

SURJECTIVE ISOMETRIES ON GRASSMANN SPACES

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ABSTRACT. Let \mathcal{H} be a complex Hilbert space, n a given positive integer and let $P_n(\mathcal{H})$ be the set of all projections on \mathcal{H} with rank n . Under the condition $\dim \mathcal{H} \geq 4n$, we describe the surjective isometries of $P_n(\mathcal{H})$ with respect to the gap metric (the metric induced by the operator norm).

1. INTRODUCTION

The study of isometries of function spaces or operator algebras is an important research area in functional analysis with history dating back to the 1930's. The most classical results in the area are the Banach-Stone theorem describing the surjective linear isometries between Banach spaces of complex-valued continuous functions on compact Hausdorff spaces and its noncommutative extension for general C^* -algebras due to Kadison. This has the surprising and remarkable consequence that the surjective linear isometries between C^* -algebras are closely related to algebra isomorphisms. More precisely, every such surjective linear isometry is a Jordan $*$ -isomorphism followed by multiplication by a fixed unitary element. We point out that in the aforementioned fundamental results the underlying structures are algebras and the isometries are assumed to be linear. However, descriptions of isometries of non-linear structures also play important roles in several areas. We only mention one example which is closely related with the subject matter of this paper. This is the famous Wigner's theorem that describes the structure of quantum mechanical symmetry transformations and plays a fundamental role in the probabilistic aspects of quantum theory. To formulate Wigner's theorem, let \mathcal{H} be a complex Hilbert space and denote by $P_1(\mathcal{H})$ the set of all rank-1 projections on \mathcal{H} . (The elements of $P_1(\mathcal{H})$ represent the pure states of a quantum system to which the Hilbert space \mathcal{H} is associated.) The transition probability between the elements $p, q \in P_1(\mathcal{H})$ is the quantity $\operatorname{tr} pq$, where tr stands for the usual trace functional. Wigner's theorem characterizes all bijective maps of $P_1(\mathcal{H})$ which preserve the transition probability. Before giving the precise formulation we remark that the numerical quantity $\operatorname{tr} pq$ can be interpreted in several ways. In fact, it is easily seen to be the square of the cosine of the angle between the ranges of p and q (as one-dimensional subspaces of \mathcal{H}). On the other hand, we also have $\|p - q\| = \sqrt{1 - \operatorname{tr} pq}$, where $\|\cdot\|$ stands for the operator norm (see, e.g., [7], p. 127). Therefore, the quantum mechanical symmetry transformations can be viewed either as bijective maps on the space of all lines in \mathcal{H} going through the origin which preserve the angle between any two lines or, alternatively, as bijective maps on $P_1(\mathcal{H})$ which preserve the norm distance between any two rank 1 projections. Now we can formulate Wigner's theorem in the following way.

Theorem 1.1. (cf. [7], p. 12) *The bijective transformation $\Phi : P_1(\mathcal{H}) \rightarrow P_1(\mathcal{H})$ satisfies*

$$\|\Phi(p) - \Phi(q)\| = \|p - q\|, \quad \forall p \in P_1(\mathcal{H})$$

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if and only if there exists either a unitary or an antiunitary operator U on \mathcal{H} such that

$$\Phi(p) = UpU^*, \quad \forall p \in P_1(\mathcal{H}).$$

Denote by $B(\mathcal{H})$ the C^* -algebra of all bounded operators on \mathcal{H} . It is well known that the $*$ -automorphisms of $B(\mathcal{H})$ are all of the form $A \mapsto UAU^*$, for some unitary operator U on \mathcal{H} , and its $*$ -antiautomorphisms are all of the form $A \mapsto UA^*U^*$, for some antiunitary operator U on \mathcal{H} .

Consequently, the theorem above says that every surjective isometry of $P_1(\mathcal{H})$ extends either to a $*$ -automorphism or to a $*$ -antiautomorphism of the algebra $B(\mathcal{H})$.

We proceed by pointing out the fact that $P_1(\mathcal{H})$ is a particular Grassmann space. In fact, one usually defines a Grassmann space as the collection of all subspaces of a Hilbert space with a fixed finite dimension. However, closed subspaces and projections (self-adjoint idempotents) are in a one-to-one correspondence. In this paper we prefer to consider the spaces $P_n(\mathcal{H})$ of all projections on \mathcal{H} of rank n (n is a given positive integer) as Grassmann spaces.

The operator norm defines a metric on $P_n(\mathcal{H})$ which is usually called the gap metric. This metric was first investigated by B. Székelyfalvi-Nagy and independently by M.G. Krein and M.A. Krasnoselski under the name ‘‘aperture’’ (see [1], Section 34). The gap metric has a wide range of applications from pure mathematics to engineering. One can easily find a large number of references demonstrating this broad applicability, among others, we list the following fields: perturbation theory of linear operators, perturbation analysis of invariant subspaces, optimization, robust control, multi-variable control, system identification, and signal processing.

