TWO-DIMENSIONAL LANDSBERG MANIFOLDS WITH VANISHING DOUGLAS TENSOR

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Abstract. The present paper can be considered a continuation of [9] and deals with two-dimensional Landsberg manifolds with vanishing Douglas tensor. Using coordinate-free methods only, we first elaborate a general theory of two-dimensional Finsler manifolds, and derive their fundamental equations, including E. Cartan's "permutation formulas", intrinsically. Second, we express the Douglas tensor of a two-dimensional Landsberg manifold in terms of the main scalar and the Berwald frame. Finally, we determine the iterated Lie derivatives, up to the fifth order, of the main scalar with respect to the members of the Berwald frame in a two-dimensional Landsberg manifold with vanishing Douglas tensor, and deduce that such a manifold is a Berwald manifold.

Introduction

In his outstanding work [2], Ludwig Berwald showed that a two-dimensional Landsberg manifold reduces to a Berwald manifold if its Douglas tensor vanishes. In his own words: "*The Landsberg spaces, the extremals of which form a quasigeodesic system of curves, are identical with the two-dimensional affinely connected Finsler spaces.*" ([2], p. 110.) The notion of a "quasigeodesic system of curves" was introduced by W. Blaschke and G. Bol in their book [3]. In modern language, the condition on the extremals states that the geodesics of a Landsberg manifold coincide with those of a linear connection on the underlying manifold. Since by the Douglas-Shen theorem ([7], 6.6) this property is equivalent to the vanishing of the Douglas tensor, our formulation is indeed a translation of Berwald's theorem.

The analogous result in the higher dimensional case was proved in [9] with the help of the projection of the Douglas tensor onto the indicatrix bundle. Unfortunately, this elegant method does not work in two dimensions, since the Douglas tensor then has components only along the Liouville vector field by formula (26a) of Remark 2.4 below. Since the Liouville vector field is orthogonal to the unit sphere bundle, we infer immediately that *the projected Douglas tensor of a two-dimensional Finsler manifold vanishes identically.* So we have to search for a completely different plan of attack in the two-dimensional case. Our approach is based on Berwald's original ideas, and it may be considered as an intrinsic version of them.

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We shall adopt throughout the notations, terminology and conventions of [9], with one restriction. In this paper (M, E) will denote a *positive definite two-dimensional Finsler manifold*. (There is a generalization of the theorem for two-dimensional Finsler manifolds with nondegenerate Riemann-Finsler metric. It also needs a little modification of Berwald's method as we can see in [1] using the machinery of classical tensor calculus.)

1. Two-dimensional Finsler manifolds

1.1 The Berwald frame. Starting from the Liouville vector field C and the canonical spray S of (M, E), let us first consider the unit vector fields

$$C_0 := \frac{1}{\sqrt{2E}} C \quad \text{and} \quad S_0 := \frac{1}{\sqrt{2E}} S.$$

Next, using the Gram-Schmidt process we construct, at least locally, a g-orthonormal basis (C_0, X_0) of $\mathfrak{X}^v(\mathcal{T}M)$, where $g \in \mathcal{T}_2^0(\mathcal{T}M)$ is the metric tensor given by (20) in [9]. Applying the almost complex structure F associated with the Barthel endomorphism of (M, E), we obtain a local g-orthonormal basis (FX_0, S_0) of $\mathfrak{X}^h(\mathcal{T}M)$. The quadruple

$$(C_0, X_0, FX_0, S_0)$$

constructed in this way is a (local) orthonormal basis of $\mathfrak{X}(\mathcal{T}M)$; it is called the *Berwald frame* of the Finsler manifold (M, E).

Note that

$$X_0 E = 0. \tag{1}$$

This can be seen by a straightforward calculation:

$$\begin{aligned} 0 = g(C, X_0) &\overset{(20)/[9]}{=} \omega(C, FX_0) \overset{(15)/[9]}{=} d_J E(FX_0) \\ &= dE[JF(X_0)] \overset{(13)/[6]}{=} dE[v(X_0)] = dE(X_0) = X_0 E. \end{aligned}$$

1.2 Proposition. *The members of the Berwald frame have the following homogeneity properties:*

$$[C, C_0] = -C_0, \tag{2}$$

$$[C, S_0] = 0, \tag{3}$$

$$[C, X_0] = -X_0, \tag{4}$$

$$[C, FX_0] = 0. \tag{5}$$

Proof. Taking into account the fact that the function $\frac{1}{\sqrt{2E}}$ is homogeneous of degree -1, i.e., that

$$C\left(\frac{1}{\sqrt{2E}}\right) = -\frac{1}{\sqrt{2E}}, \tag{6}$$

we obtain

$$[C, C_0] = [C, \frac{1}{\sqrt{2E}}C] = C\left(\frac{1}{\sqrt{2E}}\right)C = -\frac{1}{\sqrt{2E}}C = -C_0,$$

which shows (2). Similarly, since S is a spray and hence $[C, S] = S$,

$$[C, S_0] = \frac{1}{\sqrt{2E}}[C, S] + C\left(\frac{1}{\sqrt{2E}}\right)S = \frac{1}{\sqrt{2E}}S + C\left(\frac{1}{\sqrt{2E}}\right)S \overset{(6)}{=} 0,$$

so (3) is also true.

