-algebra and* Tr *is a faithful normalized trace on $\mathcal{A}$. For any $a \in \mathcal{A}_s$ we have $a \geq 0$ if and only if $\operatorname{Tr} ax \geq 0$ holds for all $x \in \mathcal{A}_+^{-1}$. In particular, for any given $y \in \mathcal{A}$ satisfying $\operatorname{Tr} xy = 0$ for all $x \in \mathcal{A}_+^{-1}$, we have $y = 0$.*

Proof. The necessity part of the first statement is obvious. Indeed, if $a \geq 0$, then for every $x \in \mathcal{A}_+^{-1}$ we have $x^{1/2}ax^{1/2} \geq 0$ and hence $\operatorname{Tr} ax = \operatorname{Tr} x^{1/2}ax^{1/2} \geq 0$. As for the sufficiency, if $\operatorname{Tr} ax \geq 0$ holds for all $x \in \mathcal{A}_+^{-1}$, the same is true for all $x \in \mathcal{A}_+$ by the continuity of the trace functional. Writing $a = a_+ - a_-$, where $a_+, a_- \in \mathcal{A}_+$ with $a_+a_- = 0$, we have $0 \leq \operatorname{Tr} aa_- = -\operatorname{Tr} a_-^2 \leq 0$. This implies $\operatorname{Tr} a_-^2 = 0$. By the faithfulness of Tr we have $a_-^2 = 0$ yielding $a_- = 0$ and hence we can conclude that a is positive.

Assume $y \in \mathcal{A}$ is such that $\operatorname{Tr} xy = 0$ holds for all $x \in \mathcal{A}_+^{-1}$. Then we easily have that the same holds for the real and imaginary parts of y, i.e., for the elements

$$\frac{y + y^*}{2}, \frac{y - y^*}{2i},$$

too. Therefore, we can assume that y is self-adjoint. Let y_+, y_- be positive elements in $\mathcal{A}$ with $y = y_+ - y_-$ and $y_+y_- = 0$. For any $x \in \mathcal{A}_+^{-1}$ we have

$$\operatorname{Tr}(y_+^3 x) = \operatorname{Tr}(y_+ y y_+ x) = \operatorname{Tr} y(y_+ x y_+) = 0$$

which, by the faithfulness of Tr implies $y_+^3 = 0$ and then $y_+ = 0$ holds. In a similar way we obtain $y_- = 0$. This gives $y = 0$ completing the proof of the lemma. □

The proof of Theorem 1 rests heavily on a celebrated theorem of Kadison which asserts that every unital linear order isomorphism between C^*-algebras is a Jordan *-isomorphism. See [2, Corollary 5].

Proof of Theorem 1. First observe that for given $a, a' \in \mathcal{A}_+^{-1}$ we obtain

$$S_r^w(a, b) \geq S_r^w(a', b), \quad b \in \mathcal{A}_+^{-1}$$

if and only if

$$\operatorname{Tr}(\log a)w^*bw = \operatorname{Tr} w(\log a)w^*b \leq \operatorname{Tr} w(\log a')w^*b = \operatorname{Tr}(\log a')w^*bw$$

holds for all $b \in \mathcal{A}_+^{-1}$. Since w^*bw runs through the set $\mathcal{A}_+^{-1}$, by Lemma 6 this is equivalent to

$$\log a \leq \log a'.$$

It follows that by the preserver property (5) of the transformation ϕ we have $\log a \leq \log a'$ if and only if $\log \phi(a) \leq \log \phi(a')$. This means that the bijective map $\psi : \mathcal{A}_s \to \mathcal{A}_s$ defined by

$$\psi(x) = \log \phi(\exp x), \quad x \in \mathcal{A}_s \tag{23}$$

is an order automorphism of $\mathcal{A}_s$. We claim that ψ is affine, too. Indeed, we have

$$\begin{aligned} &\operatorname{Tr}(ww^*(\exp x)x - wyw^* \exp x) \\ &= \operatorname{Tr}(ww^*(\exp \psi(x))\psi(x) - w\psi(y)w^* \exp \psi(x)) \end{aligned} \tag{24}$$

