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Isometries of spaces of normalized positive operators under the operator norm

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Abstract. In this paper a former result of ours [13, Theorem 2] is completed. It asserts that for all real numbers $p > 1$ the p-norm isometries of the space of elements with p-norm 1 in the cone of positive operators on a finite dimensional complex Hilbert space are unitary or antiunitary conjugations. The purpose of this paper is to provide an analogous statement in the case $p = \infty$, i.e. the case of the operator norm.

1. Introduction and statement of the main result

In this paper we investigate a particular case of the following problem. Let $\mathcal{H}$ be a complex Hilbert space. Determine the structure of isometries ϕ of the set formed by all positive operators in the unit sphere of the pth Schatten ideal on $\mathcal{H}$ under the condition $1 \leq p \leq \infty$. In what follows, we mention former results concerning this question. It was answered in [11, Theorem 1] and in a result of [12] for the particular case $p = 1$. Those statements give us the general form of non-surjective, resp. surjective isometries of the set formed by all density operators (positive operators of trace 1) on $\mathcal{H}$ relative to the 1-norm in the case where H is of finite, resp. of arbitrary dimension. One can observe that those operators – which play a basic role in the mathematical foundations of quantum mechanics – are exactly the positive ones of unit 1-norm. In the case $1 < p < \infty$, we presented the solution of the problem above in our statements [13, Theorems

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1 and 2] under the condition that ϕ is surjective, resp. $\mathcal{H}$ is finite dimensional. Motivated by the previous results, in the present paper we give a partial solution in the missing case $p = \infty$. We remark that the Schatten ideal corresponding to the parameter $p = \infty$ is the space $\mathcal{L}(\mathcal{H})$ of all bounded linear operators acting on $\mathcal{H}$ endowed with the operator norm.

Our investigations here concern isometries of "the positive part" of the unit sphere in certain normed spaces. Such maps of the whole set of normalized elements were studied in several papers. The article [18] is one of them in which the following problem of Tingley was formulated. Given real Banach spaces X and Y, is it true that each onto isometry between their unit spheres can be extended to an affine one from X to Y? This question was answered for several particular spaces, e.g. classes of Hilbert space operators, sets of continuous functions, $\mathbb{L}^p$ spaces and sequence spaces. For results on Tingley's problem in different settings the reader can consult, e.g. the survey paper [3] of Ding. Recently, considerable progress has been made concerning that problem. For example, in [16] and [17] Tanaka has given a positive answer to it for finite dimensional C*-algebras and finite von Neumann algebras. In [14], Peralta and Tanaka have solved the problem for compact C*-algebras and weakly compact JB*-triples of rank not less than 5. The answer to it in the case of weakly compact JB*-triples is given by Peralta and Fernández-Polo in [5]. The problem has been solved also for $\mathcal{L}(\mathcal{H})$ and for atomic C*-algebras and JB*-triples by the latter authors, see [6, 7]. Fernández-Polo, Garcés, Peralta and Villanueva have answered it for spaces of trace class operators (c.f. [4]).

We also mention the nice statement [15, Theorem 6.1] of Tanaka which is closely related to the main result of this paper. It describes the general form of surjective isometries of the unit sphere in the space $M_n(\mathbb{C})$ of $n \times n$ complex matrices endowed with the spectral norm and solves Tingley's problem in the affirmative in the case $X = Y = M_n(\mathbb{C})$. Here we provide a local version of [15, Theorem 6.1], which concerns isometries of the positive part of that sphere, i.e. the space of normalized positive semidefinite matrices in $M_n(\mathbb{C})$. We remark that the latter set is much smaller than the sphere under consideration, therefore we can not use the method of proof of that theorem in our arguments.

Turning to the main result of the paper, we introduce the following notation which will be used hereafter. The operator norm is denoted by $||.||$ and d stands for the metric induced by it. We denote by $\mathcal{L}(\mathcal{H})_1^+$ the space of positive operators A on $\mathcal{H}$ with $||A|| = 1$. We mention that, clearly unitary or antiunitary similarity transformations of $\mathcal{L}(\mathcal{H})_1^+$ are isometries with respect to d. Our main result tells us that if $\mathcal{H}$ is finite dimensional, then the reverse statement also holds, i.e.

any isometry of $(\mathcal{L}(\mathcal{H})_1^+, d)$ is such a transformation. It shows that in the case $\dim \mathcal{H} < \infty$, distance preserving maps of that structure can be extended to affine isometries of the space of linear operators on $\mathcal{H}$. The main statement of the paper reads as follows.