In view of the definition (21) and the property (27) in [9] of the first Cartan tensor $\mathcal{C}$,

$$\begin{aligned} 0 &= 2g(\mathcal{C}(S, FX_0), X_0) = \mathcal{L}_C(J^*g)(FX_0, FX_0) \\ &= Cg(X_0, X_0) - 2g(J[C, FX_0], X_0) \overset{1.1}{=} -2g(J[C, FX_0], X_0). \end{aligned}$$

In the same way we find that

$$\begin{aligned} 0 &= 2g(\mathcal{C}(S, FX_0), C) = Cg(X_0, C) - g(J[C, FX_0], C) - g(X_0, J[C, S]) \\ &\overset{1.1}{=} -g(J[C, FX_0], C) - g(X_0, C) \overset{1.1}{=} -g(J[C, FX_0], C). \end{aligned}$$

It follows from the last two relations that $J[C, FX_0] = 0$, and so $[C, FX_0]$ is vertical. Thus,

$$X_0 = vX_0 \overset{(13)/[6]}{=} JF(X_0) \overset{(9)/[6]}{=} [J, C](FX_0) = [X_0, C] - J[FX_0, C] = -[C, X_0]$$

which proves the relation (4).

Taking into account again the fact that the vector field $[FX_0, C]$ is vertical, from the homogeneity of the Barthel endomorphism (see (18) in [9]) we obtain

$$\begin{aligned} 0 &= [h, C](FX_0) = [h(FX_0), C] - h[FX_0, C] \\ &= [h(FX_0), C] \overset{(13)/[6]}{=} [F(vX_0), C] = [FX_0, C] \end{aligned}$$

whence (5). □

1.3 Proposition. *With the hypothesis above, the following relations hold:*

(7) $\mathcal{C}'(FX_0, FX_0) = v[X_0, FX_0] \quad (v := 1 - h)$,

(8) $v[X_0, S] = 0$,

(9) $\forall Y \in \mathfrak{X}(\mathcal{T}M) : D_{hY}X_0 = 0$, *where D is the Cartan connection.*

Proof. According to the definition of the second Cartan tensor $\mathcal{C}'$ (see e.g. [9], formula (23)),

$2g(\mathcal{C}'(FX_0, FX_0), X_0) = \left(\mathcal{L}_{h(FX_0)}g\right)(J(FX_0), J(FX_0)) \overset{(13)/[6]}{=} (\mathcal{L}_{FX_0}g)(X_0, X_0) = FX_0g(X_0, X_0) - 2g([FX_0, X_0], X_0) \overset{1.1}{=} 2g([X_0, FX_0], X_0) = 2g(v[X_0, FX_0], X_0) + 2g(h[X_0, FX_0], X_0) = 2g(v[X_0, FX_0], X_0)$, using the g-orthogonality of the vertical and horizontal subbundle in the last step. From this we obtain the relation

$$g(\mathcal{C}'(FX_0, FX_0) - v[X_0, FX_0], X_0) = 0. \tag{10}$$

Furthermore, by the properties (26), (27) in [9] of the second Cartan tensor, we can write

$$\begin{aligned} 0 &= 2g(\mathcal{C}'(FX_0, FX_0), C) = FX_0 g(X_0, C) - g([FX_0, X_0], C) - g(X_0, [FX_0, C]) \\ &\overset{\mathbf{1.1},\,(5)}{=} g([X_0, FX_0], C) = g(v[X_0, FX_0], C) + g(h[X_0, FX_0], C) = g(v[X_0, FX_0], C). \end{aligned}$$

This means that the equality

$$g(\mathcal{C}'(FX_0, FX_0) - v[X_0, FX_0], C) = 0 \tag{11}$$

is valid automatically. The relations (10) and (11) show that the vertical vector field $\mathcal{C}'(FX_0, FX_0) - v[X_0, FX_0]$ is orthogonal to two nowhere vanishing vertical vector fields. Hence it must be the zero vector field, which proves (7).

Similarly, on the one hand

$$\begin{aligned} 0 &= 2g(\mathcal{C}'(S, FX_0), X_0) = Sg(X_0, X_0) - 2g([S, X_0], X_0) = 2g([X_0, S], X_0) \\ &= 2g(v[X_0, S], X_0) + 2g(h[X_0, S], X_0) = 2g(v[X_0, S], X_0) \end{aligned}$$

on the other hand

$$\begin{aligned} 0 &= 2g(\mathcal{C}'(S, FX_0), C) = Sg(X_0, C) - g([S, X_0], C) - g(X_0, [S, C]) = g([X_0, S], C) \\ &\quad + g(X_0, S) = g([X_0, S], C) = g(v[X_0, S], C) + g(h[X_0, S], C) = g(v[X_0, S], C), \end{aligned}$$

which imply the relation (8).

To prove the formula (9) it is sufficient to check that $D_{FX_0} X_0$ and $D_S X_0$ vanish. But this is immediate:

$$D_{FX_0} X_0 \overset{1.6/[9]}{=} v[FX_0, X_0] + \mathcal{C}'(FX_0, FX_0) \overset{(7)}{=} 0, \quad D_S X_0 \overset{1.6/[9]}{=} v[S, X_0] \overset{(8)}{=} 0. \quad \square$$

1.4 Definition and remark. *The function*

$$\lambda := g(\mathcal{C}(FX_0, FX_0), X_0) \tag{12}$$

is said to be the main scalar *of* (M, E) *with respect to the Berwald frame* (C_0, X_0, FX_0, S_0). *– Actually,* λ *depends only on the choice of* X_0 *and it is uniquely determined up to sign.*