for all $x, y \in \mathcal{A}_s$. Now, select $y, y' \in \mathcal{A}_s$. For arbitrary non-negative real numbers λ, μ with sum 1, we have

$$\begin{aligned} &\operatorname{Tr}(ww^*(\exp \psi(x))\psi(x) - w\psi(\lambda y + \mu y')w^* \exp \psi(x)) \\ &= \operatorname{Tr}(ww^*(\exp x)x - w(\lambda y + \mu y')w^* \exp x) \\ &= \lambda(\operatorname{Tr}(ww^*(\exp x)x - wyw^* \exp(x)) + \mu \operatorname{Tr}(ww^*(\exp x)x - wy'w^* \exp x) \\ &= \lambda \operatorname{Tr}(ww^*(\exp \psi(x))\psi(x) - w\psi(y)w^* \exp \psi(x)) \\ &\quad + \mu \operatorname{Tr}(ww^*(\exp \psi(x))\psi(x) - w\psi(y')w^* \exp \psi(x)) \\ &= \operatorname{Tr}(ww^*(\exp \psi(x))\psi(x) - w(\lambda\psi(y) + \mu\psi(y'))w^* \exp \psi(x)). \end{aligned}$$

From this we deduce that

$$\mathrm{Tr}(w\psi(\lambda y+\mu y')w^*\exp\psi(x))=\mathrm{Tr}(w(\lambda\psi(y)+\mu\psi(y'))w^*\exp\psi(x))$$

holds for all $x\in\mathcal{A}_s$. Since $w^*(\exp\psi(x))w$ runs through the whole set $\mathcal{A}_+^{-1}$, by Lemma 6 we obtain that

$$\psi(\lambda y+\mu y')=\lambda\psi(y)+\mu\psi(y')$$

is true for any $y,y'\in\mathcal{A}_s$ and non-negative real numbers λ,μ with $\lambda+\mu=1$. Therefore, ψ is affine and it follows that $\psi(.)-\psi(0)$ is a linear bijection on $\mathcal{A}_s$ preserving order in both directions. It is apparent that this map can be extended to a linear order automorphism of $\mathcal{A}$. Such a transformation necessarily maps the identity to a positive invertible element. By Kadison's theorem mentioned above, there is a Jordan *-isomorphism $J:\mathcal{A}\to\mathcal{A}$ such that ψ is of the form

$$\psi(x)=(\psi(1)-\psi(0))^{1/2}J(x)(\psi(1)-\psi(0))^{1/2}+\psi(0),\quad x\in\mathcal{A}_s.$$

Here $\psi(1)-\psi(0)$ is a positive invertible element of $\mathcal{A}$ and since a Jordan *-isomorphism preserves the set of invertible elements (see (13)), we can select elements $a_0\in\mathcal{A}_+^{-1}$ and $y_0\in\mathcal{A}_s$ such that $J(a_0)=(\psi(1)-\psi(0))^{1/2}$ and $J(y_0)=\psi(0)$. Therefore, by (11) we can write

$$\psi(x)=J(a_0xa_0+y_0),\quad x\in\mathcal{A}_s. \tag{25}$$

Now, from (24) we have

$$\begin{aligned}&\mathrm{Tr}(ww^*(\exp x)x-wyw^*\exp x)\\&=\mathrm{Tr}(ww^*(\exp J(a_0xa_0+y_0))J(a_0xa_0+y_0)\\&\quad-wJ(a_0ya_0+y_0)w^*\exp J(a_0xa_0+y_0))\end{aligned} \tag{26}$$

for any $x,y\in\mathcal{A}_s$. Rearranging this equality we deduce

$$\begin{aligned}&\mathrm{Tr}(ww^*(\exp J(a_0xa_0+y_0))J(a_0xa_0+y_0)-(\exp x)x))\\&=\mathrm{Tr}(wJ(a_0ya_0+y_0)w^*\exp J(a_0xa_0+y_0)-wyw^*\exp x)\end{aligned}$$

