ON DARÓCZY'S PROBLEM FOR ADDITIVE FUNCTIONS

A. VARGA[1] AND CS. VINCZE[2]

Abstract. In this paper we investigate the functional equation

$$\sum_{i=1}^{n} \alpha_i A(\beta_i x) = 0$$

which holds for all $x \in \mathbb{R}$ with an unknown additive function $A\colon \mathbb{R} \to \mathbb{R}$ and fixed real parameters α_i, β_i, where $i = 1, \dots, n$.

1. Introduction and preliminaries

Consider the functional equation

$$\sum_{i=1}^{n} \alpha_i A(\beta_i x) = 0 \tag{1.1}$$

which holds for all $x \in \mathbb{R}$ with an unknown additive function $A\colon \mathbb{R} \to \mathbb{R}$ and fixed real parameters α_i, β_i, where $i = 1, \dots, n$. Since for any additive function vanishes at $x = 0$, without loss of generality we can suppose that none of the parameters equals to the zero.

The case $n = 2$ has been investigated in Daróczy [1]. His fundamental result states that the functional equation

$$\alpha_1 A(\beta_1 x) + \alpha_2 A(\beta_2 x) = 0$$

has non-trivial solutions if and only if both the parameters

$$\lambda := -\frac{\alpha_2}{\alpha_1} \quad \text{and} \quad \mu := \frac{\beta_1}{\beta_2}$$

are transcendent or they are algebraic with the same defining polynomial. It is important to see that this condition is equivalent to the existence of a field isomorphism

$$\delta\colon \mathbb{Q}(\lambda) \to \mathbb{Q}(\mu)$$

2000 *Mathematics Subject Classification.* 39B22.

Key words and phrases. Functional equation, additive functions.

such that $\delta(\lambda) = \mu$. Consider now $\mathbb{R}$ as the vector space over $\mathbb{Q}(\lambda)$ and $\mathbb{Q}(\mu)$, respectively we can use the procedure of semilinear extension to construct an additive function

$$A\colon \mathbb{R} \to \mathbb{R}$$

such that $A(\lambda x) = \mu A(x)$ for all $x \in \mathbb{R}$. This is obviously equivalent to equation (1.1) in case of $n = 2$.

In this paper we investigate the case $n \geq 3$ which is a natural generalization of the original problem. The theory of functional equations containing weighted arithmetic means also gives important motivations. Equation (1.1) has been found as a "first order condition" for the solutions of the functional equation

$$\sum_{i=0}^{n} a_i f(\alpha_i x + (1 - \alpha_i)y) = 0$$

which holds for all $x, y \in I$. Here $I \subset \mathbb{R}$ is a non-void open interval, $f\colon I \to \mathbb{R}$ is an unknown function, the parameters $\alpha_i \in [0, 1]$ are arbitrarily fixed and $i = 0, 1, \ldots, n$. The particular case $n = 3$, $a_0 = a_1 = 1$, $a_2 = a_4 = -1$ and $\alpha_2 = 1$, $\alpha_3 = 0$ has been investigated in Daróczy-Maksa-Páles [3], Daróczy-Lajkó-Lovas-Maksa-Páles [8], and also in Maksa [9] in connection with the equivalence of certain functional equations involving means. The result have been extended for the case of arbitrary $\alpha_2,\ \alpha_3 \in (0, 1)$ in the paper [10]. The investigation of the general case can be found in [11] with proving that f always has the form

$$f(x) = A_0 + A_1(x) + \ldots + A_{n-1}(x, \ldots, x),$$

where $A_k\colon \mathbb{R}^k \to \mathbb{R}$ is a k-additive function such that the first order condition

$$a_1 A_1(\beta_1 x) + \ldots + a_{n-1} A_1(\beta_{n-1} x) + a_n A_1(\beta_n x) = 0,$$

the second order conditions

$$a_1 A_2(x, \beta_1 y) + \ldots + a_{n-1} A_2(x, \beta_{n-1} y) + a_n A_2(x, \beta_n y) = 0 \quad \text{and}$$

$$a_1 A_2(\beta_1 x, \beta_1 x) + \ldots + a_{n-1} A_2(\beta_{n-1} x, \beta_{n-1} x) + a_n A_2(\beta_n x, \beta_n x) = 0$$

and so on are also satisfied with the parameters

$$\beta_i := \frac{\alpha_i - \alpha_0}{\alpha_n - \alpha_0}$$

for all real numbers x and y. It can be easily seen that the condition for the additive function A_1 is of the same type as equation (1.1).

First of all we introduce some basic notions we need in the following.

Definition 1.1. Let m be a positive integer and consider the elements $\vec{\lambda} := (\lambda_1, \ldots, \lambda_m)$ and $\vec{\lambda} := (\mu_1, \ldots, \mu_m)$ of the real coordinate space $\mathbb{R}^m$.

(i) The ideal

$$\mathcal{I}(\vec{\lambda}) := \{ \ p \in \mathbb{Q}[x_1, \ldots, x_m] \mid p(\lambda_1, \ldots, \lambda_m) = 0 \ \}$$

of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_m]$ is called the *defining ideal* of $\vec{\lambda} := (\lambda_1, \ldots, \lambda_m)$.

(ii) If the defining ideals of $\vec{\lambda}$ and $\vec{\mu}$ are the same then we say that they are *algebraic conjugate* of each other.

Remark 1.2. An important special case when the defining ideal of

$$\vec{\lambda} = (\lambda_1, \ldots, \lambda_m)$$

contains only the zero polynomial, i.e. the coordinates are algebraic independent. Otherwise they are algebraic dependent. In the particular case $m = 1$ the ideal $\mathcal{I}(\lambda)$ can be generated by the minimal polynomial and Definition 1.1 (ii) gives back the following notion: λ and μ are algebraic conjugate if both of them are transcendent or they are algebraic and their defining polynomials are the same.

Lemma 1.3. *Supoose that $2 \leq n \in \mathbb{N}$ and let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_{n-1})$ and $\vec{\mu} := (\mu_1, \ldots, \mu_{n-1})$ be arbitrarily fixed. There exists a field isomorphism*

$$\delta \colon \mathbb{Q}(\mu_1, \ldots, \mu_{n-1}) \to \mathbb{Q}(\lambda_1, \ldots, \lambda_{n-1})$$

such that

$$\delta(\mu_i) = \lambda_i \ \text{ for all } \ i = 1, \ldots, n-1$$

if and only if $\vec{\lambda}$ and $\vec{\mu}$ are algebraic conjugate.

For the proof see A. Varga and Cs. Vincze [11].

Definition 1.4. A translate of a k - dimensional linear subspace of $\mathbb{R}^m$ is called a *k-flat*. If $k = 1$ then we speak about a *line*. In case of $k = m - 1$ we have a *hyperplane*. A k-flat F_k is called *algebraic* if there exists a not identically zero element P of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_m]$ such that P vanishes at all points of the k - flat, i.e.

$$P \in \cap_{\vec{\lambda} \in F_k} \mathcal{I}(\vec{\lambda}).$$

Remark 1.5. In case of $k = 0$, i.e. if the flat reduces to a point we can refer to Definition 1.1.