Theorem. *Suppose that $n = \dim \mathcal{H} < \infty$ and let $\phi\colon \mathcal{L}(\mathcal{H})_1^+ \to \mathcal{L}(\mathcal{H})_1^+$ be an isometry with respect to d. Then there exists a unitary or an antiunitary operator U on $\mathcal{H}$ such that ϕ can be written in the form*

$$\phi(A) = UAU^* \quad (A \in \mathcal{L}(\mathcal{H})_1^+).$$

Concerning this result, we recall the fact that if $\dim \mathcal{H} < \infty$, then $\mathcal{L}(\mathcal{H})_1^+$ is compact. Therefore referring to [2, Excercise 2.4.1] which states that any isometry of such a metric space is surjective, we deduce that the map ϕ in Theorem is onto and hence bijective.

2. Proof

In this section we shall use the following notation. The class of projections in $\mathcal{L}(\mathcal{H})_1^+$, i.e. the set of nonzero projections on $\mathcal{H}$ is denoted by $\mathcal{P}(\mathcal{H})_0$, and its rank-one elements by $\mathcal{P}_1(\mathcal{H})$. The symbol $\mathcal{L}(\mathcal{H})_1^{++}$ stands for the collection of invertible operators in $\mathcal{L}(\mathcal{H})_1^+$. Moreover I denotes the identity operator and for any element $P \in \mathcal{P}(\mathcal{H})_0$ we define $P^\perp = I - P$. Finally, we denote by $\sigma(A)$ the spectrum of a linear operator A.

Turning to the proof of Theorem, since its conclusion holds in the case $n = 1$ we assume $n \geq 2$. The following characterization will be used several times in this section.

Lemma 1. *Let $\mathcal{M} \subset \mathcal{H}$ be a subspace, A, B be positive operators on $\mathcal{M}$ with maximal eigenvalues α_M, resp. β_M, minimal eigenvalues α_m, resp. β_m and corresponding eigenspaces $\mathcal{M}_{\alpha_M}, \mathcal{M}_{\beta_M}$, $\mathcal{M}_{\alpha_m}, \mathcal{M}_{\beta_m}$. Then*

$$d(A, B) \leq \max\{|\alpha_M - \beta_m|, |\beta_M - \alpha_m|\}$$

and the following assertions hold. If $|\alpha_M - \beta_m| > |\beta_M - \alpha_m|$, then

$$d(A, B) = \max\{|\alpha_M - \beta_m|, |\beta_M - \alpha_m|\} \iff \mathcal{M}_{\alpha_M} \cap \mathcal{M}_{\beta_m} \neq \{0\}.$$

In the case $|\beta_M - \alpha_m| > |\alpha_M - \beta_m|$

$$d(A, B) = \max\{|\alpha_M - \beta_m|, |\beta_M - \alpha_m|\} \iff \mathcal{M}_{\alpha_m} \cap \mathcal{M}_{\beta_M} \neq \{0\}.$$

If $|\alpha_M - \beta_m| = |\beta_M - \alpha_m|$, *then*

$$d(A,B) = \max\{|\alpha_M - \beta_m|, |\beta_M - \alpha_m|\} \iff \mathcal{M}_{\alpha_M} \cap \mathcal{M}_{\beta_m} \neq \{0\} \text{ or } \mathcal{M}_{\alpha_m} \cap \mathcal{M}_{\beta_M} \neq \{0\}.$$

Proof. Using the fact that $\dim \mathcal{H} < \infty$ and that the operator norm of a self-adjoint operator coincides with its numerical radius, we infer that there is a unit vector $x_0 \in \mathcal{M}$ such that

$$\begin{aligned} d(A,B) &= \sup\{|\langle (A-B)x, x\rangle| \ : \ x \in \mathcal{M}, \ ||x|| = 1\} \\ &= |\langle Ax_0, x_0\rangle - \langle Bx_0, x_0\rangle|. \end{aligned}$$

Now we make use of the Rayleigh-Ritz theorem which states the following. For any self-adjoint operator Y on $\mathcal{M}$ and unit vector $x \in \mathcal{M}$ one has $\min \sigma(Y) \leq \langle Yx, x\rangle \leq \max \sigma(Y)$ and equality holds in the former, resp. latter inequality iff x is an eigenvector for Y corresponding to the minimum, resp. maximum of $\sigma(Y)$. By the previous observations, we easily conclude that the desired characterizations hold. □