for all $x,y\in\mathcal{A}_s$. Since the left hand side of this equality does not depend on y, the same is true for the right hand side and hence we infer

$$\begin{aligned}&\mathrm{Tr}(wJ(a_0ya_0+y_0)w^*\exp J(a_0xa_0+y_0)-wyw^*\exp x)\\&=\mathrm{Tr}(wJ(y_0)w^*\exp J(a_0xa_0+y_0))\end{aligned} \tag{27}$$

for any $x,y\in\mathcal{A}_s$. Subtracting the right hand side of this equation from both sides, we get

$$\mathrm{Tr}(wJ(a_0ya_0)w^*\exp J(a_0xa_0+y_0)-wyw^*\exp x)=0,$$

i.e.,

$$\mathrm{Tr}(wJ(a_0ya_0)w^*\exp J(a_0xa_0+y_0))=\mathrm{Tr}(wyw^*\exp x)$$

for all $x,y\in\mathcal{A}_s$. We can rewrite this equality in the following way

$$\mathrm{Tr}(wJ(a_0ya_0)w^*\exp J(x+y_0))=\mathrm{Tr}(wyw^*\exp(a_0^{-1}xa_0^{-1})),\quad x,y\in\mathcal{A}_s. \tag{28}$$

For any $\lambda\in\mathbb{R}$, plug $\lambda 1$ in the place of x. We then have

$$\mathrm{Tr}(wJ(a_0ya_0)w^*\exp(\lambda 1+J(y_0)))=\mathrm{Tr}(wyw^*\exp(\lambda a_0^{-2})).$$

Considering the power series of the exponential function, by the uniqueness of coefficients we deduce that

$$\mathrm{Tr}(wJ(a_0ya_0)w^*\exp J(y_0)) = \mathrm{Tr}(wyw^*(a_0^{-2})^n)$$

holds for every non-negative integer n and $y \in \mathcal{A}_s$. Since the left hand side of this equality does not depend on n, the same must be true for the right hand side, too. In particular, we obtain that

$$\mathrm{Tr}(wyw^*) = \mathrm{Tr}(wyw^*a_0^{-2})$$

holds for every $y \in \mathcal{A}_s$. By Lemma 6 we infer that $a_0^{-2} = 1$ yielding $a_0 = 1$. By (28) we now have

$$\mathrm{Tr}(wJ(y)w^*\exp J(x+y_0)) = \mathrm{Tr}(wyw^*\exp x), \quad x, y \in \mathcal{A}_s.$$

Using (14), we can rewrite this equality in the following different way

$$\mathrm{Tr}(wJ(y)w^*J(\exp(\log a + y_0))) = \mathrm{Tr}(wyw^*a)$$

for all $a \in \mathcal{A}_+^{-1}$, $y \in \mathcal{A}_s$. The right hand side of this equality is additive in the variable a, hence that must hold for the left hand side as well. Using the second statement in Lemma 6, we can easily infer that the map

$$a \mapsto \exp(\log a + y_0), \quad a \in \mathcal{A}_+^{-1}$$

is additive which implies by Lemma 4 that y_0 is a central element in $\mathcal{A}$. According to (26) we have

$$\begin{aligned} &\mathrm{Tr}(ww^*(\exp J(x+y_0))J(x+y_0) - wJ(y+y_0)w^*\exp J(x+y_0)) \\ &= \mathrm{Tr}(ww^*(\exp x)x - wyw^*\exp x), \quad x, y \in \mathcal{A}_s. \end{aligned} \tag{29}$$

The left hand side of this equation can be computed as follows.

$$\begin{aligned} &\mathrm{Tr}(ww^*(\exp J(x+y_0))(J(x)+J(y_0)) - w(J(y)+J(y_0))w^*\exp J(x+y_0)) \\ &= \mathrm{Tr}(ww^*(\exp J(x+y_0))J(x) - wJ(y)w^*\exp J(x+y_0)) \\ &+ \mathrm{Tr}(ww^*(\exp J(x+y_0))J(y_0) - wJ(y_0)w^*\exp J(x+y_0)), \quad x, y \in \mathcal{A}_s. \end{aligned} \tag{30}$$