2. An observation on the cardinal number of polynomial rings over the rationals

Throughout the paper *card A* denotes the cardinal number of the set A.

Lemma 2.1.

$$\textit{card}\ \mathbb{Q}[x_1,\dots,x_m] = \textit{card}\ \mathbb{Q}.$$

Proof. We denote $\mathcal{P}_n^m$ the set of the polynomials of degree at most n in $\mathbb{Q}[x_1,\dots,x_m]$, i.e.,

$$\mathcal{P}_n^m = \{p \in \mathbb{Q}[x_1,\dots,x_m] \mid \textit{deg}\ p \le n\}.$$

As $\mathbb{Q}[x_1,\dots,x_m] = \cup_{n\in\mathbb{N}}\mathcal{P}_n^m$, it is enough to show that

$$\text{card}\ \mathcal{P}_n^m = \text{card}\ \mathbb{Q}$$

for all $n, m \in \mathbb{N}$. We prove this by induction of m, i.e. the number of the variables.

I. By the induction of the degree we get that

$$\text{card}\ \mathcal{P}_n^1 = \text{card}\ \mathbb{Q}$$

holds for all $n \in \mathbb{N}$ as follows:

1. For the case $n = 0$ the statement is clear, because $\mathcal{P}_0^1 = \mathbb{Q}$.
2. Let assume that card $\mathcal{P}_{n-1}^1 =$ card $\mathbb{Q}$, $n \ge 1$.
3. Let $p,\ q \in \mathcal{P}_n^1$ and consider the following

$$p \equiv q \ \text{ if and only if } \ p' = q'$$

equivalence relation, where r' means the usual derivative of the polynomial r. Denote H_p the equivalence class containing the polynomial p and let

$$\mathcal{H} := \{H_p \mid p \in \mathcal{P}_n^1\}.$$

Then

$$\varphi\colon \mathcal{H} \to \mathcal{P}_{n-1}^1,\ \varphi(H_p) = p'$$

is a well-defined 1-to-1 mapping onto $\mathcal{P}_{n-1}^1$. Therefore

$$\text{card}\ \mathcal{H} = \text{card}\ \mathcal{P}_{n-1}^1 = \text{card}\ \mathbb{Q}$$

because condition 2. Finally we show that

$$\text{card}\ H_p = \text{card}\ \mathbb{Q}$$

for any polynomial p. Indeed, for all $q \in H_p$ we have $q = p + r$, where $r \in \mathbb{Q}$.

II. Let assume that card $\mathcal{P}_n^{m-1} =$ card $\mathbb{Q}$ for all $n \in \mathbb{N}$.

III. By the induction of the degree we get that

$$\text{card } \mathcal{P}_n^m = \text{card } \mathbb{Q}$$

holds for all $n \in \mathbb{N}$ as follows:

1. For the case $n = 0$ the statement is clear, because $\mathcal{P}_0^m = \mathbb{Q}$.
2. Let assume that card $\mathcal{P}_{n-1}^m =$ card $\mathbb{Q}$.
3. Let $p,\ q \in \mathcal{P}_n^m$ and consider the following

$$p \equiv q \ \text{ if and only if } \ D_m p = D_m q$$

equivalence relation, where $D_m r$ means the usual partial derivative of the polynomial r with respect to x_m. Denote H_p the equivalence class containing the polynomial p and let

$$\mathcal{H} := \{H_p \mid p \in \mathcal{P}_n^m\}.$$

Then

$$\varphi\colon \mathcal{H} \to \mathcal{P}_{n-1}^m,\ \varphi(H_p) = D_m p$$

is a well-defined 1-to-1 mapping onto $\mathcal{P}_{n-1}^m$. Therefore

$$\text{card } \mathcal{H} = \text{card } \mathcal{P}_{n-1}^m = \text{card } \mathbb{Q}$$

because condition 2. Finally we show that

$$\text{card } H_p = \text{card } \mathbb{Q}$$

for any polynomial p. Indeed, for all $q \in H_p$ we have that $q = p + r$, where $r \in \mathcal{P}_n^{m-1}$ and, by condition II, the proof is completed.

□

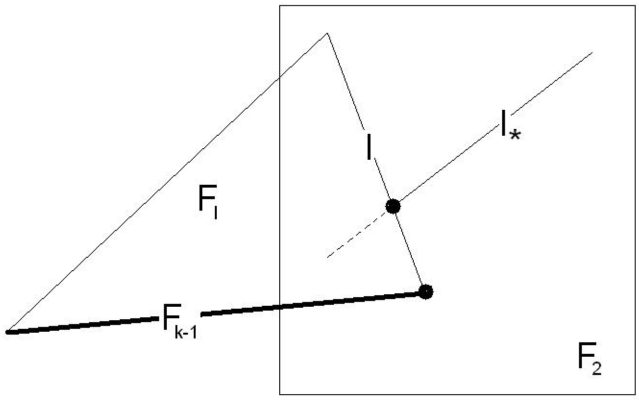
Lemma 2.2. *A k-flat F_k in $\mathbb{R}^m$ is the union of zero's of not identically zero elements of the polynomial ring $\mathbb{Q}[x_1, \dots, x_m]$ if and only if it is algebraic.*

Proof. If F_k is algebraic then the statement is trivial. The proof of the converse is by the induction of k as follows.

I. If $k = 1$ then we have a line. If we substitute the points of F_1 into the arguments of any not identically zero element P of the polynomial ring $\mathbb{Q}[x_1, \dots, x_m]$ in the parametric form $\vec{x} + t\vec{v}$, we have a polynomial p of the only variable t with real coefficients. If p is not identically zero then it has at most finite roots which means that P vanishes at most finite point of F_1. But the set of the polynomials is countable. Therefore there must be at least one identically zero polynomial p_* to get *all* the points on the line F_1 as a root. This means that P_* vanishes along the line F_1.

II. Suppose that the statement is true for the k - flats in $\mathbb{R}^m$, where $k \leq m-2$ but $k \geq 1$.

III. To prove the case of F_{k+1} let a point $\vec{x}_0 \in F_{k+1}$ be fixed and consider a flat (reducing maybe to a point) $F_{k-1} \subset F_{k+1}$ passing through $\vec{x}_0$ and its orthogonal complement $F_2 \subset F_{k+1}$ at $\vec{x}_0$. F_{k-1} and any line $l \subset F_2$ passing through $\vec{x}_0$ determine a k -



flat F_l which is algebraic by the hypothesis II. Let P_l denote a nonzero element of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_m]$ such that P_l vanishes at any point of F_l. But the set of the polynomials is countable. Therefore there exists at least one polynomial P_* such that it vanishes at the points of an at least countable but not finite set of k - flats generated by F_{k-1} and the elements of the set $L := \{l_1, \ldots, l_n, \ldots\}$ consisting of lines passing through $\vec{x}_0$ in F_2. Now we prove that P_* vanishes at any point of F_2. For $\vec{x}_0$ it is trivial. Let now $\vec{x}_0 \neq \vec{y} \in F_2$ and take a line l_* in F_2 which doesn't contain the fixed point $\vec{x}_0$ but $\vec{y} \in l_*$. Then l_* intersects the elements of L except the uniquely determined parallel one. Therefore P_* has more than finite zero's along l_* which means that P_* vanishes at any point, especially $\vec{y}$, of l_*. If $k = 1$ then the flat F_{k-1} reduces to a point and $F_2 = F_{k+1}$ thus the proof is finished. In case of $k > 1$ translate F_2 at each point of F_{k-1} to have that P_* vanishes at all the points of F_{k+1}.