1.5 Lemma. *With the help of the main scalar and the vector field* X_0*, the first Cartan tensors can be represented in the form*

$$\mathcal{C} = \lambda i_{X_0}\omega \otimes i_{X_0}\omega \otimes X_0 \quad \text{and} \quad \mathcal{C}_\flat = \lambda i_{X_0}\omega \otimes i_{X_0}\omega \otimes i_{X_0}\omega, \tag{13}$$

where ω *is the fundamental two-form.*

Proof. The vertical vector field $\mathcal{C}(FX_0, FX_0)$ can be uniquely represented as a linear combination

$$\mathcal{C}(FX_0, FX_0) = \lambda_1 X_0 + \lambda_2 C_0, \quad \lambda_1, \lambda_2 \in C^\infty(\mathcal{T}M).$$

Since on the one hand

$$g(\mathcal{C}(FX_0, FX_0), C_0) = \frac{1}{\sqrt{2E}} \mathcal{C}_\flat(FX_0, FX_0, S) \overset{(27)/[8]}{=} 0$$

on the other hand

$$g(\mathcal{C}(FX_0, FX_0), C_0) = g(\lambda_1 X_0 + \lambda_2 C_0, C_0) = \lambda_2,$$

it follows that $\lambda_2 = 0$. So

$$\lambda := g(\mathcal{C}(FX_0, FX_0), X_0) = g(\lambda_1 X_0, X_0) = \lambda_1,$$

hence $\mathcal{C}(FX_0, FX_0) = \lambda X_0$, where λ is the main scalar. From this observation we infer immediately (13). □

1.6 Corollary. *A positive definite two-dimensional Finsler manifold is a Riemannian manifold if and only if its main scalar vanishes.*

Proof. This is an immediate consequence of (13), because (M, E) is a Riemannian manifold if and only if $\mathcal{C} = 0$. □

1.7 Proposition. *A positive definite two-dimensional Finsler manifold is a Berwald manifold if and only if the horizontal vector fields annihilate the main scalar, i.e., $d_h \lambda = 0$.*

Proof. We first recall that a Finsler manifold is a Berwald manifold if and only if the h-covariant derivative, with respect to the Cartan connection, of the first Cartan tensor vanishes. (An intrinsic proof of this well-known fact is available in [8].) Thus we can argue as follows:

(M, E) *is a Berwald manifold* $\Leftrightarrow \forall Y \in \mathfrak{X}(\mathcal{T}M) : D_{hY}\mathcal{C} = 0$
$\Leftrightarrow \forall Y \in \mathfrak{X}(\mathcal{T}M) : 0 = (D_{hY}\mathcal{C})(FX_0, FX_0) = D_{hY}[\mathcal{C}(FX_0, FX_0)]$
$-2\mathcal{C}(D_{hY}FX_0, FX_0) \overset{(\mathrm{F{\small INS}}2)/[6]}{=} D_{hY}[\mathcal{C}(FX_0, FX_0)] - 2\mathcal{C}(FD_{hY}X_0, FX_0) \overset{(9)}{=}$
$D_{hY}[\mathcal{C}(FX_0, FX_0)] \overset{(13)}{=} D_{hY}(\lambda X_0) = [(hY)\lambda]X_0 + \lambda D_{hY}X_0 \overset{(9)}{=} [(hY)\lambda]X_0$
$= [d_h\lambda(Y)]X_0 \Leftrightarrow d_h\lambda = 0.$ □

1.8 Corollary. *The second Cartan tensor of (M, E) is completely determined by the formula*

$$\mathcal{C}'(FX_0, FX_0) = -S(\lambda)X_0. \tag{14}$$

Proof. Since $D_S\mathcal{C} = -\mathcal{C}'$ (see [5], Prop. (A.12)), taking into account the previous proof we obtain

$$\begin{aligned}\mathcal{C}'(FX_0, FX_0) &= -(D_S\mathcal{C})(FX_0, FX_0) = -[d_h\lambda(S)]X_0 \\ &= -[d\lambda(hS)]X_0 = [-(d\lambda)S]X_0 = -S(\lambda)X_0.\end{aligned}$$ □

1.9 Proposition and definition. *Let us consider the curvature tensor* $R = -\frac{1}{2}[h,h]$ *of* (M,E). *Then*

$$R(FX_0, S_0) = g(R(FX_0, S_0), X_0)X_0, \tag{15}$$

and R *is uniquely determined by this formula on the domain of the Berwald frame constructed in* **1.1**. *The function*

$$\kappa := g(R(FX_0, S_0), X_0) \tag{16}$$

is said to be the Gauss curvature *of* (M,E).

Proof. Since R is a semibasic tensor of type $(1,2)$, it is uniquely determined by its value on the local basis (FX_0, S_0) of $\mathfrak{X}^h(\mathcal{T}M)$. So our only task is to verify the equality (15). Starting with the definition of the Nijenhuis torsion, we obtain

$$R(FX_0, S_0) = -([hFX_0, hS_0] + h^2[FX_0, S_0] - h[FX_0, hS_0] - h[hFX_0, S_0]) = -[FX_0, S_0] - h[FX_0, S_0] + 2h[FX_0, S_0] = -v[FX_0, S_0].$$

Now, if

$$R(FX_0, S_0) = f_1X_0 + f_2C_0 \quad (f_1, f_2 \in C^\infty(\mathcal{T}M)),$$

then on the one hand

$$g(R(FX_0, S_0), C) = g(f_1X_0 + f_2C_0, C) = f_2g(C_0, C) = \frac{1}{\sqrt{2E}}f_2g(C, C) = \sqrt{2E}\, f_2$$

on the other hand, using repeatedly the fact that $d_hE = 0$,

$$\begin{aligned} g(R(FX_0, S_0), C) &= -g(v[FX_0, S_0], C) = -\omega(v[FX_0, S_0], FC) = -\omega(v[FX_0, S_0], S) \\ &= i_S\omega(v[FX_0, S_0]) \overset{(16)/[9]}{=} -v[FX_0, S_0](E) \\ &= h[FX_0, S_0](E) - [FX_0, S_0](E) = [S_0, FX_0](E) \\ &= S_0[FX_0(E)] - FX_0[S_0(E)] = 0. \end{aligned}$$