Now, recall that every Jordan *-isomorphism sends central elements to central elements, see the listed properties of Jordan *-isomorphisms at the beginning of the section. Therefore, $J(y_0)$ is central and hence the last term on the right hand side in (30) is zero. Consequently, by (29) and (30) we obtain

$$\begin{aligned} &\mathrm{Tr}(ww^*(\exp J(x+y_0))J(x) - wJ(y)w^*\exp J(x+y_0)) \\ &= \mathrm{Tr}(ww^*(\exp x)x - wyw^*\exp x), \quad x, y \in \mathcal{A}_s. \end{aligned} \tag{31}$$

Plugging $y = 0$ into (31), we infer

$$\mathrm{Tr}(ww^*(\exp J(x+y_0))J(x)) = \mathrm{Tr}(ww^*(\exp x)x)$$

and then obtain from (31) that

$$\mathrm{Tr}(wJ(y)w^*\exp J(x+y_0)) = \mathrm{Tr}(wyw^*\exp x)$$

holds for all $x, y \in \mathcal{A}_s$. Denote $c_0 = \exp J(y_0)$. It follows that

$$\mathrm{Tr}(wJ(y)w^*J(\exp x)c_0) = \mathrm{Tr}(wyw^*\exp x)$$

holds for all $x, y \in \mathcal{A}_s$ which easily implies

$$\mathrm{Tr}(wJ(y)w^*J(x)c_0) = \mathrm{Tr}(wyw^*x), \quad x, y \in \mathcal{A}_s$$

(indeed, we first have this for positive invertible and then for general self-adjoint x). By linearity we obtain the equality above for all $x, y \in \mathcal{A}$, too. This gives us that the equality (7) in the statement of the theorem holds. Finally, by (23) and (25), for any $x \in \mathcal{A}_s$ we have that

$$\phi(\exp x) = \exp(J(x) + J(y_0)) = \exp(J(x)) \exp(J(y_0)) = c_0 J(\exp x)$$

and this gives us that

$$\phi(a) = c_0 J(a), \quad a \in \mathcal{A}_+^{-1}.$$

The proof of the necessity part of the first statement in Theorem 1 is done.

As for the sufficiency part, it can be checked rather easily using the following observations. Since a Jordan *-isomorphism sends unit to unit, (7) implies that $\mathrm{Tr}(cww^* J(x)) = \mathrm{Tr}(ww^* x)$ holds for all $x \in \mathcal{A}$. Moreover, since using (12) we have that

$$J(a^n)J(a^m) = J(a)^n J(a)^m = J(a)^{n+m} = J(a^{n+m})$$

holds for all non-negative integers n, m, it follows from polynomial approximation of continuous functions that

$$J(f(a))J(g(a)) = J((fg)(a))$$

holds for any pair f, g of continuous functions defined on the spectrum of the element $a \in \mathcal{A}_+^{-1}$. In particular, we have

$$J(a)J(\log a) = J(a \log a), \quad a \in \mathcal{A}_+^{-1}.$$

Let us now see the proof of the statement concerning a finite von Neumann factor $\mathcal{A}$. By the first part of the theorem for any bijective map ϕ preserving the quasi-entropy S_r^w we have a positive invertible central element c of $\mathcal{A}$ and a Jordan *-isomorphism J on $\mathcal{A}$ such that ϕ is of the form (6), and (7) is satisfied. Since the center of the algebra $\mathcal{A}$ is trivial, c is a positive scalar multiple of the identity. We denote this scalar by the same symbol c. As $\mathcal{A}$ is a factor, it is a prime algebra and hence, by Hernstein's theorem (see the beginning of the section), we obtain that J is either an algebra *-isomorphism or an algebra *-antiisomorphism of $\mathcal{A}$. We deal only with the first case, the second one can be treated similarly. Denote the algebra *-isomorphism in question by Φ. One can see that $\mathrm{Tr}\,\Phi(.)$ is a normalized trace on $\mathcal{A}$, therefore, by its uniqueness, it follows that $\mathrm{Tr}\,\Phi(x) = \mathrm{Tr}\,x$, $x \in \mathcal{A}$, i.e., Φ is trace preserving. Let $v \in \mathcal{A}$ be an invertible element such that $\Phi(v) = w$. It follows from (7) that we have