The goal of the present paper is to determine and describe the surjective isometries of the Grassmann space $P_n(\mathcal{H})$ with respect to the gap metric. Observe that in light of Theorem 1.1 our main result, which follows, can be viewed as an extension of Wigner’s theorem from $P_1(\mathcal{H})$ to the case of $P_n(\mathcal{H})$.

Theorem 1.2. *Let \mathcal{H} be a complex Hilbert space, n a given positive integer, and $\dim \mathcal{H} \geq 4n$. Assume that the surjective map $\Phi : P_n(\mathcal{H}) \rightarrow P_n(\mathcal{H})$ is an isometry with respect to the gap metric, i.e.,*

$$\|\Phi(p) - \Phi(q)\| = \|p - q\|, \quad \forall p, q \in P_n(\mathcal{H}).$$

Then there exists either a unitary or an antiunitary operator U on \mathcal{H} such that

$$\Phi(p) = UpU^*, \quad \forall p \in P_n(\mathcal{H}).$$

Consequently, just as in the case of $P_1(\mathcal{H})$, we obtain that every surjective isometry of the Grassmann space $P_n(\mathcal{H})$ under the gap metric extends either to a $*$ -automorphism or to a $*$ -antiautomorphism of the full operator algebra $B(\mathcal{H})$ on \mathcal{H} .

We remark that in [8] (alternatively, see Section 2.1 in [7]) the third author presented an extension of Wigner’s theorem for the space of higher rank projections. In [8], Molnár considered (not necessarily surjective) transformations on $P_n(\mathcal{H})$ which preserve the collection of so-called principal angles between the elements of $P_n(\mathcal{H})$. The transformations considered in [8] preserve the complete system of principal angles (an n -tuple of scalars) but Theorem 1.2 deals with transformations that preserve only one of those principal angles, namely, the largest one. In the last section of this paper we shall discuss this further.

A few words follow about the scheme for the proof of Theorem 1.2. The main ingredient is the use of a non-commutative Mazur-Ulam type result on the local algebraic behavior of surjective isometries between substructures of metric groups. In fact, this will imply that the isometries we consider here preserve the relation of commutativity between the elements of $P_n(\mathcal{H})$. Next, using a characterization of orthogonality of rank- n projections involving the relation of commutativity and the gap topology, we show that the orthogonality of the elements of $P_n(\mathcal{H})$ is preserved under any surjective isometry of $P_n(\mathcal{H})$. Finally, we complete the proof by applying a nice result due to Györy and Šemrl describing the structure of orthogonality preserving bijections of $P_n(\mathcal{H})$.

This paper is organized as follows. In Section 2 we review all notation used throughout this paper and also collect the results needed for forthcoming arguments used in our proofs. We give the details of the

proof of our main result in Section 3. In Section 4 we present some remarks and briefly discuss surjective isometries of $P_n(\mathcal{H})$ under some other metrics.

2. BACKGROUND AND NOTATION

In what follows the symbol \mathcal{H} represents a finite or infinite dimensional complex Hilbert space. For a given positive integer n , we denote by $P_n(\mathcal{H})$ the set of all projections on \mathcal{H} with rank equal to n . The metric defined on $P_n(\mathcal{H})$ is called the “gap metric” and given by $d_g(p, q) = \|p - q\|$, $p, q \in P_n(\mathcal{H})$, where $\|\cdot\|$ denotes the usual operator norm.

Throughout this paper, $\Phi : P_n(\mathcal{H}) \rightarrow P_n(\mathcal{H})$ is a given surjective isometry, i.e., a surjective map with the property that

$$\|\Phi(p) - \Phi(q)\| = \|p - q\|, \quad \forall p, q \in P_n(\mathcal{H}).$$

We recall that for the gap metric, the distance between any two projections of different rank is equal to 1. In fact, this follows immediately from the following folk result, valid even in the context of general C^* -algebras. Its proof is omitted since it only requires standard C^* -algebra techniques and elementary computations. By a symmetry in a C^* -algebra we mean any self-adjoint unitary element (or self-adjoint involution).

Proposition 2.1. *If p, q are projections in a C^* -algebra \mathcal{A} with unit 1 such that $\|p - q\| < 1$, then the element $u = (1 - (p - q)^2)^{-1/2}(p + q - 1)$ is a symmetry intertwining p and q , i.e., $upu = q$.*

We can extend our original surjective isometry $\Phi : P_n(\mathcal{H}) \rightarrow P_n(\mathcal{H})$ from the set $P_n(\mathcal{H})$ to the set of all projections on \mathcal{H} by simply defining $\Phi(q) = q$ for every projection q with rank different from n . It is now apparent that this extension yields a surjective isometry on the set of all projections on \mathcal{H} .