These imply that $f_2 = 0$, $R(FX_0, S_0) = f_1X_0$, and

$$f_1 = g(f_1X_0, X_0) = g(R(FX_0, S_0), X_0)$$

whence (15). □

1.10 Theorem (E. Cartans*'s "permutation formulas"*)**.** *For the Lie brackets of the members of the Berwald frame we have*

$$\boxed{\begin{aligned} [X_0, FX_0] &= -\frac{1}{\sqrt{2E}}S_0 - \lambda FX_0 - S(\lambda)X_0 \\ [S_0, X_0] &= -\frac{1}{\sqrt{2E}}FX_0 \\ [FX_0, S_0] &= -\kappa X_0 \end{aligned}}, \tag{17a-c}$$

where λ *is the main scalar,* κ *is the Gauss curvature of* (M,E).

Proof. Since

$$[X_0, FX_0] = v[X_0, FX_0] + h[X_0, FX_0] \overset{(7),\,(14)}{=} -S(\lambda)X_0 + h[X_0, FX_0],$$

to prove the relation (17a) it remains to be shown that

$$h[X_0, FX_0] = -\frac{1}{\sqrt{2E}} S_0 - \lambda FX_0. \tag{18}$$

First we observe that

$$\begin{aligned} 2\lambda \overset{(12)}{=} 2g(\mathcal{C}(FX_0, FX_0), X_0) &= X_0 g(X_0, X_0) \\ - 2g(J[X_0, FX_0], X_0) &= -2g(J[X_0, FX_0], X_0) \end{aligned}$$

whence

$$g(J[X_0, FX_0], X_0) = -\lambda. \tag{19}$$

Similarly,

$$\begin{aligned} 0 &= 2g(\mathcal{C}(FX_0, FX_0), C) = X_0 g(X_0, C) - g(J[X_0, FX_0], C) - g(X_0, J[X_0, S]) \\ &= -g(J[X_0, FX_0], C) - g(X_0, J[X_0, S]) \overset{\text{Prop. I.7/[4]}}{=} -g(J[X_0, FX_0], C) - g(X_0, X_0). \end{aligned}$$

Hence it follows that

$$g(J[X_0, FX_0], C_0) = -\frac{1}{\sqrt{2E}}. \tag{20}$$

Now we use orthonormal expansion to express $J[X_0, FX_0]$ in terms of the local basis (X_0, C_0) of $\mathfrak{X}^v(\mathcal{T}M)$:

$$\begin{aligned} J[X_0, FX_0] &= g(J[X_0, FX_0], X_0)X_0 + g(J[X_0, FX_0], C_0)C_0 \\ &\overset{(19),\,(20)}{=} -\lambda X_0 - \frac{1}{\sqrt{2E}} C_0. \end{aligned}$$

In view of the identity $F \circ J = h$, from this we obtain (18) and hence (17a).

For the second formula (17b) we have

$$\begin{aligned} [S_0, X_0] &= h[S_0, X_0] + v[S_0, X_0] \overset{(8)}{=} h[S_0, X_0] = -FJ[X_0, \frac{1}{\sqrt{2E}} S] \\ &\overset{(1)}{=} -\frac{1}{\sqrt{2E}} FJ[X_0, S_0] \overset{\text{Prop. I.7/[3]}}{=} -\frac{1}{\sqrt{2E}} FX_0. \end{aligned}$$

To prove (17c), first it will be shown that the vector field $[FX_0, S_0]$ is vertical, i.e.,

$$h[FX_0, S_0] = 0. \tag{21}$$

The vanishing of the h-horizontal torsion of the Cartan connection (see [9], **1.7** or [6], (M3)) yields

$$h[FX_0, S_0] = D_{FX_0}S_0 - D_{S_0}FX_0 = D_{FX_0}S_0 - FD_{S_0}X_0 \overset{(9)}{=} D_{FX_0}S_0.$$

Using the fact that $d_hE = 0$ and the vanishing of the h-deflection of the Cartan connection ([6], (M4)) we obtain

$$\begin{aligned} D_{FX_0}S_0 = D_{FX_0}\left(\frac{1}{\sqrt{2E}}S\right) &= FX_0\left(\frac{1}{\sqrt{2E}}\right)S + \frac{1}{\sqrt{2E}}D_{FX_0}S \\ &= \frac{1}{\sqrt{2E}}D_{FX_0}FC = \frac{1}{\sqrt{2E}}FD_{FX_0}C = 0, \end{aligned}$$

thus (21) is true. By this observation and taking into account **1.9** it follows that

$$[FX_0, S_0] = v[FX_0, S_0] = -R(FX_0, S_0) = -\kappa X_0,$$

completing the proof. □

1.11 Proposition ("Bianchi identity")**.**

$$\boxed{\lambda\kappa + X_0(\kappa) + S_0(S\lambda) = 0}. \tag{22}$$

Proof. Starting with the Jacobi identity, taking into account that both FX_0 and S_0 are horizontal vector fields and $d_hE = 0$, finally using (17a-c) we obtain
$0 = [X_0, [S_0, FX_0]] + [FX_0, [X_0, S_0]] + [S_0, [FX_0, X_0]] \overset{(17a-c)}{=} [X_0, \kappa X_0]+$ $[FX_0, \frac{1}{\sqrt{2E}}FX_0] + [S_0, \frac{1}{\sqrt{2E}}S_0 + \lambda FX_0 + S(\lambda)X_0] = (X_0\kappa)X_0 + \lambda[S_0, FX_0]+$ $S_0(\lambda)FX_0 + S(\lambda)[S_0, X_0] + S_0(S\lambda)X_0 \overset{(17b-c)}{=} (X_0\kappa)X_0 + \lambda\kappa X_0 + S_0(\lambda)FX_0 -$ $\frac{1}{\sqrt{2E}}S(\lambda)FX_0 + S_0(S\lambda)X_0 = [X_0(\kappa) + \lambda\kappa + S_0(S\lambda)]X_0$
whence (22). □