Now we introduce some notation that will be used in the rest of the proof. For an element $A \in \mathcal{L}(\mathcal{H})_1^+$ the symbol $\mathrm{Fix}(A)$ stands for the eigenspace of A corresponding to its eigenvalue 1. Next we define the relation $\sim$ on $\mathcal{L}(\mathcal{H})_1^+$ in the following way. For any operators $A, B \in \mathcal{L}(\mathcal{H})_1^+$ one has

$$A \sim B \iff \mathrm{Fix}(A) \cap \ker B \neq \{0\} \text{ or } \mathrm{Fix}(B) \cap \ker A \neq \{0\}.$$

If $\mathcal{A} \subset \mathcal{L}(\mathcal{H})_1^+$ is a set, then let $S(\mathcal{A})$ be the collection of all operators $R \in \mathcal{L}(\mathcal{H})_1^+$ satisfying $R \sim X \ \forall X \in \mathcal{A}$. For any operator $A \in \mathcal{L}(\mathcal{H})_1^+$ the symbol $S(S(A))$ denotes the set $S(S(\{A\}))$. We proceed with the following assertion which is a very easy consequence of Lemma 1 and the fact that the spectral radius of a normal operator coincides with its norm.

Corollary. *For any* $A, B \in \mathcal{L}(\mathcal{H})_1^+$ *the equality* $d(A,B) = 1$ *holds if and only if* $A \sim B$.

Our next claim shows an important invariance property of ϕ.

Claim 1. *The restriction* $\phi|_{\mathcal{P}(\mathcal{H})_0}$ *is a bijection of* $\mathcal{P}(\mathcal{H})_0$.

Proof. We show that the next assertion holds: For any operator $A \in \mathcal{L}(\mathcal{H})_1^+$ the set $S(S(A))(= S(S(\{A\})))$ coincides with

$$\{B \in \mathcal{L}(\mathcal{H})_1^+ : \mathrm{Fix}(A) \subset \mathrm{Fix}(B), \ker A \subset \ker B\}. \tag{1}$$

Clearly, this set is contained in the former one. As for the other inclusion, let $B \in S(S(A))$ be an operator. We pick an element $P \in \mathcal{P}_1(\mathcal{H})$ with $\operatorname{rng} P \subset \ker A$. Then one has $P + (1/2)P^{\perp} \sim A$, thus $B \sim P + (1/2)P^{\perp}$ implying $\operatorname{rng} P \subset \ker B$. It follows that $\ker A \subset \ker B$. Now let $\mathcal{L} \subset \operatorname{Fix}(A)$ be an arbitrary one-dimensional subspace. Then either there exists a linear space $\mathcal{M}_0 \subset \mathcal{L}^{\perp}$ such that $\dim \mathcal{M}_0 = 1$, $\mathcal{M}_0 \cap \ker B = \{0\}$ or for all one-dimensional subspaces $\mathcal{M}$ of $\mathcal{L}^{\perp}$ one has that $\mathcal{M} \subset \ker B$. In the first case let $P \in \mathcal{P}_1(\mathcal{H})$ be the projection with $\operatorname{rng} P = \mathcal{M}_0$ and Q be the projection on $\mathcal{H}$ whose range is $(\mathcal{L} + \mathcal{M}_0)^{\perp}$. Then $P + (1/2)Q \sim A$, therefore $B \sim P + (1/2)Q$, which by the properties of $\mathcal{M}_0$ yields that $\mathcal{L} \subset \operatorname{Fix}(B)$. In the second case we deduce that $\mathcal{L}^{\perp} \subset \ker B$ which implies that these sets coincide. We derive that B is the projection onto $\mathcal{L}$, hence $\mathcal{L} \subset \operatorname{Fix}(B)$. The previous discussion yields $\operatorname{Fix}(A) \subset \operatorname{Fix}(B)$, thus by what we have proved above, we see that B belongs to the set (1). Now we arrive at the conclusion that $S(S(A))$ is contained in that set, completing the verification of their equality.