□

Remark 2.3. The result says that a k-flat F_k is algebraic if and only if the coordinates of any point of F_k are algebraic dependent.

Lemma 2.4. *The hyperplane in* $\mathbb{R}^m$ *defined by the equation*

$$\lambda_1 x_1 + \ldots \lambda_{m-1} x_{m-1} + \lambda_m = x_m$$

is algebraic if and only if all of the coefficients $\lambda_1, \dots, \lambda_m$ *is algebraic.*

Proof. Suppose that the hyperplane is algebraic. Then we have a not identically zero elements P of the polynomial ring $\mathbb{Q}[x_1, \dots, x_m]$ which vanishes at the points of the hyperplane. Therefore

$$P(x_1, x_2, \dots, x_{m-1}, \lambda_1 x_1 + \dots + \lambda_{m-1} x_{m-1} + \lambda_m) = 0. \tag{2.1}$$

Substituting $x_1 = \dots = x_{m-1} = 0$ we have that λ_m is the root of the polynomial

$$p(t) := P(0, \dots, 0, t).$$

Suppose, in contrary, that λ_m is not algebraic. Then

$$P(0, \dots, 0, t) = 0 \ \Rightarrow \ D_m P(0, \dots, 0, t) = 0 \text{ and so on.}$$

Differentiating (2.1.) by x_i, for any indeces $i = 1, \dots, m-1$ we have that

$$D_i P(0, \dots, 0, \lambda_m) + \lambda_i D_m P(0, \dots, 0, \lambda_m) = 0$$

and, consequently,

$$D_i P(0, \dots, 0, \lambda_m) = 0.$$

Because λ_m is not algebraic

$$D_i P(0, \dots, 0, t) = 0 \ \Rightarrow \ D_m D_i P(0, \dots, 0, t) = 0$$

for any indeces $i = 1, \dots, m-1$ and, of course, the same is true if $i = m$. Differentiating again (2.1) by x_i and x_j, we have that

$$D_j D_i P(0, \dots, 0, \lambda_m) + \lambda_j D_m D_i P(0, \dots, 0, \lambda_m) +$$
$$\lambda_i D_j D_m P(0, \dots, 0, \lambda_m) + \lambda_i \lambda_j D_m D_m P(0, \dots, 0, \lambda_m) = 0$$

and, consequently

$$D_j D_i P(0, \dots, 0, \lambda_m) = 0.$$

Since λ_m is not algebraic, it follows that

$$D_i D_j P(0, \dots, 0, t) = 0$$

where i, j are arbitrary indeces including the cases $i = m$ or $j = m$ too. According to the Taylor formula at $(0, 0, \dots, 0)$, the proccess shows that P must be identically zero, which is a contradiction. Let now, for example, $x_1 := 1$, $x_2 = \dots = x_m = 0$. Then we have

$$P(1, 0, \dots, 0, \lambda_1 + \lambda_m) = 0,$$

i.e. $\lambda_1 + \lambda_m$ is the root of the polynomial

$$p(t) := P(1, 0, \dots, 0, t).$$

A similar reasoning as above gives that $\lambda_1 + \lambda_m$ is algebraic. Because the algebraic numbers form a field it follows that

$$\lambda_1 = \lambda_1 + \lambda_m - \lambda_m$$

is algebraic. The proof is similar for any further coefficients.

Conversely suppose that the coefficients are algebraic numbers and consider their defining polynomials in the form

$$\Omega_1(t) := (t-\lambda_{11})(t-\lambda_{12})\cdots(t-\lambda_{1k_1}),$$
$$\Omega_2(t) := (t-\lambda_{21})(t-\lambda_{22})\cdots(t-\lambda_{2k_2}),$$
$$\vdots$$
$$\Omega_{m-1}(t) := (t-\lambda_{m-11})(t-\lambda_{m-12})\cdots(t-\lambda_{m-1k_{m-1}}),$$
$$V_m(t) := (t-\lambda_{m1})(t-\lambda_{m2})\cdots(t-\lambda_{mk_m}),$$

where $\lambda_{i1} := \lambda_i$ and for any further indeces j, λ_{ij} are the algebraic conjugates of λ_i. Let

$$P(x_1,\ldots,x_m) :=$$
$$\prod_{i_1=1}^{k_1}\prod_{i_2=1}^{k_2}\cdots\prod_{i_m=1}^{k_m}(x_m-\lambda_{1i_1}x_1-\ldots-\lambda_{m-1i_{m-1}}x_{m-1}-\lambda_{mi_m})$$

which is obviously vanishes at the points of the hyperplane. On the other hand, for any fixed $x_1,\ldots,x_m$, it can be considered as a symmetric polynomial of the variables $\lambda_{11},\ldots,\lambda_{1k_1}$. Using the fundamental theorem of symmetric polynomials it has a unique representation as the polynomial of the elementary symmetric polynomials

$$E_0(\lambda_{11},\ldots,\lambda_{1k_1}) = 1,$$
$$E_1(\lambda_{11},\ldots,\lambda_{1k_1}) = \lambda_{11}+\ldots+\lambda_{1k_1},$$
$$E_2(\lambda_{11},\ldots,\lambda_{1k_1}) = \lambda_{11}\lambda_{12}+\ldots+\lambda_{11}\lambda_{1k_1}+$$
$$+\lambda_{12}\lambda_{13}+\ldots+\lambda_{12}\lambda_{1k_1}+\ldots+\lambda_{1k_1-1}\lambda_{1k_1},$$
$$\vdots$$
$$E_{k_1}(\lambda_{11},\ldots,\lambda_{1k_1}) = \lambda_{11}\lambda_{12}\ldots\lambda_{1k_1}.$$

According to the relations between the coefficients of the polynomials and its roots we have that P is a polynomial of the variables $x_1, x_2, \ldots, x_m$, $\lambda_{21},\ldots,\lambda_{2k_2},\ldots,\lambda_{mk_m}$ with rational coefficients. Repeating the procedure as above we have that P is an element of the polynomial ring $\mathbb{Q}[x_1,\ldots,x_m]$. □

Example 2.5. Consider the line

$$x_2 = \sqrt{3}x_1 + \sqrt{2}$$

in $\mathbb{R}^2$. The we have that

$$P(x_1, x_2) =$$

$$(x_2 - \sqrt{3}x_1 - \sqrt{2})(x_2 - \sqrt{3}x_1 + \sqrt{2})(x_2 + \sqrt{3}x_1 - \sqrt{2})(x_2 + \sqrt{3}x_1 + \sqrt{2}).$$

An easy calculation shows that

$$P(x_1, x_2) = x_2^4 - 6x_1^2x_2^2 - 4x_2^2 + 9x_1^4 - 12x_1^2 + 4.$$

3. Sufficient conditions for the existence of non-trivial solutions

Before we can state our main theorems their final forms we need the following lemma.