The set of all projections are in a bijective correspondence with the set of all symmetries in $B(\mathcal{H})$ that we denote by $S(\mathcal{H})$. In fact, this correspondence is given by associating with any projection p the symmetry a defined by $a = 1 - 2p$ (1 represents the identity operator on \mathcal{H}). Clearly, the map Φ determines a transformation $\Psi : S(\mathcal{H}) \rightarrow S(\mathcal{H})$ given by

$$\Psi(a) = 1 - 2\Phi\left(\frac{1 - a}{2}\right).$$

It is trivial to check that Ψ is a surjective isometry of $S(\mathcal{H})$ with respect to the metric defined from the operator norm, i.e., $\|\Psi(a) - \Psi(b)\| = \|a - b\|$ holds for all pairs a, b in $S(\mathcal{H})$.

A Mazur-Ulam type result plays a fundamental role in the proof of our main result. The statement of this result, the forthcoming Proposition 2.3, needs some preliminary definitions that we formulate first.

Let G be a group with unit 1. We call a subset X of G a twisted subgroup if $1 \in X$ and $ba^{-1}b \in X$, for all $a, b \in X$. If d is a metric on G , we say that it is translation and inverse invariant if, for all $a, b, c, d \in G$, we have $d(cad, cbd) = d(a, b)$ and $d(a^{-1}, b^{-1}) = d(a, b)$, respectively.

It is clear that the unitary group $U(\mathcal{H})$ of all unitary operators on the Hilbert space \mathcal{H} equipped with the metric determined by the operator norm is a metric group with translation and inverse invariant metric and $S(\mathcal{H})$ is a twisted subgroup of $U(\mathcal{H})$.

One of the main tools in the proof of Theorem 1.2 is a general Mazur-Ulam type theorem on the local algebraic behavior of the surjective isometries of twisted subgroups of groups with translation and inverse invariant metrics. The original Mazur-Ulam theorem states that every surjective isometry between normed real-linear spaces is automatically affine. Motivated by a miraculous proof of the Mazur-Ulam theorem given by Väisälä [11], the authors in [5], presented results concerning surjective isometries of general non-commutative metric groups showing that those transformations (under given conditions) locally preserve the inverted Jordan triple product $ba^{-1}b$. Our argument in the proof of Theorem 1.2 relies on a result of the same type. However, the result needed in our proof cannot be deduced from the results presented in [5], hence we give its formulation in our forthcoming Proposition 2.3 and also include its proof. First, we prove a preliminary lemma (cf. Lemma 2.3 in [5]).

Lemma 2.2. *Let M be a bounded metric space, $\varphi : M \rightarrow M$ a surjective isometry. Assume $c \in M$ and $k > 1$ is a constant such that*

$$d(\varphi(x), x) \geq kd(x, c),$$

for every $x \in M$. Then $T(c) = c$, for every surjective isometry $T : M \rightarrow M$.

Proof. Let

$$\lambda = \sup\{d(T(c), c) : T : M \rightarrow M \text{ is a surjective isometry}\}.$$

Clearly, $\lambda < \infty$. Select a surjective isometry $T : M \rightarrow M$ and consider $\tilde{T} = T^{-1} \circ \varphi \circ T$, which is also a surjective isometry of M . We have

$$\lambda \geq d(T^{-1}(\varphi(T(c))), c) = d(\varphi(T(c)), T(c)) \geq kd(T(c), c).$$

Since this holds for every surjective isometry $T : M \rightarrow M$, we obtain $\lambda \geq k\lambda$ which implies $\lambda = 0$. This completes the proof. \square

We introduce some additional notation. Given X a subset of a metric group, and two elements a, b in X , we denote by $L_{a,b}$ the following subset of X :

$$L_{a,b} = \{x \in X : d(x, a) = d(x, ba^{-1}b) = d(a, b)\}.$$

We now formulate the result which plays an important role in the proof of Theorem 1.2.

Proposition 2.3. *Let X be a twisted subgroup of a metric group G with translation and inverse invariant metric d . Let $a, b \in X$ and let $k > 1$ be such that*

$$(1) \quad d(bx^{-1}b, x) \geq kd(x, b),$$

for all $x \in L_{a,b}$. If $T : X \rightarrow X$ is a surjective isometry and there exists $c \in X$ such that $c(Ta)^{-1}c = T(ba^{-1}b)$, then $c(Tb)^{-1}c = Tb$.

Proof. Define

$$H = \{y \in X : d(y, T(a)) = d(y, T(ba^{-1}b)) = d(a, b)\}.$$

One can easily check that $H = T(L_{a,b})$. The maps $\varphi, \psi : X \rightarrow X$ defined by $\varphi(t) = bt^{-1}b$, $\psi(t) = ct^{-1}c$, $t \in X$ are surjective isometries of X . It is straightforward to check that $\varphi(L_{a,b}) = L_{a,b}$ and $\psi(H) = H$. Therefore, the transformation $\tilde{T} = T^{-1} \circ \psi \circ T$, when restricted to $L_{a,b}$, is a surjective isometry. Since $L_{a,b}$ is a bounded metric space, applying Lemma 2.2 we obtain $\tilde{T}(b) = b$. This yields $c(Tb)^{-1}c = Tb$ and completes the proof. \square

We note that a similar result from [5] has been applied in [6] to determine the surjective isometries of the unitary group $U(\mathcal{H})$.