$$\mathrm{Tr}(cvxv^*y) = \mathrm{Tr}\,\Phi(cvxv^*y) = \mathrm{Tr}(cw\Phi(x)w^*\Phi(y)) = \mathrm{Tr}(wxw^*y)$$

for all $x, y \in \mathcal{A}$. This gives us that

$$cvxv^* = wxw^*, \quad x \in \mathcal{A}. \tag{32}$$

Now observe the following. If u is an invertible element of $\mathcal{A}$ such that $uxu^* = x$ holds for all $x \in \mathcal{A}$, then we trivially have that u is unitary. Next, from $ux = xu$, $x \in \mathcal{A}$ it follows that u is in the center and hence it is a scalar multiple of the identity, the scalar being of modulus 1.

Using this observation, from (32) we deduce that for some complex number λ of modulus 1 we have $v = (\lambda/\sqrt{c})w$. Applying the transformation Φ on both sides, it follows that $\Phi(w) = (\sqrt{c}/\lambda)w$. As it was mentioned among the properties of Jordan *-isomorphisms, Φ is an isometry and therefore we obtain that $c = 1$. The proof of the theorem can now be completed easily. □

After this the proof of the corollary is very simple.

Proof of Corollary 2. Apparently, the algebra $\mathbb{M}_n$ is a finite factor, every algebra *-isomorphism of $\mathbb{M}_n$ is a unitary similarity transformation and every algebra *-antiisomorphism is the composition of a unitary similarity transformation with the transposition. The statement now follows trivially from the second part of Theorem 1. □

We finally present the proof of our second main result concerning the structure of bijections preserving the quasi-entropy S^w_α.

Proof of Theorem 3. Assume the real number α satisfies $\alpha \neq 0, 1/2, 1$ and the element $w \in \mathcal{A}$ is invertible. Let $\phi : \mathcal{A}_+^{-1} \to \mathcal{A}_+^{-1}$ be a bijective map fulfilling (8), i.e., suppose

$$S^w_\alpha(\phi(a), \phi(b)) = S^w_\alpha(a, b), \quad a, b \in \mathcal{A}_+^{-1}.$$

Define $\psi : \mathcal{A}_+^{-1} \to \mathcal{A}_+^{-1}$ by $\psi(b) = \phi(b^{1/\alpha})^\alpha$, $b \in \mathcal{A}_+^{-1}$. We have

$$\operatorname{Tr}(w\phi(a)^{1-\alpha} w^* \psi(b)) = \operatorname{Tr}(wa^{1-\alpha}w^*b), \quad a, b \in \mathcal{A}_+^{-1}.$$

Since the right hand side of this equality is additive in b and $w\phi(a)^{1-\alpha}w^*$ runs through the whole set $\mathcal{A}_+^{-1}$, it follows that ψ is an additive map on $\mathcal{A}_+^{-1}$. Clearly, ψ is bijective. The additive bijective maps on $\mathcal{A}_+^{-1}$ were determined in [1]. By [1, Lemma 8] (or, alternatively, applying the slightly more general statement [8, Proposition 1] on the structure of bijective Jensen maps on $\mathcal{A}_+^{-1}$ what we have used in the proof of Lemma 4), there is a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{A}$ such that

$$\psi(b) = \psi(1)^{1/2} J(b) \psi(1)^{1/2}, \quad b \in \mathcal{A}_+^{-1}, \tag{33}$$

or, equivalently,

$$\phi(b) = (\phi(1)^{\alpha/2} J(b^\alpha) \phi(1)^{\alpha/2})^{1/\alpha}, \quad b \in \mathcal{A}_+^{-1}. \tag{34}$$