Lemma 3.1. *Let $3 \leq n \in \mathbb{N}$ be arbitrarily fixed and $\beta_i, \delta_i, \beta_n \in \mathbb{R}$ be nonzero real numbers, $i = 1, \ldots, n-1$. If there exists a field isomorphism*

$$\delta \colon \mathbb{Q}\left(\frac{\beta_1}{\beta_n}, \ldots, \frac{\beta_{n-1}}{\beta_n}\right) \to \mathbb{Q}\left(\delta_1, \ldots, \delta_{n-1}\right)$$

such that

$$\delta\left(\frac{\beta_i}{\beta_n}\right) = \delta_i \quad (i = 1, \ldots, n-1) \tag{3.1}$$

and

$$\frac{\alpha_1}{\alpha_n}\delta_1 + \ldots + \frac{\alpha_{n-1}}{\alpha_n}\delta_{n-1} = -1, \tag{3.2}$$

then there exists a not identically zero additive function $A \colon \mathbb{R} \to \mathbb{R}$ such that

$$\sum_{i=1}^{n} \alpha_i \; A(\beta_i x) = 0 \quad (x \in \mathbb{R}).$$

Proof. Consider $\mathbb{R}$ as the vector space over $\mathbb{Q}\left(\frac{\beta_1}{\beta_n}, \ldots, \frac{\beta_{n-1}}{\beta_n}\right)$ with the basis $\mathcal{H}$. Define $A \colon \mathbb{R} \to \mathbb{R}$ as follows: on the elements of $\mathcal{H}$ we define it arbitrarily and for $x = \sum_j c_j h_j$, where $c_j \in \mathbb{Q}\left(\frac{\beta_1}{\beta_n}, \ldots, \frac{\beta_{n-1}}{\beta_n}\right)$ and $h_j \in \mathcal{H}$ let

$$A(x) := \sum_j \delta(c_j) A(h_j) \quad (x \in \mathbb{R});$$

it is easy to see that for any $i = 1, \dots, n-1$

$$A\left(\frac{\beta_i}{\beta_n}x\right) = \delta_i A(x) \quad (x \in \mathbb{R}).$$

Indeed,

$$A\left(\frac{\beta_i}{\beta_n}x\right) = A\left(\sum_j \frac{\beta_i}{\beta_n}c_j h_j\right) = \sum_j \delta\left(\frac{\beta_i}{\beta_n}c_j\right)A(h_j) =$$

$$= \sum_j \delta_i \delta(c_j)A(h_j) = \delta_i \sum_j \delta(c_j)A(h_j) = \delta_i A(x)$$

holds for all $x \in \mathbb{R}$, where $i = 1, \dots, n-1$. Therefore

$$\frac{\alpha_1}{\alpha_n}A\left(\frac{\beta_1}{\beta_n}x\right) + \ldots + \frac{\alpha_{n-1}}{\alpha_n}A\left(\frac{\beta_{n-1}}{\beta_n}x\right) + A(x) =$$

$$= \left(\frac{\alpha_1}{\alpha_n}\delta_1, \dots, + \frac{\alpha_{n-1}}{\alpha_n}\delta_{n-1} + 1\right)A(x) = 0$$

for all $x \in \mathbb{R}$, i.e.

$$\sum_{i=1}^{n} \frac{\alpha_i}{\alpha_n} A\left(\frac{\beta_i}{\beta_n}x\right) = 0 \quad (x \in \mathbb{R}).$$

Multiplying by α_n and substituting x by $\beta_n x$ this equation is equivalent to

$$\sum_{i=1}^{n} \alpha_i \; A(\beta_i x) = 0 \quad (x \in \mathbb{R})$$

which was to be stated. □

Theorem 3.2. *Suppose that $n \geq 3$. If the parameters*

$$\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$$

are algebraic independent and at least one of the parameters $\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$ is transcendent then equation (1.1) always has nontrivial additive solution which is semi-homogeneous in the sense that

$$A\left(\frac{\beta_i}{\beta_n}x\right) = \delta_i A(x)$$

for some δ_i's, where $x \in \mathbb{R}$ and $i = 1, \dots n-1$.

Proof. Let $\frac{\beta_1}{\beta_n}, \ldots, \frac{\beta_{n-1}}{\beta_n}$ be fixed such that they are algebraic independent. According to Lemma 1.3 and Lemma 3.1 equation (1.1) has nontrivial additive solutions if there exist $\delta_1, \ldots, \delta_{n-1} \in \mathbb{R}^{n-1}$ such that they satisfy the equation

$$\frac{\alpha_1}{\alpha_n} x_1 + \ldots + \frac{\alpha_{n-1}}{\alpha_n} x_{n-1} = -1 \tag{3.3}$$

and the defining ideal contains only the zero polynomial. Because of lemma 2.4 the hyperplane defined by (3.3) is not algebraic which means, by lemma 2.3 that it is not the union of zero's of nontrivial elements of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_m]$. This implies the existence of δ_i's as was to be stated. □

Theorem 3.3. *Suppose that $n \geq 3$. If the parameters*

$$\frac{\alpha_1}{\alpha_n}, \ldots, \frac{\alpha_{n-1}}{\alpha_n}$$

are algebraic independent and at least one of the parameters $\frac{\beta_1}{\beta_n}, \ldots, \frac{\beta_{n-1}}{\beta_n}$ is transcendent then equation (1.1) always has nontrivial additive solution which is semi-homogeneous in the sense that

$$A(\delta_i x) = \frac{\alpha_i}{\alpha_n} A(x)$$

for some δ_i's, where $x \in \mathbb{R}$ and $i = 1, \ldots n-1$.

Proof. The proof is similar as that of Theorem 3.2 using the field isomorphism

$$\delta \colon \mathbb{Q}\Big(\delta_1, \ldots, \delta_{n-1}\Big) \to \mathbb{Q}\Big(\frac{\alpha_1}{\alpha_n}, \ldots, \frac{\alpha_{n-1}}{\alpha_n}\Big)$$

such that

$$\delta(\delta_i) = \frac{\alpha_i}{\alpha_n} \quad (i = 1, \ldots, n-1)$$

and

$$\delta_1 \frac{\beta_1}{\beta_n} + \ldots + \delta_{n-1} \frac{\beta_{n-1}}{\beta_n} = -1. \tag{3.4}$$

□

Remark 3.4. First of all note that β_n and α_n can be substituted with any other coefficients β_i and α_i, respectively, where $i = 1, \ldots, n-1$. On the other hand the reasoning in the proofs shows that points with algebraic independent coordinates can be find almost anywhere on the hyperplanes defined by (3.3) and (3.4).