2. Two-dimensional landsberg manifolds with vanishing Douglas tensor

2.1 Proposition. *A positive definite two-dimensional Finsler manifold is a Landsberg manifold if and only if the main scalar is a first integral of the canonical spray, i.e.,*

$$S(\lambda) = 0. \tag{23}$$

Proof. This is an immediate consequence of (14) and **2.1** in [9]. □

2.2 Lemma. *Suppose that (M, E) is a (positive definite, two-dimensional) Landsberg manifold. Then the mixed curvature and the mixed Ricci tensor of the Berwald connection are completely determined by the formulas*

$$\overset{\circ}{\mathbb{P}}(FX_0, FX_0)FX_0 = -FX_0(\lambda)X_0, \tag{24}$$

$$\overset{\tilde{\circ}}{\mathbb{P}}(FX_0, FX_0) = -FX_0(\lambda), \tag{25}$$

where λ is the main scalar of (M, E).

Proof. (25) is a trivial consequence of (24). To prove (24), we first recall that the relation

$$\overset{\circ}{\mathbb{P}}(X, Y)Z = -\left(D_{hX}\mathcal{C}\right)(Y, Z) \quad (X, Y, Z \in \mathfrak{X}(\mathcal{T}M))$$

holds in any Landsberg manifold (see e.g. [9], **2.1**/(iv)). Thus, in our case

$$\begin{aligned}\overset{\circ}{\mathbb{P}}(FX_0, FX_0)FX_0 &= -\left(D_{FX_0}\mathcal{C}\right)(FX_0, FX_0) = -D_{FX_0}\mathcal{C}(FX_0, FX_0) \\ &\quad + 2\mathcal{C}\left(D_{FX_0}FX_0, FX_0\right) \overset{(13)}{=} -D_{FX_0}\lambda X_0 \\ &\quad + 2\mathcal{C}\left(FD_{FX_0}X_0, FX_0\right) \overset{(9)}{=} -FX_0(\lambda)X_0\end{aligned}$$

whence (25). □

2.3 Proposition. *The Douglas tensor of a positive definite two-dimensional Landsberg manifold is completely determined by*

$$\boxed{\mathbb{D}(FX_0, FX_0)FX_0 = \frac{1}{3}[X_0(FX_0(\lambda) + 2\lambda FX_0(\lambda)]C}\,. \tag{26}$$

Proof. As we know from **6.2** and **6.3** in [7], $\mathbb{D}$ is semibasic, symmetric, and for any semispray S_0, $i_{S_0}\,\mathbb{D} = 0$. Thus, in two dimensions $\mathbb{D}$ is completely determined by its value on the triplet (FX_0, FX_0, FX_0). According to 6.2/(b) in [7],
$\mathbb{D}(FX_0, FX_0)FX_0 = \overset{\circ}{\mathbb{P}}(FX_0, FX_0)FX_0 - \frac{1}{3}\left(\overset{\circ}{D}_J\overset{\tilde{\circ}}{\mathbb{P}}\right)(FX_0, FX_0, FX_0)C -$
$\overset{\tilde{\circ}}{\mathbb{P}}(FX_0, FX_0)X_0 \overset{(24),\,(25)}{=} -\frac{1}{3}\left(\overset{\circ}{D}_{X_0}\overset{\tilde{\circ}}{\mathbb{P}}\right)(FX_0, FX_0)C = -\frac{1}{3}\left[X_0\left(\overset{\tilde{\circ}}{\mathbb{P}}(FX_0, FX_0)\right) -\right.$
$\left.2\overset{\tilde{\circ}}{\mathbb{P}}\left(\overset{\circ}{D}_{X_0}FX_0, FX_0\right)\right]C \overset{(25)}{=} \frac{1}{3}\left[X_0(FX_0(\lambda)) + 2\overset{\tilde{\circ}}{\mathbb{P}}\left(\overset{\circ}{D}_{X_0}FX_0, FX_0\right)\right]C.$

There remains only to calculate the second member of the right hand side. Applying the rules of calculation of the Berwald connection ((27) and (Brw1–4) in [6]) we obtain

$$\begin{aligned}\overset{\circ}{D}_{X_0}FX_0 &= F\overset{\circ}{D}_{JFX_0}JFX_0 = FJ[X_0, FX_0] = h[X_0, FX_0] \\ &\overset{(17a)}{=} -\frac{1}{\sqrt{2E}}S_0 - \lambda(h \circ F)X_0 - S(\lambda)hX_0 = -\frac{1}{\sqrt{2E}}S_0 - \lambda FX_0.\end{aligned}$$