In a similar fashion, we can deduce that there exists a Jordan *-isomorphism $J' : \mathcal{A} \to \mathcal{A}$ such that

$$\phi(a) = (\phi(1)^{(1-\alpha)/2} J'(a^{1-\alpha}) \phi(1)^{(1-\alpha)/2})^{1/(1-\alpha)}, \quad a \in \mathcal{A}_+^{-1}.$$

Applying (11), (14), from the equality of these two forms of ϕ we conclude that with $J^{-1}(\phi(1)) = c$ and $J'' = J^{-1} \circ J'$ we have

$$(c^{\alpha/2}(x^\alpha)c^{\alpha/2})^{1/\alpha} = (c^{(1-\alpha)/2} J''(x^{1-\alpha}) c^{(1-\alpha)/2})^{1/(1-\alpha)}, \quad x \in \mathcal{A}_+^{-1}.$$

This implies that

$$c^{(\alpha-1)/2}(c^{\alpha/2}xc^{\alpha/2})^{(1-\alpha)/\alpha} c^{(\alpha-1)/2} = J''(x^{(1-\alpha)/\alpha}), \quad x \in \mathcal{A}_+^{-1}.$$

Since Jordan *-isomorphisms preserve the inverse operation, see (13), we have

$$\begin{aligned} &c^{(\alpha-1)/2}(c^{\alpha/2}x^{-1}c^{\alpha/2})^{(1-\alpha)/\alpha} c^{(\alpha-1)/2} \\ &= c^{-(\alpha-1)/2}(c^{-\alpha/2}x^{-1}c^{-\alpha/2})^{(1-\alpha)/\alpha} c^{-(\alpha-1)/2}, \quad x \in \mathcal{A}_+^{-1}. \end{aligned}$$

Multiplying both sides by $c^{(\alpha-1)/2}$, from this equality we obtain

$$c^{(\alpha-1)}(c^{\alpha/2}yc^{\alpha/2})^{(1-\alpha)/\alpha} c^{(\alpha-1)} = (c^{-\alpha/2}yc^{-\alpha/2})^{(1-\alpha)/\alpha}, \quad y \in \mathcal{A}_+^{-1}.$$

Substituting $c^{-\alpha/2}yc^{-\alpha/2} = x$, we have

$$c^{(\alpha-1)}(c^\alpha x c^\alpha)^{(1-\alpha)/\alpha} c^{(\alpha-1)} = x^{(1-\alpha)/\alpha} \tag{35}$$

and then

$$(c^\alpha x c^\alpha)^{(1-\alpha)/\alpha} = c^{(1-\alpha)} x^{(1-\alpha)/\alpha} c^{(1-\alpha)}, \quad x \in \mathcal{A}_+^{-1}.$$

Setting $d = c^\alpha$ and $t = (1-\alpha)/\alpha$ we deduce

$$(dxd)^t = d^t x^t d^t, \quad x \in \mathcal{A}_+^{-1}.$$

Clearly, $t \neq -1, 0, 1$ holds by the conditions on α. Using Lemma 5 we infer that d is in the center of $\mathcal{A}$ and hence the same holds for c, too. We have already learned that Jordan *-isomorphisms map central elements to central elements. It follows that $\phi(1) = J(c)$ is central. Therefore, using (14), from (34) we obtain

$$\phi(x) = \phi(1)J(x^\alpha)^{1/\alpha} = eJ(x), \quad x \in \mathcal{A}_+^{-1}$$

with $e = \phi(1)$. Applying (8) one can easily verify that

$$\operatorname{Tr} eJ(a)w^*J(b)w = \operatorname{Tr} aw^*bw, \quad a, b \in \mathcal{A}_+^{-1}.$$

This trivially gives us the necessity part of the first statement in the theorem. The sufficiency part can be checked straightforwardly.