Remark 3.5. If an additive function A is semihomogeneous in the sense that

$$A\left(\frac{\beta_i}{\beta_n}x\right) = \delta_i A(x)$$

for some δ_i's, where $x \in \mathbb{R}$ and $i = 1, \dots n-1$ then, it can be easily seen that for any $\beta \in \mathbb{Q}(\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n})$

$$A(\beta x) = \delta A(x)$$

for some $\delta \in \mathbb{Q}(\delta_1, \dots, \delta_{n-1})$. Moreover, if

$$\beta = \frac{w\left(\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}\right)}{k\left(\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}\right)},$$

then

$$\delta = \frac{w(\delta_1, \dots, \delta_{n-1})}{k(\delta_1, \dots, \delta_{n-1})},$$

where w, k are the elements of the polynomial ring $\mathbb{Q}[x_1, \dots, x_{n-1}]$, provided that A is not identically zero. According to lemma 3.1 in [11] the fields $\mathbb{Q}(\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n})$ and $\mathbb{Q}(\delta_1, \dots, \delta_{n-1})$ are isomorphic to each other.

Like the *homogeneity field* [5] we can define the *inner*

$$IS(A) := \{\beta \in \mathbb{R} \mid A(\beta x) = \delta A(x) \text{ for some } \delta \text{ and for all real number } x\}$$

and the *outer semihomogeneity field*

$$OS(A) := \{\delta \in \mathbb{R} \mid A(\beta x) = \delta A(x) \text{ for some } \beta \text{ and for all real number } x\}.$$

The semihomogeneity field is unique in the sense that $IS(A)$ and $OS(A)$ is isomorphic provided A is not identically zero.

In terms of the semihomogeneity the following converse of Theorem 3.2 is natural.

Theorem 3.6. *Suppose that $n \geq 3$. If the parameters*

$$\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$$

are algebraic independent and equation (1.1) has a nontrivial semihomogeneous additive solution with

$$\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$$

in its inner semihomogeneity field, then at least one of the parameters

$$\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$$

is transcendent.

In a similar way we have the following result.

Theorem 3.7. *Suppose that $n \geq 3$. If the parameters*

$$\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$$

are algebraic independent and equation (1.1) has a nontrivial semihomogeneous additive solution with

$$\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$$

in its outer semihomogeneity field then at least one of the parameters $\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$ is transcendent.

4. Examples for the complementer cases

In the previous section we formulated existence theorems for the nontrivial solution of equation (1.1) under the conditions

(i) $\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$ are algebraic independent and at least one of the parameters $\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$ is transcendent,

(ii) $\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$ are algebraic independent and at least one of the parameters $\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$ is transcendent.

To complete the discussion we need to investigate the following complementer cases:

(iii) both of the collections of the parameters $\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$ and $\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$ are algebrically dependent,

(iv) $\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$ are algebraic independent and all of the parameters $\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$ are algebraic, or

(v) $\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$ are algebraic independent and all of the parameters $\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$ are algebraic.

This section is devoted to the investigations of examples for the cases (iv) and (v). The **first example** is the functional equation

$$\sqrt{d_1} A\left(\frac{\beta_1}{\beta_3} x\right) + \sqrt{d_2} A\left(\frac{\beta_2}{\beta_3} x\right) + A(x) = 0,$$

where d_1 and d_2 are positive rationals and the parameters $\frac{\beta_1}{\beta_3}.\frac{\beta_2}{\beta_3}$ are algebraic independent. Note that it is equivalent equation (1.1) under $n = 3$,

$$\frac{\alpha_1}{\alpha_3} = \sqrt{d_1}, \quad \frac{\alpha_2}{\alpha_3} = \sqrt{d_2}.$$

To solve the functional equation consider the following procedure. Multiplying by $\sqrt{d_1}$

$$A\left(d_1\frac{\beta_1}{\beta_3}x\right) + \sqrt{d_1 d_2}A\left(\frac{\beta_2}{\beta_3}x\right) + \sqrt{d_1}A(x) = 0.$$

Substituting $\frac{\beta_1}{\beta_3}x$ we have

$$A\left(d_1\left(\frac{\beta_1}{\beta_3}\right)^2 x\right) + \sqrt{d_1 d_2}A\left(\frac{\beta_1\beta_2}{\beta_3^2}x\right) + \sqrt{d_1}A\left(\frac{\beta_1}{\beta_3}x\right) = 0. \tag{4.1}$$

In a similar way

$$\sqrt{d_1 d_2}A\left(\frac{\beta_1}{\beta_3}x\right) + A\left(d_2\frac{\beta_2}{\beta_3}x\right) + \sqrt{d_2}A(x) = 0$$

and

$$\sqrt{d_1 d_2}A\left(\frac{\beta_1\beta_2}{\beta_3^2}x\right) + A\left(d_2\left(\frac{\beta_2}{\beta_3}\right)^2 x\right) + \sqrt{d_2}A\left(\frac{\beta_2}{\beta_3}x\right) = 0 \tag{4.2}$$

follows immediately. Taking the difference of (4.2) and (4.1)

$$\sqrt{d_2}A\left(\frac{\beta_2}{\beta_3}x\right) - \sqrt{d_1}A\left(\frac{\beta_1}{\beta_3}x\right) = A\left(\left(d_1\left(\frac{\beta_1}{\beta_3}\right)^2 - d_2\left(\frac{\beta_2}{\beta_3}\right)^2\right)x\right).$$

On the other hand

$$\sqrt{d_1}A\left(\frac{\beta_1}{\beta_3}x\right) + \sqrt{d_2}A\left(\frac{\beta_2}{\beta_3}x\right) = -A(x)$$

as the original functional equation shows. Taking the sum

$$2\sqrt{d_2}A\left(\frac{\beta_2}{\beta_3}x\right) = A\left(\left(d_1\left(\frac{\beta_1}{\beta_3}\right)^2 - d_2\left(\frac{\beta_2}{\beta_3}\right)^2 - 1\right)x\right).$$

Substituting $\frac{\beta_3}{2\beta_2}x$ we have that

$$\sqrt{d_2}A(x) = A\left(\left(d_1\left(\frac{\beta_1}{\beta_3}\right)^2 - d_2\left(\frac{\beta_2}{\beta_3}\right)^2 - 1\right)\frac{\beta_3}{2\beta_2}x\right)$$

which means, by Daróczy theorem, that

$$\sqrt{d_2} \text{ and } \left(d_1\left(\frac{\beta_1}{\beta_3}\right)^2 - d_2\left(\frac{\beta_2}{\beta_3}\right)^2 - 1\right)\frac{\beta_3}{2\beta_2},$$

are algebraic conjugate to each other, i.e.

$$\pm\sqrt{d_2} = \left(d_1\left(\frac{\beta_1}{\beta_3}\right)^2 - d_2\left(\frac{\beta_2}{\beta_3}\right)^2 - 1\right)\frac{\beta_3}{2\beta_2}$$

provided that A is not identically zero. In a similar way (taking the difference instead of the sum) it follows that

$$-2\sqrt{d_1}A\left(\frac{\beta_1}{\beta_3}x\right) = A\left(\left(d_1\left(\frac{\beta_1}{\beta_3}\right)^2 - d_2\left(\frac{\beta_2}{\beta_3}\right)^2 + 1\right)x\right)$$

and, consequently,

$$\pm\sqrt{d_1} = \left(d_1\left(\frac{\beta_1}{\beta_3}\right)^2 - d_2\left(\frac{\beta_2}{\beta_3}\right)^2 + 1\right)\frac{\beta_3}{2\beta_1}$$

provided that A is not identically zero. It can be easily seen that both of the final results contradicts to the condition of algebrically independence of the parameters $\frac{\beta_1}{\beta_3}$ and $\frac{\beta_2}{\beta_3}$. Therefore the only solution is the identically zero function.