We infer from this equation that an operator $A \in \mathcal{L}(\mathcal{H})_1^+$ is a projection if and only if $S(S(A))$ is a singleton. By Corollary we obtain that the bijective isometry ϕ preserves the relation $\sim$ and hence satisfies the equality $\phi(S(S(A))) = S(S(\phi(A)))$ for all $A \in \mathcal{L}(\mathcal{H})_1^+$. The previous observations give us that ϕ leaves the set $\mathcal{P}(\mathcal{H})_0$ invariant. Since it is also bijective, the proof is now complete. □

We remark that by Claim 1, $\phi|_{\mathcal{P}(\mathcal{H})_0}$ is a surjective isometry of $\mathcal{P}(\mathcal{H})_0$. However this fact does not contribute to the proof of Theorem, since such maps do not have any regular form. To see it, we deduce from Corollary that if $d(P,Q) < 1$ for some projections $P, Q \in \mathcal{P}(\mathcal{H})_0$, then their ranks coincide. It yields that if $P, Q \in \mathcal{P}(\mathcal{H})_0$ are operators with different ranks, then $d(P,Q) = 1$. It follows that a map of $\mathcal{P}(\mathcal{H})_0$ which acts as a surjective isometry on each of the classes of elements in $\mathcal{P}(\mathcal{H})_0$ with fixed rank is an onto distance preserving transformation and it can be chosen to be of no regular form. We proceed with the assertion below in which we determine the form of $\phi|_{\mathcal{P}(\mathcal{H})_0}$.

Claim 2. *There exists either a unitary or an antiunitary operator U on $\mathcal{H}$ such that one has the equality*

$$\phi(P) = UPU^* \quad (P \in \mathcal{P}(\mathcal{H})_0).$$

Proof. It can be checked that among the projections in $\mathcal{L}(\mathcal{H})_1^+$ the identity is the only one whose distance from any element of their set is 0 or 1. We infer that $\phi(I) = I$ and since the collection of invertible operators in $\mathcal{L}(\mathcal{H})_1^+$ is the open unit ball centered at I, the map ϕ preserves this set. Using the above method which was employed to verify that for any operators $A, B \in \mathcal{L}(\mathcal{H})_1^+$ such that $B \in S(S(A))$ one has $\ker A \subset \ker B$, it can be shown that the following claim holds: If $P, Q \in \mathcal{P}(\mathcal{H})_0$ are projections such that the invertible operators in $S(P)$ belong to $S(Q)$, then $\ker P \subset \ker Q$. The reverse implication is also true, therefore the invariance properties of ϕ imply that it preserves the inclusion between the kernels of elements in $\mathcal{P}(\mathcal{H})_0$. Since $\ker P \subset \ker Q$ holds exactly when $Q \leq P$ for any such operators P, Q, we get that the bijection $\phi|_{\mathcal{P}(\mathcal{H})_0}$ is an order automorphism (in this paper $\leq$ denotes the usual order between projections on $\mathcal{H}$). It induces a lattice automorphism Φ, i.e. an inclusion preserving bijective transformation of the set of all subspaces $\{0\} \neq \mathcal{M} \subset \mathcal{H}$ via the one-to-one correspondence between those spaces and the elements of $\mathcal{P}(\mathcal{H})_0$.

Now we have two cases. First suppose that $n \geq 3$. The fundamental theorem of projective geometry (see, e.g. [1, p.44]) tells us that lattice automorphisms of the family of all subspaces in a linear space with dimension at least 3 are induced by bijective semi-linear operators on that space. Applying it to Φ we deduce that there is an operator W on H of this kind such that for any subspace $\{0\} \neq \mathcal{M} \subset \mathcal{H}$ one has $\Phi(\mathcal{M}) = W(\mathcal{M})$. This means that the equality

$$\operatorname{rng}\phi(P) = W(\operatorname{rng} P) \quad (P \in \mathcal{P}(\mathcal{H})_0) \tag{2}$$

holds. In what follows, we are going to show that W is a scalar multiple of a unitary or an antiunitary operator. To see this, first we recall that the semi-linearity of W means – beside its additivity – the existence of a ring automorphism $h\colon \mathbb{C} \to \mathbb{C}$ for which

$$W(\lambda x) = h(\lambda)Wx \quad (x \in \mathcal{H},\ \lambda \in \mathbb{C}).$$

Observe that on the one hand, for any unit vector $x \in \mathcal{H}$ the set $\operatorname{rng}\phi(P_x)$ - $P_x \in \mathcal{P}_1(\mathcal{H})$ being the projection whose range is the space spanned by x - is the linear hull of $\{Wx\}$. On the other hand, denoting by tr the trace functional, we have the very well-known equality

$$d(P_x, P_y) = ||P_x - P_y|| = \sqrt{1 - |\langle x, y\rangle|^2} = \sqrt{1 - \operatorname{tr} P_x Q_y} \tag{3}$$

for each unit vectors $x, y \in \mathcal{H}$. The previous observations yield that for such vectors x, y the equation

$$\left|\left\langle \frac{1}{||Wx||}Wx, \frac{1}{||Wy||}Wy \right\rangle\right| = |\langle x, y\rangle|$$

holds true.