The proof of the second statement concerning finite von Neumann factors goes in the same way as in the corresponding part of the proof of Theorem 1, we do not present the details.

Let us discuss the remaining case where $\alpha = 1/2$. As in (34) above, we have

$$\phi(b) = (\phi(1)^{1/4}J(b^{1/2})\phi(1)^{1/4})^2, \quad b \in \mathcal{A}_+^{-1}$$

and we know

$$\operatorname{Tr}(\phi(1)^{1/4}J(a^{1/2})\phi(1)^{1/4}w^*\phi(1)^{1/4}J(b^{1/2})\phi(1)^{1/4}w) = \operatorname{Tr}(a^{1/2}w^*b^{1/2}w), \quad a, b \in \mathcal{A}_+^{-1}$$

or, equivalently,

$$\operatorname{Tr}(\phi(1)^{1/4}J(a)\phi(1)^{1/4}w^*\phi(1)^{1/4}J(b)\phi(1)^{1/4}w) = \operatorname{Tr}(aw^*bw), \quad a, b \in \mathcal{A}_+^{-1}.$$

Clearly, there is $f \in \mathcal{A}_+^{-1}$ such that $J(f) = \phi(1)^{1/4}$. Since J is either an algebra *-isomorphism or an algebra *-antiisomorphism, there exists $z \in \mathcal{A}$ such that $w = J(z)$ holds in the former case and $w = J(z^*)$ in the latter case. We have already learned in the proof of Theorem 1 that J is trace preserving. Assuming J is an algebra *-isomorphism, we can compute

$$\begin{gathered}\operatorname{Tr}(fafz^*fbfz) = \operatorname{Tr}(J(fafz^*fbfz)) = \\ \operatorname{Tr}(J(f)J(a)J(f)J(z^*)J(f)J(b)J(f)J(z)) \\ = \operatorname{Tr}(\phi(1)^{1/4}J(a)\phi(1)^{1/4}w^*\phi(1)^{1/4}J(b)\phi(1)^{1/4}w) = \operatorname{Tr}(aw^*bw)\end{gathered}$$

for all $a, b \in \mathcal{A}_+^{-1}$. It follows that

$$fz^*fbfzf = w^*bw$$

and hence

$$w^{*-1}fz^*fbfzfw^{-1} = b, \quad b \in \mathcal{A}_+^{-1}.$$

As in the last part of the proof of Theorem 1, from this we can infer that $fzfw^{-1}$ equals the identity multiplied by a scalar λ of modulus 1. We have $fzf = \lambda w$ from which we obtain

$$\phi(1)^{1/4}w\phi(1)^{1/4} = \lambda J(w).$$

The case where J is an algebra *-antiisomorphism can be treated in a similar way. This proves the necessity part of the last statement in the theorem. The sufficiency can be checked easily using the trace preserving property of algebra *-(anti)isomorphisms. The proof is complete. □

Remark 7. We close the paper with a few open problems. The first natural question is, of course, the problem of preservers of general quasi-entropies, that is, where the function parameter and the operator parameter of the quasi-entropy are both general. As mentioned in the introduction, that problem seems very hard and challenging.

The second problem we pose here is related to the characterizations of the central elements of C^*-algebras given in Lemmas 4 and 5. As for the latter one, we believe that a stronger version of the statement holds true. Namely, we suspect the following. For any real number t different from $-1, 0, 1$, if a, b are positive invertible elements of a C^*-algebra (equivalently, positive invertible operators on a complex Hilbert space) such that $(aba)^t = a^t b^t a^t$ holds, then $ab = ba$. For $t = 2$ this follows as a consequence of [1, Proposition 1] which result we have already applied at the end of the proof of Lemma 4. Next, concerning Lemma 4, we conjecture that instead of assuming the Jensen property, we have the same conclusion supposing only concavity. We mean, if for a given self-adjoint element c in a C^*-algebra the map $a \mapsto \exp(\log a + c)$ is concave on the positive definite cone, then c is necessarily a central element. This question is apparently connected to the famous Lieb's concavity theorem [3, Theorem 6].

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