The method can be easily generated for solving the functional equation

$$\frac{\alpha_1}{\alpha_3}A(\frac{\beta_1}{\beta_3}x) + \frac{\alpha_2}{\alpha_3}A(\frac{\beta_2}{\beta_3}x) + A(x) = 0$$

if the parameters

$$\frac{\alpha_1}{\alpha_3}, \quad \text{and} \quad \frac{\alpha_2}{\alpha_3}$$

is algebraic of degree at most 2. Then all of them has the form

$$\frac{\alpha_i}{\alpha_3} = r_i + s_i\sqrt{d_i},$$

where r_i, s_i and d_i are rationals and $i = 1, 2$.

Exercise 4.1. Ommitting the requirements of algebrically independence finish the procedure to solve the functional equation

$$\sqrt{d_1}A\left(\frac{\beta_1}{\beta_3}x\right) + \sqrt{d_2}A\left(\frac{\beta_2}{\beta_3}x\right) + A(x) = 0$$

as an example for the case (iii).

The **second example** is

$$\frac{\alpha_1}{\alpha_3}A(\sqrt{d_1}x) + \frac{\alpha_2}{\alpha_3}A(\sqrt{d_2}x) + A(x) = 0,$$

where $A\colon \mathbb{R} \to \mathbb{R}$ is an unknown additive function, $x \in \mathbb{R}$, $\frac{\alpha_1}{\alpha_3}$ and $\frac{\alpha_2}{\alpha_3}$ are arbitrarily fixed parameters, d_1 and d_2 are positive rationals. If at least one of $\sqrt{d_1}$ and $\sqrt{d_2}$ is rational the equation reduces to the case $n = 2$ because of the rational homogeneity of additive functions.

Therefore we can suppose that $\sqrt{d_1}$ and $\sqrt{d_2}$ are irrational numbers. Substituting $\sqrt{d_1}x$, $\sqrt{d_2}x$ and $\sqrt{d_1 d_2}x$, respectively we get the system of equations

$$\frac{\alpha_1}{\alpha_3}A(\sqrt{d_1}x) + \frac{\alpha_2}{\alpha_3}A(\sqrt{d_2}x) + A(x) = 0,$$
$$d_1\frac{\alpha_1}{\alpha_3}A(x) + \frac{\alpha_2}{\alpha_3}A(\sqrt{d_1 d_2}x) + A(\sqrt{d_1}x) = 0,$$
$$\frac{\alpha_1}{\alpha_3}A(\sqrt{d_1 d_2}x) + d_2\frac{\alpha_2}{\alpha_3}A(x) + A(\sqrt{d_2}x) = 0.$$
$$d_1\frac{\alpha_1}{\alpha_3}A(\sqrt{d_2}x) + d_2\frac{\alpha_2}{\alpha_3}A(\sqrt{d_1}x) + A(\sqrt{d_1 d_2}x) = 0.$$

Taking the matrix

$$M := \begin{pmatrix} 1 & \frac{\alpha_1}{\alpha_3} & \frac{\alpha_2}{\alpha_3} & 0 \\ d_1\frac{\alpha_1}{\alpha_3} & 1 & 0 & \frac{\alpha_2}{\alpha_3} \\ d_2\frac{\alpha_2}{\alpha_3} & 0 & 1 & \frac{\alpha_1}{\alpha_3} \\ 0 & d_2\frac{\alpha_2}{\alpha_3} & d_1\frac{\alpha_1}{\alpha_3} & 1 \end{pmatrix}$$

we can write that

$$M \begin{pmatrix} A(x) \\ A(\sqrt{d_1}x) \\ A(\sqrt{d_2}x) \\ A(\sqrt{d_1 d_2}x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Theorem 4.2. *If the parameters $\frac{\alpha_1}{\alpha_3}$ and $\frac{\alpha_2}{\alpha_3}$ are algebrically independent then the identically zero additive function is the only solution of the funcional equation*

$$\frac{\alpha_1}{\alpha_3}A(\sqrt{d_1}x) + \frac{\alpha_2}{\alpha_3}A(\sqrt{d_2}x) + A(x) = 0,$$

where d_1, d_2 are positive real numbers.

Proof. We are going to prove that $\det M \neq 0$ which means that the linear transzformation represented by M is regular and the kernel contains only the zero vector. This means that $A(x) = 0$ for any $x \in \mathbb{R}$. Suppose, in contrary that $\det M = 0$. Then the polynomial

$$P(x,y) := \det \begin{pmatrix} 1 & x & y & 0 \\ d_1 x & 1 & 0 & y \\ d_2 y & 0 & 1 & x \\ 0 & d_2 y & d_1 x & 1 \end{pmatrix}$$

with rational coefficients must be identically zero. But this is a contradiction because $P(0,0) = 1$. □

In what follows we omit the requirement of the independence of the parameters $\frac{\alpha_1}{\alpha_3}$ and $\frac{\alpha_2}{\alpha_3}$. Since

$$\det\begin{pmatrix} \frac{\alpha_1}{\alpha_3} & \frac{\alpha_2}{\alpha_3} & 0 \\ 1 & 0 & \frac{\alpha_2}{\alpha_3} \\ 0 & 1 & \frac{\alpha_1}{\alpha_3} \end{pmatrix} = -2\frac{\alpha_1\alpha_2}{\alpha_3^2}$$

we have that the rank of M is at least 3 unless $\alpha_1 = 0$ or $\alpha_2 = 0$ when the equation reduces to the case $n = 2$. If the rank is maximal then the only solution is the identically zero additive function. So we can suppose that the rank of M is 3 which implies that the kernel of the transformation represented by M is one-dimensional, i.e. it is generated by the vector

$$\vec{v} = (v_0, v_1, v_2, v_3).$$

This means that

$$\begin{pmatrix} A(x) \\ A(\sqrt{d_1}x) \\ A(\sqrt{d_2}x) \\ A(\sqrt{d_1d_2}x) \end{pmatrix} = \lambda(x)\begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

for some additive function $\lambda\colon \mathbb{R} \to \mathbb{R}$. If $v_0 = 0$ then we have that the only solution is the identically zero function. Otherwise

$$A(\sqrt{d_1}x) = v_1\lambda(x) = \frac{v_1}{v_0}A(x)$$

and, in a similar way

$$A(\sqrt{d_2}x) = v_2\lambda(x) = \frac{v_2}{v_0}A(x)$$

and, consequently,

$$A(\sqrt{d_1d_2}) = \frac{v_3}{v_0}A(x), \quad \textit{where} \quad \frac{v_3}{v_0} = \frac{v_1v_2}{v_0^2}.$$

Using Daróczy's result these equations have non-zero solutions if and only if

$$\frac{v_1}{v_0} = \pm\sqrt{d_1} \ \text{ and } \ \frac{v_2}{v_0} = \pm\sqrt{d_2}$$

and we have the following result.