Then it is straightforward to see that W preserves orthogonality, i.e. if $x, y \in \mathcal{H}$ are arbitrary vectors, then $< Wx, Wy >= 0$ if and only if $< x, y >= 0$. Pick two mutually orthogonal unit vectors $u, v \in \mathcal{H}$ and a number $\lambda \in \mathbb{C}$. Now it is trivial that $\langle \lambda u+v, u-\overline{\lambda}v\rangle = 0$, hence $\langle h(\lambda)Wu+Wv, Wu-h(\overline{\lambda})Wv\rangle = 0$. Making use of the equalities $h(1) = 1$ and $\langle Wu, Wv\rangle = 0$, we infer that $||Wu|| = ||Wv||$ yielding that $h(\overline{\lambda}) = \overline{h(\lambda)}$ which implies $h(\mathbb{R}) \subset \mathbb{R}$. Referring to the fact that the only nonzero ring endomorphism of $\mathbb{R}$ is the identity, we obtain that h is either the identity or the conjugation on $\mathbb{C}$ and thus it follows that W is either linear or conjugate-linear. The orthogonality preserving property of W gives us that for any vectors $x, y \in \mathcal{H}$ with $\langle x, y\rangle = 0$ the relation $\langle W^*Wx, y\rangle = 0$ holds meaning that W^*W is a scalar operator or, equivalently W is a scalar multiple of a unitary or antiunitary operator U. Then using also the equation (2), it is straightforward to see that Claim 2 holds in the case $n \geq 3$.

Now assume that $n = 2$. Then the elements of $\mathcal{P}_1(\mathcal{H})$ are exactly the projections in $\mathcal{P}(\mathcal{H})_0$ different from I, hence we get that $\phi|_{\mathcal{P}_1(\mathcal{H})}$ is a bijection of $\mathcal{P}_1(\mathcal{H})$. Since (3) is valid also in this case, it follows that ϕ preserves the transition probability (the trace of the product) between the operators in $\mathcal{P}_1(\mathcal{H})$. A famous theorem of Wigner (see, e.g. [10, p. 7]) describes the structure of those bijections of the set formed by all rank-one projections on a Hilbert space which leave the transition probability invariant. Applying this statement to $\phi|_{\mathcal{P}_1(\mathcal{H})} \colon \mathcal{P}_1(\mathcal{H}) \to \mathcal{P}_1(\mathcal{H})$, we arrive at the conclusion that there exists either a unitary or an antiunitary operator U on $\mathcal{H}$ such that this restriction can be written in the form $\phi(P) = UPU^* \quad (P \in \mathcal{P}_1(\mathcal{H}))$. Then we see that ϕ has this form on the whole set $\mathcal{P}(\mathcal{H})_0$. The proof of Claim 2 is complete. □

Let $\psi \colon \mathcal{L}(\mathcal{H})_1^+ \to \mathcal{L}(\mathcal{H})_1^+$ be the transformation defined by

$$\psi(A) = U^*\phi(A)U \quad (A \in \mathcal{L}(\mathcal{H})_1^+).$$

Observe that ψ is an isometry which acts as the identity on $\mathcal{P}(\mathcal{H})_0$ and this implies that

$$d(\psi(A), P) = d(A, P) \quad (A \in \mathcal{L}(\mathcal{H})_1^+,\ P \in \mathcal{P}(\mathcal{H})_0). \tag{4}$$

Then referring to Corollary, we easily obtain that for any operator $A \in \mathcal{L}(\mathcal{H})_1^{++}$ and projection $P \in \mathcal{P}(\mathcal{H})_0$ with rank $n-1$ the distance $d(A, P)$ is 1 if and only if $\operatorname{rng} P^{\perp} \subset \operatorname{Fix}(A)$, and the same holds for $\psi(A)$. This together with the equation (4) implies that the assertion below holds.

Claim 3. *For any* $A \in \mathcal{L}(\mathcal{H})_1^{++}$ *one has* $\operatorname{Fix}(\psi(A)) = \operatorname{Fix}(A)$.

In the rest of the proof, we will make use of the following auxiliary lemma.