Hence

$$\begin{aligned}\overset{\circ}{\mathbb{P}}\left(\overset{\circ}{D}_{X_0}FX_0, FX_0\right) &= -\frac{1}{2E}\overset{\tilde{\circ}}{\mathbb{P}}(S, FX_0) - \lambda\overset{\tilde{\circ}}{\mathbb{P}}(FX_0, FX_0) \\ &\overset{(4.4a)/[7]}{=} -\lambda\overset{\tilde{\circ}}{\mathbb{P}}(FX_0, FX_0) \overset{(25)}{=} \lambda FX_0(\lambda),\end{aligned}$$

and the result follows. □

2.4 Remark. Using the same technique, it may be proved that the effect of the tensors $\overset{\circ}{\mathbb{P}}$, $\overset{\tilde{\circ}}{\mathbb{P}}$ and $\mathbb{D}$ can be described analogously in *any* (positive definite, two-dimensional) Finsler manifold. More precisely, the following relations are fulfilled in the general case:

$$\overset{\circ}{\mathbb{P}}(FX_0, FX_0)FX_0 = -[FX_0(\lambda) + X_0(S\lambda)]X_0 + 2\frac{S(\lambda)}{\sqrt{2E}}C_0, \tag{24a}$$

$$\overset{\tilde{\circ}}{\mathbb{P}}(FX_0, FX_0)FX_0 = -FX_0(\lambda) - X_0(S\lambda), \tag{25a}$$

$$\begin{aligned}\mathbb{D}(FX_0, FX_0)FX_0 = \frac{1}{3}\Big[&X_0(FX_0(\lambda)) + X_0(X_0(S\lambda)) \\ &+ 2\lambda FX_0(\lambda) + 2X_0(S\lambda) + 3\frac{S(\lambda)}{E}\Big]C.\end{aligned} \tag{26a}$$

2.5 Lemma. *Suppose that (M, E) is a positive definite two-dimensional Landsberg manifold with vanishing Douglas tensor. The iterated Lie derivatives of the main scalar with respect to the vector fields X_0, FX_0, S_0 (up to fifth order) can be expressed as follows:*

$$\mathcal{L}_{FX_0}\mathcal{L}_{X_0}\lambda = -\lambda\mathcal{L}_{FX_0}\lambda, \tag{27}$$

$$\mathcal{L}_{S_0}\mathcal{L}_{X_0}\lambda = -\frac{1}{\sqrt{2E}}\mathcal{L}_{FX_0}\lambda, \tag{28}$$

$$\mathcal{L}_{FX_0}\mathcal{L}^2_{X_0}\lambda = \left(\lambda^2 - \mathcal{L}_{X_0}\lambda - \frac{1}{2E}\right)\mathcal{L}_{FX_0}\lambda, \tag{29}$$

$$\mathcal{L}_{S_0}\mathcal{L}^2_{X_0}\lambda = \frac{3}{\sqrt{2E}}\lambda\mathcal{L}_{FX_0}\lambda, \tag{30}$$

$$\mathcal{L}_{FX_0}\mathcal{L}^3_{X_0}\lambda = \left(-\lambda^3 + 3\lambda\mathcal{L}_{X_0}\lambda - \mathcal{L}^2_{X_0}\lambda + \frac{2}{E}\lambda\right)\mathcal{L}_{FX_0}\lambda, \tag{31}$$

$$\mathcal{L}_{S_0}\mathcal{L}^3_{X_0}\lambda = \frac{1}{\sqrt{2E}}\left(4\mathcal{L}_{X_0}\lambda - 7\lambda^2 + \frac{1}{2E}\right)\mathcal{L}_{FX_0}\lambda, \tag{32}$$

$$\begin{aligned}\mathcal{L}_{FX_0}\mathcal{L}^4_{X_0}\lambda = \Big(&-\mathcal{L}^3_{X_0}\lambda + 4\lambda\mathcal{L}^2_{X_0}\lambda + 3(\mathcal{L}_{X_0}\lambda)^2 - 6\lambda^2\mathcal{L}_{X_0}\lambda \\ &+ \frac{4}{E}\mathcal{L}_{X_0}\lambda - \frac{11}{2E}\lambda^2 + \lambda^4 + \frac{1}{(2E)^2}\Big)\mathcal{L}_{FX_0}\lambda\end{aligned} \tag{33}$$

($\mathcal{L}^n_{X_0} := \mathcal{L}_{X_0} \circ \ldots \circ \mathcal{L}_{X_0}$ (n times)).

Proof. We shall verify only the first three formulas, the remaining ones can be handled in the same way. First we observe that the vanishing of the Douglas tensor implies by (26) the relation

$$\boxed{X_0(FX_0(\lambda)) = -2\lambda FX_0(\lambda)}. \tag{34}$$

From now on we calculate.

(a) $\mathcal{L}_{FX_0}\mathcal{L}_{X_0}\lambda = [FX_0, X_0]\lambda + \mathcal{L}_{X_0}\mathcal{L}_{FX_0}\lambda \overset{(34)}{=} [FX_0, X_0]\lambda - 2\lambda FX_0(\lambda) \overset{(17a),\ (23)}{=}$ $\frac{1}{\sqrt{2E}}(S_0\lambda) + \lambda(FX_0)\lambda - 2\lambda(FX_0)\lambda \overset{(23)}{=} -\lambda(FX_0)\lambda = -\lambda\mathcal{L}_{FX_0}\lambda,$ thus (27) is proved.

(b) $\mathcal{L}_{S_0}\mathcal{L}_{X_0}\lambda = [S_0, X_0]\lambda + X_0(S_0\lambda) \overset{(17b),\ (23)}{=} -\frac{1}{\sqrt{2E}}(FX_0)\lambda = -\frac{1}{\sqrt{2E}}\mathcal{L}_{FX_0}\lambda,$ so we have obtained (28).