Theorem 4.3. *Suppose that $\frac{\alpha_1}{\alpha_3} \neq 0$ and $\frac{\alpha_2}{\alpha_3} \neq 0$. The functional equation*

$$\frac{\alpha_1}{\alpha_3}A(\sqrt{d_1}x) + \frac{\alpha_2}{\alpha_3}A(\sqrt{d_2}x) + A(x) = 0$$

has non-zero solutions if and only if one of the following conditions are satisfied:

(i)
$$1 + \frac{\alpha_1}{\alpha}\sqrt{d_1} + \frac{\alpha_2}{\alpha_3}\sqrt{d_2} = 0,$$
(ii)
$$1 + \frac{\alpha_1}{\alpha}\sqrt{d_1} - \frac{\alpha_2}{\alpha_3}\sqrt{d_2} = 0,$$
(iii)
$$1 - \frac{\alpha_1}{\alpha}\sqrt{d_1} + \frac{\alpha_2}{\alpha_3}\sqrt{d_2} = 0,$$
(iv)
$$1 - \frac{\alpha_1}{\alpha}\sqrt{d_1} - \frac{\alpha_2}{\alpha_3}\sqrt{d_2} = 0$$
provided that both $\sqrt{d_1}$ *and* $\sqrt{d_2}$ *are irrational numbers.*

Proof. If A is a not identically zero solution then, as we have seen above, it is semihomogeneous in the sense that
$$A(\sqrt{d_1}x) = v_1\lambda(x) = \frac{v_1}{v_0}A(x)$$
and
$$A(\sqrt{d_2}x) = v_2\lambda(x) = \frac{v_2}{v_0}A(x).$$
Using Daróczy's theorem we have that one of the conditions (i), (ii), (iii) and (iv) is satisfied. Conversely if (i) is satisfied then, for example, let $A(x) := x$ or $A(x) := -x$. As an illustration consider the case when (ii) is satisfied. We are going to prove that
$$\vec{\mu} = (\sqrt{d_1}, \sqrt{d_2}) \quad \text{and} \quad \vec{\nu} = (\sqrt{d_1}, -\sqrt{d_2})$$
have the same definig ideal. Suppose that
$$P(\sqrt{d_1}, \sqrt{d_2}) = 0.$$
Then we can write
$$P(\sqrt{d_1}, y) = f(y)(y^2 - d_2)$$
because the degree of $\sqrt{d_2}$ is 2. Therefore $P(\sqrt{d_1}, -\sqrt{d_2}) = 0$. The method is similar for any elements $Q[x, y]$ of the defining ideal of $\vec{\nu}$. This means that we can use the procedure of semilinear extension to construct a not identically zero additive function A such that
$$A(\sqrt{d_1}x) = \sqrt{d_1}A(x)$$
and
$$A(\sqrt{d_2}x) = -\sqrt{d_2}A(x).$$
The function A is obviously a solution of the functional equation. The case of (iii) and (iv) is similar. □

Remark 4.4. The method can be easily generated for solving the functional equation

$$\frac{\alpha_1}{\alpha_3}A(\frac{\beta_1}{\beta_3}x)+\frac{\alpha_2}{\alpha_3}A(\frac{\beta_2}{\beta_3}x)+A(x)=0$$

if the parameters

$$\frac{\beta_1}{\beta_3} \text{ and } \frac{\beta_2}{\beta_3}$$

are algebraic of degree at most 2. Then all of them has the form

$$\frac{\beta_i}{\beta_3}=r_i+s_i\sqrt{d_i},$$

where r_i, s_i and d_i are rationals and $i=1,2$.

The **third example** is the functional equation

$$\frac{\alpha_1}{\alpha_4}A(\sqrt{2}x)+\frac{\alpha_2}{\alpha_4}A(\sqrt{5}x)+\frac{\alpha_3}{\alpha_4}A(\sqrt{7}x)+A(x)=0.$$

Substituting $\sqrt{2}x$, $\sqrt{5}x$, $\sqrt{7}x$, $\sqrt{2}\sqrt{5}x$, $\sqrt{2}\sqrt{7}x$, $\sqrt{5}\sqrt{7}x$ and $\sqrt{2}\sqrt{5}\sqrt{7}x$, respectively we get the following system of equations

$$\frac{\alpha_1}{\alpha_4}A(\sqrt{2}x)+\frac{\alpha_2}{\alpha_4}A(\sqrt{5}x)+\frac{\alpha_3}{\alpha_4}A(\sqrt{7}x)+A(x)=0,$$

$$2\frac{\alpha_1}{\alpha_4}A(x)+\frac{\alpha_2}{\alpha_4}A(\sqrt{2}\sqrt{5}x)+\frac{\alpha_3}{\alpha_4}A(\sqrt{2}\sqrt{7}x)+A(\sqrt{2}x)=0,$$

$$\frac{\alpha_1}{\alpha_4}A(\sqrt{2}\sqrt{5}x)+5\frac{\alpha_2}{\alpha_4}A(x)+\frac{\alpha_3}{\alpha_4}A(\sqrt{7}\sqrt{5}x)+A(\sqrt{5}x)=0,$$

$$\frac{\alpha_1}{\alpha_4}A(\sqrt{2}\sqrt{7}x)+\frac{\alpha_2}{\alpha_4}A(\sqrt{5}\sqrt{7}x)+7\frac{\alpha_3}{\alpha_4}A(x)+A(\sqrt{7}x)=0,$$

$$2\frac{\alpha_1}{\alpha_4}A(\sqrt{5}x)+5\frac{\alpha_2}{\alpha_4}A(\sqrt{2}x)+\frac{\alpha_3}{\alpha_4}A(\sqrt{7}\sqrt{2}\sqrt{5}x)+A(\sqrt{2}\sqrt{5}x)=0,$$

$$2\frac{\alpha_1}{\alpha_4}A(\sqrt{7}x)+\frac{\alpha_2}{\alpha_4}A(\sqrt{5}\sqrt{2}\sqrt{7}x)+7\frac{\alpha_3}{\alpha_4}A(\sqrt{2}x)+A(\sqrt{2}\sqrt{7}x)=0,$$

$$\frac{\alpha_1}{\alpha_4}A(\sqrt{2}\sqrt{5}\sqrt{7}x)+5\frac{\alpha_2}{\alpha_4}A(\sqrt{7}x)+7\frac{\alpha_3}{\alpha_4}A(\sqrt{5}x)+A(\sqrt{5}\sqrt{7}x)=0,$$

$$2\frac{\alpha_1}{\alpha_4}A(\sqrt{5}\sqrt{7}x)+5\frac{\alpha_2}{\alpha_4}A(\sqrt{2}\sqrt{7}x)+7\frac{\alpha_3}{\alpha_4}A(\sqrt{2}\sqrt{5}x)+A(\sqrt{2}\sqrt{5}\sqrt{7}x)=0.$$