Lemma 2. *Let $\mathcal{M} \subset \mathcal{H}$ be a subspace and A be a positive operator on $\mathcal{M}$ with $||A|| \leq 1$. If the value $d(A, P)$ is constant for all nontrivial projections P on $\mathcal{M}$, then A is a scalar operator.*

Proof. Since the assertion is trivial in the case where $\mathcal{M}$ is at most one-dimensional, we assume that $r = \dim \mathcal{M} > 1$. Now let $\lambda_1 \geq \ldots \geq \lambda_r$ be the eigenvalues of A counted according to multiplicities and $e_1, \ldots, e_r$ be an orthonormal basis of $\mathcal{M}$ consisting of the corresponding eigenvectors for A. Then for an arbitrary number $i = 1, \ldots, r-1$ we define P_i, resp. Q_i to be the projection on $\mathcal{M}$ onto the subspace generated by $\{e_1, \ldots, e_i\}$, resp.

$$\{e_l : l \in \mathbb{N},\ 1 \leq l \leq i-1\} \cup \{e_{i+1}\}.$$

Since the quantity $d(A, P)$ does not depend on the nontrivial projection P on $\mathcal{M}$ the equality

$$\max\{1 - \lambda_i, \lambda_{i+1}\} = d(A, P_i) = d(A, Q_i) = \max\{1 - \lambda_{i+1}, \lambda_i\}$$

is valid implying $\lambda_i = \lambda_{i+1}$. Now it follows that all eigenvalues of A are the same and this completes the proof. □

We finish this section with the following assertion which immediately implies the statement of Theorem.

Claim 4. *The transformation ψ is the identity.*

Proof. We prove this assertion in an inductive way. Accordingly, first observe that ψ fixes I, the only element in $\mathcal{L}(\mathcal{H})_1^{++}$ whose spectrum contains exactly 1 number. Now let $A \in \mathcal{L}(\mathcal{H})_1^{++}$ be an operator with 2 eigenvalues and let $\sigma(A) = \{1, \lambda\}$, where $\lambda \in]0, 1[$ is a number. Then by Claim 3 one has $\mathrm{Fix}(\psi(A)) = \mathrm{Fix}(A)$. Define $\mathcal{M} = \mathrm{Fix}(A)^\perp$ and let $\tilde{P} \in \mathcal{P}(\mathcal{M})_0$ be a nontrivial projection. Then plugging the element $P \in \mathcal{P}(\mathcal{H})_0$ with $\mathrm{rng}\, P = \mathrm{Fix}(A) + \mathrm{rng}\, \tilde{P}$ into (4), we deduce that for the operator $\tilde{B} = \psi(A)|_{\mathcal{M}}$ one has

$$d(\tilde{B}, \tilde{P}) = d(\lambda I, \tilde{P}) = \max\{1 - \lambda, \lambda\}.$$

It follows that $d(\tilde{B}, \tilde{P})$ is constant for all nontrivial projections $\tilde{P} \in \mathcal{P}(\mathcal{M})_0$. Then referring to Lemma 2, we obtain that $\tilde{B}$ is a scalar operator implying card $\sigma(\tilde{B}) = 1$, i.e. card $\sigma(\psi(A)) = 2$. Moreover, plugging the projection P to $\mathrm{Fix}(A)$ in (4) we see that the smaller eigenvalues of A and $\psi(A)$ coincide. Hence we conclude that $\psi(A)$ and A have the same eigenvalues and this holds also for their corresponding eigenspaces, therefore $\psi(A) = A$. To sum up, for any operator $A \in \mathcal{L}(\mathcal{H})_1^{++}$ with card $\sigma(A) \leq 2$ one has $\psi(A) = A$. Since the collection of such elements is dense in the set of all operators in $\mathcal{L}(\mathcal{H})_1^{+}$ which have at most 2 eigenvalues and ψ is continuous, it is the identity on that set.

It remains to show the next implication. If $r = 2, \dots, n-1$ is a number such that $\psi(A) = A$ for all elements $A \in \mathcal{L}(\mathcal{H})_1^{+}$ with card $\sigma(A) \leq r$, then this holds for any operator $A \in \mathcal{L}(\mathcal{H})_1^{+}$ with card $\sigma(A) \leq r+1$. So assume that the former hypothesis is valid and let $A \in \mathcal{L}(\mathcal{H})_1^{++}$ be an element which possesses the latter property, moreover set $B = \psi(A)$. The inductive hypothesis gives us that for any element $X \in \mathcal{L}(\mathcal{H})_1^{+}$ with card $\sigma(X) \leq r$ one has

$$d(B, X) = d(A, X). \tag{5}$$

Let the spectral decomposition of A, resp. B be $\sum_{i=1}^{r+1} \lambda_i P_i$, resp. $\sum_{j=1}^{s} \mu_j Q_j$ with numbers $1 = \lambda_1 > \ldots > \lambda_{r+1}$, resp. $1 = \mu_1 > \ldots > \mu_s$. We are going to show in an inductive way that the following assertion is valid.