(c) $\mathcal{L}_{FX_0}\mathcal{L}^2_{X_0}\lambda = \mathcal{L}_{FX_0}\mathcal{L}_{X_0}(X_0\lambda) = [FX_0, X_0](X_0\lambda) + X_0[FX_0(X_0\lambda)] \overset{(17a),\ (23)}{=}$ $\frac{1}{\sqrt{2E}}S_0(X_0\lambda) + \lambda(FX_0)(X_0\lambda) + X_0[FX_0(X_0\lambda)] \overset{(28),\ (27)}{=} -\frac{1}{2E}\mathcal{L}_{FX_0}\lambda-$ $\lambda^2\mathcal{L}_{FX_0}\lambda + X_0[FX_0(X_0\lambda)] \overset{(27)}{=} \left(-\frac{1}{2E} - \lambda^2\right)\mathcal{L}_{FX_0}\lambda + X_0(-\lambda(FX_0)\lambda) =$ $\left(-\frac{1}{2E} - \lambda^2\right)\mathcal{L}_{FX_0}\lambda - (X_0\lambda)\mathcal{L}_{FX_0}\lambda - \lambda\mathcal{L}_{X_0}\mathcal{L}_{FX_0}\lambda \overset{(34)}{=}$ $\left(-\frac{1}{2E} - \lambda^2 - \mathcal{L}_{X_0}\lambda\right)\mathcal{L}_{FX_0}\lambda + 2\lambda^2\mathcal{L}_{FX_0}\lambda = \left(\lambda^2 - \mathcal{L}_{X_0}\lambda - \frac{1}{2E}\right)\mathcal{L}_{FX_0}\lambda,$ showing that (29) is also valid. □

2.6 Theorem. *If a positive definite two-dimensional Landsberg manifold has a vanishing Douglas tensor, then it is a Berwald manifold.*

Proof.
(**A**) In the next, quite tedious calculations our aim is to show that

$$\mathcal{L}_{FX_0}\lambda = 0. \tag{35}$$

Then on the one hand

$$0 = (FX_0)\lambda = [(F \circ v)X_0]\lambda = [h(FX_0)]\lambda = (d_h\lambda)(FX_0)$$

on the other hand

$$(d_h\lambda)(S) = (d\lambda)(hS) = (d\lambda)S = S(\lambda) \overset{(23)}{=} 0,$$

so it follows that $d_h\lambda = 0$ and, in view of Proposition 1.7, (M, E) is a Berwald manifold.

Notice that our subsequent reasoning relies heavily on the fact that

$$\lambda\kappa = -X_0(\kappa). \tag{36}$$

This relation is an immediate consequence of the Bianchi identity (22) and the property (23).

(**B**) We start with the "permutation formula" (17c) and apply both its sides to the main scalar. Taking into account (23), we obtain

$$\mathcal{L}_{S_0}\mathcal{L}_{FX_0}\lambda = \kappa X_0(\lambda). \tag{37}$$

Now we evaluate the vector field $[S_0, X_0]$ on the function $FX_0(\lambda)$.
$[S_0, X_0](FX_0(\lambda)) = S_0[X_0(FX_0(\lambda))] - X_0[S_0(FX_0(\lambda))] \overset{(34),\ (37)}{=}$
$-2S_0[\lambda(FX_0)(\lambda)] - X_0(\kappa X_0(\lambda)) \overset{(23)}{=} -2\lambda S_0[(FX_0)\lambda] - X_0(\kappa)X_0(\lambda)$

$-\kappa X_0(X_0\lambda) \overset{(37),\,(36)}{=} -2\lambda\kappa X_0(\lambda) + \lambda\kappa X_0(\lambda) - \kappa X_0(X_0\lambda) =$
$-\kappa(\lambda X_0(\lambda) + X_0(X_0\lambda))$.

Since, on the other side, $[S_0, X_0] \overset{(17b)}{=} -\frac{1}{\sqrt{2E}} FX_0$, it follows that

$$\frac{1}{\sqrt{2E}}\mathcal{L}^2_{FX_0}\lambda = \kappa(\lambda X_0(\lambda) + X_0(X_0\lambda)). \tag{38}$$

Applying the vector field X_0 to both sides of (38), owing to (1) we obtain

$$\frac{1}{\sqrt{2E}}\mathcal{L}_{X_0}\mathcal{L}^2_{FX_0}\lambda = X_0(\kappa)(\lambda X_0(\lambda) + X_0(X_0\lambda)) + \kappa\Big[(\mathcal{L}_{X_0}\lambda)^2 + \lambda\mathcal{L}^2_{X_0}\lambda + \mathcal{L}^3_{X_0}\lambda\Big] \overset{(36)}{=} \kappa(-\lambda^2\mathcal{L}_{X_0}\lambda + (\mathcal{L}_{X_0}\lambda)^2 + \mathcal{L}^3_{X_0}\lambda),$$

i.e.,

$$\frac{1}{\sqrt{2E}}\mathcal{L}_{X_0}\mathcal{L}^2_{FX_0}\lambda = \kappa(-\lambda^2\mathcal{L}_{X_0}\lambda + (\mathcal{L}_{X_0}\lambda)^2 + \mathcal{L}^3_{X_0}\lambda). \tag{39}$$

The Lie derivatives of the two sides of (34) with respect to FX_0 yield

$$\frac{1}{\sqrt{2E}}\mathcal{L}_{FX_0}\mathcal{L}_{X_0}\mathcal{L}_{FX_0}\lambda = \frac{1}{\sqrt{2E}}[-2(\mathcal{L}_{FX_0}\lambda)^2 - 2\lambda\mathcal{L}^2_{FX_0}\lambda]. \tag{40}$$