As the last preliminary step we introduce the following notion. Let X be a twisted subgroup of a group, $T : X \rightarrow X$ a transformation, and a and b elements of X . We say that T is (a, b) -multiplicative if $T(ba^{-1}b) = T(b)T(a)^{-1}T(b)$.

3. PROOF OF THE MAIN THEOREM

In this section we present the detailed proof of our main theorem.

Recall from Section 2 that the original surjective isometry $\Phi : P_n(\mathcal{H}) \rightarrow P_n(\mathcal{H})$ is now extended to the whole set of projections on \mathcal{H} . We denote the extension which is also a surjective isometry by the same symbol Φ , and Ψ stands for the corresponding surjective isometry of the space $S(\mathcal{H})$ of all symmetries in $B(\mathcal{H})$.

We first reformulate Proposition 2.1 for symmetries. This result will be used several times throughout the paper.

Proposition 3.1. *If a, b are symmetries in a unital C^* -algebra \mathcal{A} such that $\|a - b\| < 2$ then the element $s = (1 - ((a - b)/2)^2)^{-1/2}((a + b)/2)$ is a symmetry and we have $sas = b$.*

The next statement establishes a multiplicative property of Ψ for symmetries that are close in distance. In the proof we shall use that the norm $\|\cdot\|$ is unitarily invariant and also that $s^{-1} = s$ holds for each $s \in S(\mathcal{H})$.

Lemma 3.2. *If $a, b \in S(\mathcal{H})$ are such that $\|a - b\| < \frac{\sqrt{3}}{2}$ then Ψ is (a, b) -multiplicative.*

Proof. We first observe that there exists ϵ , a positive number less than $\frac{1}{2}$, such that $\|a - b\| < \frac{\sqrt{3}}{2} - \epsilon$. Then we apply Proposition 2.3. We show that there exists $k > 1$ such that

$$\|bxb - x\| \geq k\|x - b\|, \text{ for every } x \in L_{a,b}.$$

Given $x \in L_{a,b}$, we have

$$\|bxb - x\| = \|(xb)^2 - 1\| = \sup_{\lambda \in \sigma(xb)} |\lambda^2 - 1| = \sup_{\lambda \in \sigma(xb)} |\lambda - 1| |\lambda + 1|.$$

Since the element xb is a unitary operator on \mathcal{H} , we have $\sigma(xb) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Therefore, given λ in the spectrum of xb we conclude that

$$|\lambda - 1| \leq \|xb - 1\| = \|x - b\| \leq \|x - a\| + \|a - b\| = 2\|a - b\| < \sqrt{3} - 2\epsilon.$$

By the Pythagorean theorem $|\lambda + 1|^2 + |\lambda - 1|^2 = 4$, hence $|\lambda + 1|^2 > 4 - (\sqrt{3} - 2\epsilon)^2 > 1$. We set $k = \sqrt{4 - (\sqrt{3} - 2\epsilon)^2} > 1$. Then

$$\sup_{\lambda \in \sigma(xb)} |\lambda - 1| |\lambda + 1| \geq k\|xb - 1\| = k\|x - b\|.$$

Consequently, the condition displayed in (1) holds for every $x \in L_{a,b}$.

Since $\|\Psi(a) - \Psi(bab)\| = \|a - bab\| \leq \|a - b\| + \|b - bab\| = 2\|a - b\| < \sqrt{3} < 2$, by Proposition 3.1 we have that

$$c = \frac{1}{\sqrt{1 - \left(\frac{\Psi(a) - \Psi(bab)}{2}\right)^2}} \frac{\Psi(a) + \Psi(bab)}{2}$$

is a symmetry and

$$(2) \quad c\Psi(a)c = \Psi(bab).$$

Applying Proposition 2.3, we obtain that $c\Psi(b)c = \Psi(b)$. This implies that c commutes with $\Psi(b)$. We now show that the distance between c and $\Psi(b)$ is less than 2. In fact,

$$\begin{aligned}
\|c - \Psi(b)\| &= \\
&= \left\| \left[\frac{1}{\sqrt{1 - \left(\frac{\Psi(a) - \Psi(bab)}{2}\right)^2}} - 1 \right] \frac{\Psi(a) + \Psi(bab)}{2} + \frac{\Psi(a) - \Psi(b)}{2} + \frac{\Psi(bab) - \Psi(b)}{2} \right\| \\
&\leq \left\| \frac{1}{\sqrt{1 - \left(\frac{\Psi(a) - \Psi(bab)}{2}\right)^2}} - 1 \right\| \left\| \frac{\Psi(a) + \Psi(bab)}{2} \right\| + \left\| \frac{\Psi(a) - \Psi(b)}{2} \right\| + \left\| \frac{\Psi(bab) - \Psi(b)}{2} \right\| \\
&\leq \left(\frac{1}{\sqrt{1 - \left(\frac{\sqrt{3}}{2}\right)^2}} - 1 \right) + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} < 2.
\end{aligned}$$