(*) The equality $s \geq r-1$ holds and for each scalar $l \in \mathbb{N}$ with $1 \leq l \leq r-1$ one has $\lambda_l = \mu_l$, $P_l = Q_l$.

To this end, first observe that $\lambda_1 = \mu_1$ and by Claim 3 we have the equality $\mathrm{Fix}(A) = \mathrm{Fix}(B)$ which means $P_1 = Q_1$ yielding that $s \geq 2$.

Next, assume that k is a natural number such that

$$1 \leq k \leq \min\{r-2, s\}, \ \lambda_i = \mu_i, \ P_i = Q_i \quad (i = 1, \dots, k).$$

Then $s \geq k+1$ and substituting $X = \sum_{i=1}^{k} \lambda_i P_i$ in (5) we derive that $\lambda_{k+1} = \mu_{k+1}$. Next, for an arbitrary projection $P \in \mathcal{P}_1(\mathcal{H})$ with $P \leq \sum_{i=k+1}^{r+1} P_i$ insert

$$X = \sum_{i=1}^{k} \lambda_i P_i + \lambda_{k+1} \left(\sum_{i=k+1}^{r+1} P_i - P \right)$$

in (5). In that way, denoting the space $\mathrm{rng} \left(\sum_{i=k+1}^{r+1} P_i \right)$ by $\mathcal{M}_k$ we get that for each operator $\tilde{P} \in \mathcal{P}_1(\mathcal{M}_k)$ the relation $d(A|_{\mathcal{M}_k}, \lambda_{k+1}\tilde{P}^{\perp}) = d(B|_{\mathcal{M}_k}, \lambda_{k+1}\tilde{P}^{\perp})$ holds. Referring to Lemma 1, we easily obtain that $d(A|_{\mathcal{M}_k}, \lambda_{k+1}\tilde{P}^{\perp}) = \lambda_{k+1}$ if and only if $\mathrm{rng}\, \tilde{P} \subset \mathrm{rng}\, P_{k+1}$ and $d(B|_{\mathcal{M}_k}, \lambda_{k+1}\tilde{P}^{\perp}) = \lambda_{k+1}$ exactly when

$\operatorname{rng} \tilde{P} \subset \operatorname{rng} Q_{k+1}$. It follows that the last two inclusions are equivalent meaning that $P_{k+1} = Q_{k+1}$. To sum up, for any number $k \in \mathbb{N}$ satisfying $k \leq r-2$, the conditions

$$s \geq k,\ \lambda_1 = \mu_1, \dots, \lambda_k = \mu_k,\ P_1 = Q_1, \dots, P_k = Q_k$$

imply $s \geq k+1,\ \lambda_{k+1} = \mu_{k+1}$ and $P_{k+1} = Q_{k+1}$.

By what we have proved in the last two paragraphs, we conclude that (*) holds true. It follows that $s \geq r$ and substituting $X = \sum_{i=1}^{r-1} \lambda_i P_i$ in (5), we infer that $\lambda_r = \mu_r$. Now given an arbitrary operator $P \in \mathcal{P}_1(\mathcal{H})$ for which $P \leq P_r + P_{r+1}$, plug

$$X = \sum_{i=1}^{r-1} \lambda_i P_i + \lambda_{r-1} P + \lambda_r (P_r + P_{r+1} - P)$$

in (5). In a very similar way as in the proof of the equality $P_{k+1} = Q_{k+1}$ we arrive at the relations $\lambda_{r+1} = \mu_s,\ P_{r+1} = Q_s$. Next let $\tilde{P} \in \mathcal{P}(\operatorname{rng} P_r)_0$ be a nontrivial projection, P be the operator on $\mathcal{H}$ which is $\tilde{P}$ on $\operatorname{rng} P_r$ and 0 on $\operatorname{rng} P_r^{\perp}$, and plug the operator $X = \sum_{i=1}^{r-1} P_i + P$ in (5). We shall compute the distances $d(A, X)$ and $d(B, X)$ using also Weyl's inequality which gives us that for any self-adjoint operators X, Y on a d-dimensional Hilbert space with eigenvalues $\alpha_1 \geq \ldots \geq \alpha_d$, resp. $\beta_1 \geq \ldots \geq \beta_d$ one has