Taking the difference of (39) and (40), and then substituting the term $\frac{1}{\sqrt{2E}}\mathcal{L}^2_{FX_0}\lambda$ from the right hand side of (38) we obtain

$$\frac{1}{\sqrt{2E}}[X_0, FX_0](FX_0(\lambda)) = \kappa(\lambda^2\mathcal{L}_{X_0}\lambda + (\mathcal{L}_{X_0}\lambda)^2 + \mathcal{L}^3_{X_0}\lambda + 2\lambda\mathcal{L}^2_{X_0}\lambda) + \frac{2}{\sqrt{2E}}(\mathcal{L}_{FX_0}\lambda)^2.$$

The left hand side of this equality can also be written in the form

$$\frac{1}{\sqrt{2E}}[X_0, FX_0](FX_0(\lambda)) \overset{(17a),\,(23)}{=} -\frac{1}{2E}S_0[FX_0(\lambda)] - \frac{1}{\sqrt{2E}}\lambda(FX_0)(FX_0(\lambda)) \overset{(37),\,(38)}{=} -\kappa\left[\frac{1}{2E}X_0(\lambda) + \lambda^2 X_0(\lambda) + \lambda X_0(X_0\lambda)\right],$$

so it follows that

$$\frac{2}{\sqrt{2E}}(\mathcal{L}_{FX_0}\lambda)^2 + \kappa(\mathcal{L}^3_{X_0}\lambda + 3\lambda\mathcal{L}^2_{X_0}\lambda + (\mathcal{L}_{X_0}\lambda)^2 + 2\lambda^2\mathcal{L}_{X_0}\lambda + \frac{1}{2E}\mathcal{L}_{X_0}\lambda) = 0. \tag{41}$$

Now we apply the vector field X_0 to (41). Taking into account that

$$X_0\left[\frac{2}{\sqrt{2E}}(\mathcal{L}_{FX_0}\lambda)^2\right] \overset{(1)}{=} \frac{2}{\sqrt{2E}}\mathcal{L}_{X_0}(\mathcal{L}_{FX_0}\lambda)^2 = \frac{4}{\sqrt{2E}}(\mathcal{L}_{FX_0}\lambda)\mathcal{L}_{X_0}\mathcal{L}_{FX_0}\lambda \overset{(34)}{=} -\frac{8\lambda}{\sqrt{2E}}(\mathcal{L}_{FX_0}\lambda)^2 \overset{(41)}{=} 4\kappa\lambda\Big(\mathcal{L}^3_{X_0}\lambda + 3\lambda\mathcal{L}^2_{X_0}\lambda + (\mathcal{L}_{X_0}\lambda)^2 + 2\lambda^2\mathcal{L}_{X_0}\lambda + \frac{1}{2E}\mathcal{L}_{X_0}\lambda\Big),$$

we obtain the relation

$$(X_0\kappa)\left(\mathcal{L}^3_{X_0}\lambda + 3\lambda\mathcal{L}^2_{X_0}\lambda + (\mathcal{L}_{X_0}\lambda)^2 + 2\lambda^2\mathcal{L}_{X_0}\lambda + \frac{1}{2E}\mathcal{L}_{X_0}\lambda\right)$$
$$+\kappa\Big(\mathcal{L}^4_{X_0}\lambda + 5(\mathcal{L}_{X_0}\lambda)\mathcal{L}^2_{X_0}\lambda + 7\lambda\mathcal{L}^3_{X_0}\lambda + 8\lambda(\mathcal{L}_{X_0}\lambda)^2$$
$$+ 14\lambda^2\mathcal{L}^2_{X_0}\lambda + \frac{1}{2E}\mathcal{L}^2_{X_0}\lambda + 8\lambda^3\mathcal{L}_{X_0}\lambda + \frac{2\lambda}{E}\mathcal{L}_{X_0}\lambda\Big) = 0.$$

Using (36), this takes the form

$$\kappa\Big[\mathcal{L}^4_{X_0}\lambda + 6\lambda\mathcal{L}^3_{X_0}\lambda + \left(5\mathcal{L}_{X_0}\lambda + 11\lambda^2 + \frac{1}{2E}\right)\mathcal{L}^2_{X_0}\lambda + \left(7\mathcal{L}_{X_0}\lambda + 6\lambda^2 + \frac{3}{2E}\right)\lambda\mathcal{L}_{X_0}\lambda\Big] = 0. \tag{42}$$

(**C**) To conclude the proof, we finally discuss the relation (42).

(a) If $\kappa = 0$, then we see from (41) that $\mathcal{L}_{FX_0}\lambda = 0$. This means by (A) that (M, E) is a Berwald manifold.

(b) In the case $\kappa \neq 0$ the second factor has to vanish on the left hand side of (42). Then we take the Lie derivative of this factor with respect to the vector field FX_0. Appling the relations (27)–(33), after a somewhat lengthy but quite straightforward calculation we obtain

$$\left[\mathcal{L}^3_{X_0}\lambda + 3\lambda\mathcal{L}^2_{X_0}\lambda + (\mathcal{L}_{X_0}\lambda)^2 + 2\lambda^2\mathcal{L}_{X_0}\lambda + \frac{1}{2E}\mathcal{L}_{X_0}\lambda\right]\mathcal{L}_{FX_0}\lambda = 0. \tag{43}$$

If $\mathcal{L}_{FX_0}\lambda = 0$, then the process ends. Otherwise the first factor on the left hand side of (43) is zero, but, owing to (41), this also yields the desired relation $\mathcal{L}_{FX_0}\lambda = 0$. □

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