It is easy to see that the difference of two commuting projections has norm 1 unless they coincide. Since the symmetries c and $\Psi(b)$ commute and their norm-distance is less than 2, we conclude that $c = \Psi(b)$. Consequently, (2) becomes $\Psi(bab) = \Psi(b)\Psi(a)\Psi(b)$, i.e., Ψ is (a, b) -multiplicative. This completes the proof. \square

Our goal is to increase the bound in the Lemma 3.2 in order to have the multiplicative property of Ψ on a larger domain. To prove this, we need the following technical result which is a particular case of Lemma 7 in [6]. Its proof is elementary and we include it for sake of completeness.

Proposition 3.3. *Let X be a twisted subgroup of a group, and $T : X \rightarrow X$ a mapping. Let x_0, x_1, x_2, x_3 and x_4 be elements in X such that*

$$x_2 = x_1x_0^{-1}x_1, \quad x_3 = x_2x_1^{-1}x_2, \quad \text{and} \quad x_4 = x_3x_2^{-1}x_3.$$

If T is (x_i, x_{i+1}) -multiplicative for $i = 0, 1, 2$ then T is (x_0, x_2) -multiplicative.

Proof. Since T is (x_i, x_{i+1}) -multiplicative, we have

$$\begin{aligned}
T(x_2) &= T(x_1x_0^{-1}x_1) = T(x_1)T(x_0)^{-1}T(x_1) \\
T(x_3) &= T(x_2x_1^{-1}x_2) = T(x_2)T(x_1)^{-1}T(x_2) \\
T(x_4) &= T(x_3x_2^{-1}x_3) = T(x_3)T(x_2)^{-1}T(x_3).
\end{aligned}$$

We notice that

$$\begin{aligned}
x_4 &= x_3x_2^{-1}x_3 = x_2x_1^{-1}x_2x_2^{-1}x_2x_1^{-1}x_2 \\
&= x_2x_1^{-1}x_2x_1^{-1}x_2 = \\
&= x_1x_0^{-1}x_1x_1^{-1}x_1x_0^{-1}x_1x_1^{-1}x_1x_0^{-1}x_1 \\
&= x_1x_0^{-1}x_1x_0^{-1}x_1x_0^{-1}x_1 \\
&= x_2x_0^{-1}x_2.
\end{aligned}$$

Similarly, one can show that $T(x_4) = T(x_2)T(x_0)^{-1}T(x_2)$. We have $T(x_2x_0^{-1}x_2) = T(x_2)T(x_0)^{-1}T(x_2)$ which completes the proof. \square

In the next lemma we enlarge the domain over which Ψ is multiplicative.

Lemma 3.4. *If $a, b \in S(\mathcal{H})$ are such that $\|a - b\| \leq 1.2$ then Ψ is (a, b) -multiplicative.*

Proof. We define a symmetry s that intertwines a and b , i.e., $sas = b$. In fact, applying Proposition 3.1 we set

$$s = \frac{1}{\sqrt{1 - \left(\frac{a-b}{2}\right)^2}} \frac{a+b}{2}.$$

We know that s is a symmetry and $sas = b$. Moreover, we have

$$\|s - a\| = \left\| \left[\frac{1}{\sqrt{1 - \left(\frac{a-b}{2}\right)^2}} - 1 \right] \frac{a+b}{2} + \frac{b-a}{2} \right\| \leq \left(\frac{1}{\sqrt{1 - \left(\frac{1.2}{2}\right)^2}} - 1 \right) + \frac{1.2}{2} = \frac{1}{0.8} - 1 + 0.6 < \frac{\sqrt{3}}{2}.$$

An application of Lemma 3.2 implies that Ψ is (a, s) -multiplicative, i.e., $\Psi(b) = \Psi(sas) = \Psi(s)\Psi(a)\Psi(s)$. We set $x_0 = a$, $x_1 = s$, $x_2 = b = x_1x_0x_1$, $x_3 = x_2x_1x_2$ and $x_4 = x_3x_2x_3$. One can easily check that $\|a - s\| = \|x_0 - x_1\| = \|x_1 - x_2\| = \|x_2 - x_3\| = \|x_3 - x_4\| < \frac{\sqrt{3}}{2}$. Therefore Ψ is (x_i, x_{i+1}) -multiplicative with $i = 0, 1, 2$. An application of Proposition 3.3 yields that Ψ is (a, b) -multiplicative, hence $\Psi(bab) = \Psi(b)\Psi(a)\Psi(b)$. \square

We apply the same procedure to refine further the constant in Lemma 3.4.