$$\max\{|\alpha_l - \beta_l| : l = 1, \dots, d\} \leq ||X - Y||.$$

Having this relation in mind, an elementary calculation shows that $d(A, X) = d(A|_{\operatorname{rng} P_r}, \tilde{P})$ and $d(B, X) = d(B|_{\operatorname{rng} P_r}, \tilde{P})$, therefore

$$\max\{\lambda_r, 1 - \lambda_r\} = d(A|_{\operatorname{rng} P_r}, \tilde{P}) = d(B|_{\operatorname{rng} P_r}, \tilde{P}),$$

which implies that $d(B|_{\operatorname{rng} P_r}, \tilde{P})$ is constant for all nontrivial projections $\tilde{P} \in \mathcal{P}(\operatorname{rng} P_r)_0$. By Lemma 2 we deduce that $B|_{\operatorname{rng} P_r}$ is a scalar operator and since its largest eigenvalue is $\mu_r = \lambda_r$, one has $B|_{\operatorname{rng} P_r} = \lambda_r I$. This yields that P_r is the eigenspace of B that corresponds to μ_r, i.e. $P_r = Q_r$. By what we have proved so far, we arrive at the conclusion that $s = r+1$ and $\lambda_j = \mu_j$ $(j = 1, \dots, r)$, moreover $P_i = Q_i$ $(i = 1, \dots, r+1)$. We see that $A = B = \psi(A)$ and since A was an arbitrary element of $\mathcal{L}(\mathcal{H})_1^{++}$ with card $\sigma(A) \leq r+1$, we get that ψ is the identity on the set of such operators. This set is dense in the collection of all elements of $\mathcal{L}(\mathcal{H})_1^{+}$ whose spectra contains to a maximum $r+1$ scalars, so it follows that $\psi(A) = A$ for all operators $A \in \mathcal{L}(\mathcal{H})_1^{+}$ having at most $r+1$ eigenvalues. Now the proof is complete. $\square$

3. Remarks

We close the paper with two remarks. The first one concerns the natural question, are there any infinite dimensional versions of Theorem. As far as we know, there are no such results in the literature. We conjecture that in infinite dimension the statement of Theorem holds true for surjective isometries ϕ. However, to be honest we do not know how to prove or disprove it. Observe that the above proof of Theorem can not be applied for that purpose, since in the case $\dim \mathcal{H} = \infty$ there is no infinite dimensional version of Claim 1.

Our second remark concerns another question which also arises naturally in relation with Theorem. This is the problem of describing the general form of isometries of the set $\mathcal{L}(\mathcal{H})^+$ formed by all positive operators on $\mathcal{H}$ under the condition that $\mathcal{H}$ is finite dimensional. It can be solved using known results in the following way. Assume $\dim \mathcal{H} < \infty$ and let $\phi\colon \mathcal{L}(\mathcal{H})^+ \to \mathcal{L}(\mathcal{H})^+$ be an isometry. Then by continuity and injectivity, the range of the restriction Φ of ϕ to the open connected set of positive invertible operators on $\mathcal{H}$ is connected and open in the space $\mathcal{L}_s(\mathcal{H})$ of self-adjoint operators on $\mathcal{H}$. The latter property of that range follows from the invariance of domain theorem. Now we need a result of Mankiewicz [9, Theorem 5] which states that each onto isometry between open and connected subsets of normed spaces can be extended to a bijective affine distance preserving map between those spaces. Applying this statement to Φ, we infer that it is the restriction of a surjective affine isometry of $\mathcal{L}_s(\mathcal{H})$ which is clearly an extension of ϕ and – up to a translation – an onto linear isometry. By [8, Theorem 2] the latter map and therefore also ϕ can be written in the form $\phi(A) = \tau U A U^* + X$ $(A \in \mathcal{L}(\mathcal{H})^+)$ where $\tau \in \{-1, 1\}$ is a number, $X \in \mathcal{L}(\mathcal{H})^+$ is an operator and U is a unitary or an antiunitary operator on $\mathcal{H}$. It follows that $\tau = 1$, thus ϕ is the composition of the conjugation by U and the translation by X.

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