Lemma 3.5. *If $a, b \in S(\mathcal{H})$ are such that $\|a - b\| \leq \sqrt{2}$ then Ψ is (a, b) -multiplicative.*

Proof. Just as in the proof of the previous lemma, we define

$$s = \frac{1}{\sqrt{1 - \left(\frac{a-b}{2}\right)^2}} \frac{a+b}{2}.$$

Then s is a symmetry that intertwines a and b , $b = sas$. We compute the distance between s and a ,

$$\|s - a\| = \left\| \left[\frac{1}{\sqrt{1 - \left(\frac{a-b}{2}\right)^2}} - 1 \right] \frac{a+b}{2} + \frac{b-a}{2} \right\| \leq (\sqrt{2} - 1) + \frac{\sqrt{2}}{2} < 1.2.$$

Lemma 3.4 asserts that Ψ is (a, s) -multiplicative. Considering the elements $x_0 = a$, $x_1 = s$, $x_2 = b$, $x_3 = x_2x_1x_2$ and $x_4 = x_3x_2x_3$ the proof now follows a similar reasoning presented for the previous lemma. \square

We now show that the transformation Ψ preserves the commutativity between those symmetries that correspond to projections in $P_n(\mathcal{H})$. This is the most important step in our proof.

Lemma 3.6. *Let $p, q \in P_n(\mathcal{H})$ and set $a = 1 - 2p, b = 1 - 2q$. If $ab = ba$ then $\Psi(a)\Psi(b) = \Psi(b)\Psi(a)$.*

Proof. Since a and b commute, so does the corresponding projections, $p = \frac{1-a}{2}$ and $q = \frac{1-b}{2}$. Therefore there exists an orthonormal basis for \mathcal{H} such that p and q have the following block matrix representations

$$p = \begin{bmatrix} I_1 & O & O & O \\ O & O_2 & O & O \\ O & O & I_3 & O \\ O & O & O & O_4 \end{bmatrix}, \quad q = \begin{bmatrix} O_1 & O & O & O \\ O & I_2 & O & O \\ O & O & I_3 & O \\ O & O & O & O_4 \end{bmatrix}.$$

Hence

$$a = \begin{bmatrix} -I_1 & O & O & O \\ O & I_2 & O & O \\ O & O & -I_3 & O \\ O & O & O & I_4 \end{bmatrix}, \quad b = \begin{bmatrix} I_1 & O & O & O \\ O & -I_2 & O & O \\ O & O & -I_3 & O \\ O & O & O & I_4 \end{bmatrix}.$$

Since p, q are of the same rank, the sizes of the identity matrices I_1 and I_2 in the representation above are the same. Hence we can form the following block matrix

$$s = \begin{bmatrix} O & I_1 & O & O \\ I_1 & O & O & O \\ O & O & -I_3 & O \\ O & O & O & I_4 \end{bmatrix}.$$

It is easy to check that s is a symmetry that intertwines a and b , i.e., $sas = b$. Moreover, we have

$$\|s - a\| = \left\| \begin{bmatrix} I_1 & I_1 \\ I_1 & -I_1 \end{bmatrix} \right\| = \sqrt{2}.$$

Now, Lemma 3.5 implies that Ψ is (a, s) -multiplicative. Set $x_0 = a, x_1 = s, x_2 = b = x_1x_0x_1, x_3 = x_2x_1x_2$ and $x_4 = x_3x_2x_3$. One can easily check that $\|a - s\| = \|x_0 - x_1\| = \|x_1 - x_2\| = \|x_2 - x_3\| = \|x_3 - x_4\| = \sqrt{2}$. Lemma 3.5 implies that Ψ is (x_i, x_{i+1}) -multiplicative for $i = 0, 1, 2$. An application of Proposition 3.3 yields that Ψ is (a, b) -multiplicative, $\Psi(bab) = \Psi(b)\Psi(a)\Psi(b)$. Moreover, the commutativity of a and b implies that $bab = a$ and thus we have $\Psi(a) = \Psi(b)\Psi(a)\Psi(b)$. This implies that $\Psi(b)\Psi(a) = \Psi(a)\Psi(b)$ and completes the proof. \square

We observe that the previous lemma implies that our original transformation $\Phi : P_n(\mathcal{H}) \rightarrow P_n(\mathcal{H})$ maps commuting projections to commuting projections. This was a consequence of the fact that Φ is a surjective isometry on $P_n(\mathcal{H})$. However, Φ^{-1} is also a surjective isometry and hence it has the same preserving property. Consequently, we obtain that Φ preserves commutativity in both directions.

We next present a characterization of orthogonality among rank- n projections as a stepping stone for the proof that Φ preserves orthogonality in both directions. Given two projections p and q in $P_n(\mathcal{H})$ the symbol $\{p, q\}^c$ represents the (relative) commutant of $\{p, q\}$ in $P_n(\mathcal{H})$, i.e., the set of all projections in $P_n(\mathcal{H})$ that commute with both p and q . A family $\{p_i\}_i$ of projections on \mathcal{H} is called a resolution of the identity if $\sum_i p_i = 1$ and any two distinct elements are orthogonal, i.e., $p_i p_j = 0$ for $i \neq j$.

Proposition 3.7. *Let \mathcal{H} be a Hilbert space either infinite dimensional or of finite dimension with $\dim \mathcal{H} \geq 4n$. For any two commuting projections p, q in $P_n(\mathcal{H})$ we have that p and q are orthogonal if and only if the set $\{p, q\}^c$ as a subspace of the metric space $P_n(\mathcal{H})$ has a pathwise connected component K such that the maximal number of pairwise commuting projections of rank n in K^c is exactly $\binom{2n}{n}$.*

Proof. We consider two commuting projections p and q in $P_n(\mathcal{H})$. We define the following resolution of the identity obtained from p, q :

$$\mathcal{P} = \{p - pq, pq, q - pq, 1 - (p + q - pq)\}.$$

Let r be a projection of rank n that commutes with p and q , i.e., $r \in \{p, q\}^c$. Then $r = \sum_{i=1}^4 r p_i$ with $p_1 = p - pq, p_2 = pq, p_3 = q - pq$ and $p_4 = 1 - (p + q - pq)$. Setting $n_i = \text{rank}(r p_i)$ we associate with r a 4-tuple (n_1, n_2, n_3, n_4) of nonnegative integers such that $n_1 + n_2 + n_3 + n_4 = n$. We call (n_1, n_2, n_3, n_4) the rank representation of r relative to \mathcal{P} . We observe that any two projections in $\{p, q\}^c$ are pathwise connected through a path in $\{p, q\}^c$ if and only if they have the same rank representations relative to \mathcal{P} . To see this, one may recall Proposition 2.1, and invokes the following two facts: Projections of the same rank are unitarily equivalent, and the unitary group is pathwise connected in the operator norm.

Given a projection p_0 in $P_n(\mathcal{H})$ with range contained in the range of p_4 , then p_0 is orthogonal to both p and q and its rank representation relative to \mathcal{P} is $(0, 0, 0, n)$. We say that this projection is supported in

p_4 . Let K be the component of $\{p, q\}^c$ which consists of all projections in $P_n(\mathcal{H})$ supported in p_4 . Then K^c is the set of all rank- n projections that commute with all rank- n projections which are supported in p_4 . The rank of p_4 is greater than n . Selecting an element r in K^c , r commutes with every rank- n projection supported in p_4 . Then r also commutes with every rank-1 projection supported in p_4 . It implies rather easily that the range of p_4 is an eigenspace of r . Hence $rp_4 = 0$. This shows that K^c is equal to the set of all elements of $P_n(\mathcal{H})$ which are orthogonal to p_4 . The maximal number of commuting elements of this set is clearly $\binom{\text{rank}(1-p_4)}{n}$. If p, q are orthogonal, then this number is $\binom{2n}{n}$ and we obtain that the condition for orthogonality stated in the proposition is necessary.

Conversely, assume that p and q are not orthogonal. Then the range of $p \vee q = p + q - pq = 1 - p_4$ has dimension strictly less than $2n$. We consider a component K of $\{p, q\}^c$ labeled with (n_1, n_2, n_3, n_4) , $n_4 \neq 0$. We can see that every element r of K^c commutes with the difference of any two rank- n_4 projections supported in p_4 . This implies that such a projection r commutes with every rank-1 projection supported in p_4 . We infer that the range of p_4 is an eigenspace of r and hence $rp_4 = 0$. Therefore, the elements of K^c are supported in $1 - p_4$. Hence the maximal number of commuting elements in K^c is strictly less than $\binom{2n}{n}$. Finally, we consider a component K of $\{p, q\}^c$ labeled with $(n_1, n_2, n_3, 0)$. The commutant of this component contains all projections which are supported in p_4 . Since the dimension of \mathcal{H} is greater or equal to $4n$ and the rank of $p \vee q$ is less than $2n$, the rank of p_4 is strictly greater than $2n$. This implies that the maximal number of pairwise commuting projections in K^c is greater than $\binom{2n}{n}$. The proof is complete. \square

We are now in a position to prove our main theorem.

Proof for Theorem 1.2. Proposition 3.7 gives a characterization for orthogonality between the elements of $P_n(\mathcal{H})$ based on topological concepts and commutativity. Since the surjective isometry $\Phi : P_n(\mathcal{H}) \rightarrow P_n(\mathcal{H})$ preserves the topological properties as well as the commutativity in both directions, we find that the bijective map Φ preserves the orthogonality in both directions. We can now apply a result of Györy [4] and Šemrl [9] on the structure of such transformations which extends Uhlhorn's famous theorem [10] from one-dimensional subspaces to the case of higher dimensional subspaces. Šemrl proved his result for infinite dimensional Hilbert spaces \mathcal{H} , while Györy considered also the finite dimensional case for $\dim \mathcal{H} > 3n$. In either case, the conclusion is that any bijective map on $P_n(\mathcal{H})$ which preserves orthogonality in both directions is implemented either by a unitary or an antiunitary operator. This means that we have a unitary or antiunitary operator U on \mathcal{H} such that $\Phi(p) = UpU^*$, $p \in P_n(\mathcal{H})$. The proof of the theorem is complete. \square

4. REMARKS

We conclude the paper with a few remarks. First, one may ask if continuing the above process of "pumping up" the value of the constant in Lemma 3.2 as we did in Lemmas 3.4 and 3.5 we could finally reach the constant 2, i.e., we could prove that for any pair a, b of symmetries with $\|a - b\| \leq 2$ we have that Ψ is (a, b) -multiplicative. In fact, this would mean that Ψ is a Jordan triple automorphism of $S(\mathcal{H})$, a bijective map satisfying $\Psi(bab) = \Psi(b)\Psi(a)\Psi(b)$ for all $a, b \in S(\mathcal{H})$. However, we have found that this process leads to a limit which is less than 2 (using Maple we have obtained that this limit is approximately 1.67857351).

Above we have described the isometries of the Grassmann space $P_n(\mathcal{H})$ with respect to the gap metric, the metric coming from the operator norm. However, there are several other metrics on $P_n(\mathcal{H})$ which are used in different areas of mathematics. Most of those metrics are related to the concept of principal angles between higher dimensional subspaces of a Hilbert space. One way to define that concept is the following. Let M, N be n -dimensional subspaces of \mathcal{H} and denote by p, q the projections in $P_n(\mathcal{H})$ that project onto M and N , respectively. Consider the decreasing sequence of eigenvalues of the positive

finite rank operator qpq (of rank at most n) counted according multiplicity and take its first n elements $\lambda_1 \geq \dots \geq \lambda_n$. Define

$$\Theta_1(M, N) = \arccos \sqrt{\lambda_n} \geq \dots \geq \Theta_n(M, N) = \arccos \sqrt{\lambda_1}.$$

The angles $0 \leq \Theta_i(M, N) \leq \pi/2$, $i = 1, \dots, n$ are called the principal angles between M and N . We remark that there is a geometrical approach to define those angles due to C. Jordan (1875) see, e.g., [2], p. 226. The original definition was given in the setting of real spaces, the complex case is analyzed in [3], for example.

As one can see in [2], pp. 226-227, definitions of several metrics on $P_n(\mathcal{H})$ are based on the concept of principal angles. As for the gap metric, we have $\|p - q\| = \sin \Theta_1(M, N)$. The geodesic distance

$$\sqrt{\sum_{i=1}^n \Theta_i(M, N)^2}$$

corresponding to a natural Riemannian structure on $P_n(\mathcal{H})$ was determined by Y.C. Wong. The Asimov distance which comes from a Finsler geometrical structure on $P_n(\mathcal{H})$ is just the largest principal angle $\Theta_1(M, N)$. Next, the Frobenius distance between M and N is

$$\sqrt{2 \sum_{i=1}^n \sin^2 \Theta_i(M, N)}$$

which is easily seen to be the same as the Hilbert-Schmidt norm of $p - q$. Its slight variation

$$\sqrt{\sum_{i=1}^n \sin^2 \Theta_i(M, N)}$$

is called chordal distance which appears, among others, in packing problems.

The main result of the present paper gives (under a certain dimensionality constraint) the description of all surjective maps on $P_n(\mathcal{H})$ which preserve the largest principal angle (as already mentioned, in [8] the third author determined all transformations on $P_n(\mathcal{H})$ which preserve the full collection of principal angles). Hence, as a byproduct, Theorem 1.2 describes the structure of all isometries of $P_n(\mathcal{H})$ with respect to the Asimov metric. As for the geodesic distance, it is obvious that this quantity equals $\sqrt{n\pi^2/4}$ if and only if the subspaces in question are orthogonal. Therefore, any isometry with respect to this metric preserves the orthogonality in both directions and hence the Györy-Šemrl theorem can be applied to determine the corresponding isometries. In a similar fashion, one can describe the isometries of $P_n(\mathcal{H})$ with respect to the Frobenius metric. In all cases we find that the isometries are implemented by unitary or antiunitary operators and thus extend to *-automorphisms or to *-antiautomorphisms of $B(\mathcal{H})$.

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