Taking the matrix

$$M := \begin{pmatrix} 1 & \frac{\alpha_1}{\alpha_4} & \frac{\alpha_2}{\alpha_4} & \frac{\alpha_3}{\alpha_4} & 0 & 0 & 0 & 0 \\ 2\frac{\alpha_1}{\alpha_4} & 1 & 0 & 0 & \frac{\alpha_2}{\alpha_4} & \frac{\alpha_3}{\alpha_4} & 0 & 0 \\ 5\frac{\alpha_2}{\alpha_4} & 0 & 1 & 0 & \frac{\alpha_1}{\alpha_4} & 0 & \frac{\alpha_3}{\alpha_4} & 0 \\ 7\frac{\alpha_3}{\alpha_4} & 0 & 0 & 1 & 0 & \frac{\alpha_1}{\alpha_4} & \frac{\alpha_2}{\alpha_4} & 0 \\ 0 & 5\frac{\alpha_2}{\alpha_4} & 2\frac{\alpha_1}{\alpha_4} & 0 & 1 & 0 & 0 & \frac{\alpha_3}{\alpha_4} \\ 0 & 7\frac{\alpha_3}{\alpha_4} & 0 & 2\frac{\alpha_1}{\alpha_4} & 0 & 1 & 0 & \frac{\alpha_2}{\alpha_4} \\ 0 & 0 & 7\frac{\alpha_3}{\alpha_4} & 5\frac{\alpha_2}{\alpha_4} & 0 & 0 & 1 & \frac{\alpha_1}{\alpha_4} \\ 0 & 0 & 0 & 0 & 7\frac{\alpha_3}{\alpha_4} & 5\frac{\alpha_2}{\alpha_4} & 2\frac{\alpha_1}{\alpha_4} & 1 \end{pmatrix}$$

we can write that

$$M \begin{pmatrix} A(x) \\ A(\sqrt{2}x) \\ A(\sqrt{5}x) \\ A(\sqrt{7}x) \\ A(\sqrt{2}\sqrt{5}x) \\ A(\sqrt{2}\sqrt{7}x) \\ A(\sqrt{5}\sqrt{7}x) \\ A(\sqrt{2}\sqrt{5}\sqrt{7}x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Theorem 4.5. *If the parameters $\frac{\alpha_1}{\alpha_4}$, $\frac{\alpha_2}{\alpha_4}$ and $\frac{\alpha_3}{\alpha_4}$ are algebrically independent, then the identically zero additive function is the only solution of the funcional equation*

$$\frac{\alpha_1}{\alpha_4}A(\sqrt{2}x) + \frac{\alpha_2}{\alpha_4}A(\sqrt{5}x) + \frac{\alpha_3}{\alpha_4}A(\sqrt{7}x) + A(x) = 0.$$

Proof. We are going to prove that $\det M \neq 0$ which means that the linear transzformation represented by M is regular and the kernel contains only the zero vector. This means that $A(x) = 0$ for any $x \in \mathbb{R}$. Suppose, in contrary that $\det M = 0$. Then the polynomial

$$P(x,y,z) := \det \begin{pmatrix} 1 & x & y & z & 0 & 0 & 0 & 0 \\ 2x & 1 & 0 & 0 & y & z & 0 & 0 \\ 5y & 0 & 1 & 0 & x & 0 & z & 0 \\ 7z & 0 & 0 & 1 & 0 & x & y & 0 \\ 0 & 5y & 2x & 0 & 1 & 0 & 0 & z \\ 0 & 7z & 0 & 2x & 0 & 1 & 0 & y \\ 0 & 0 & 7z & 5y & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 7z & 5y & 2x & 1 \end{pmatrix}$$

with rational coefficients must be identically zero. But this is a contradiction because $P(0,0,0) = 1$. $\square$

In what follows we omit the requirement of the independence of the parameters $\frac{\alpha_1}{\alpha_4}$, $\frac{\alpha_2}{\alpha_4}$ and $\frac{\alpha_1}{\alpha_4}$.

Lemma 4.6. *If x, y and z are nonzero real numbers and the rank of the matrix*

$$M(x,y,z) := \begin{pmatrix} 1 & x & y & z & 0 & 0 & 0 & 0 \\ 2x & 1 & 0 & 0 & y & z & 0 & 0 \\ 5y & 0 & 1 & 0 & x & 0 & z & 0 \\ 7z & 0 & 0 & 1 & 0 & x & y & 0 \\ 0 & 5y & 2x & 0 & 1 & 0 & 0 & z \\ 0 & 7z & 0 & 2x & 0 & 1 & 0 & y \\ 0 & 0 & 7z & 5y & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 7z & 5y & 2x & 1 \end{pmatrix}$$

is less than 6 then $x^2 = \frac{1}{2}$, $y^2 = 1$ and $z^2 = \frac{5}{7}$.

Proof. The subdeterminant

$$\det \begin{pmatrix} z & 0 & 0 & 0 & 0 \\ 0 & y & z & 0 & 0 \\ 0 & x & 0 & z & 0 \\ 1 & 0 & x & y & 0 \\ 0 & 1 & 0 & 0 & z \end{pmatrix} = -2z^3xy$$

shows that the rank is at least 5. Suppose that the rank is less than 6. Then the vanishing of the subdeterminant

$$\det \begin{pmatrix} y & z & 0 & 0 & 0 & 0 \\ 0 & 0 & y & z & 0 & 0 \\ 1 & 0 & x & 0 & z & 0 \\ 0 & 1 & 0 & x & y & 0 \\ 2x & 0 & 1 & 0 & 0 & z \\ 0 & 2x & 0 & 1 & 0 & y \end{pmatrix} = 4z^2y^2(1-2x^2)$$

shows that $x^2 = \frac{1}{2}$. The vanishing of the subdeterminant

$$\det \begin{pmatrix} x & y & z & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & y & z & 0 & 0 \\ 0 & 1 & 0 & x & 0 & z & 0 \\ 0 & 0 & 1 & x & 0 & y & 0 \\ 5y & 2x & 0 & 1 & 0 & 0 & z \\ 7z & 0 & 2x & 0 & 1 & 0 & y \\ 0 & 7z & 5y & 0 & 0 & 1 & x \end{pmatrix} = -8x^5yz + 56z^3yx^3 - 8zyx^3 +$$

$$+40zy^3x^3 - 98xyz^5 + 140xz^3y^3 - 28xz^3y + 6zyx - 50xzy^5 - 20xzy^3 =$$

$$= xyz(-8x^4 + 56z^2x^2 - 8x^2 + 40y^2x^2 - 98z^4 +$$

$$+140z^2y^2 - 28z^2 + 6 - 50y^4 - 20y^2)$$

shows, by the substitution $x^2 = \frac{1}{2}$, that

$$0 = -98z^4 + 140z^2y^2 - 50y^4 = -2(7z^2 - 5y^2)^2$$

and, consequently, $y^2 = \frac{7}{5}z^2$. The vanishing of the determinant

$$\det M(x,y,z) = 1 + 16x^8 + 560z^2y^2x^4 + 2401z^8 +$$
$$+392z^4x^2 - 500y^6 - 28z^2 + 112z^2x^4 + 24x^4 - 6860z^6y^2 +$$
$$+7350z^4y^4 - 3500z^2y^6 - 2800z^2y^2x^2 + 625y^8 + 1960z^4y^2x^2 +$$
$$1400z^2y^4x^2 - 160x^6y^2 - 224x^6z^2 + 600x^4y^4 + 80x^4y^2 +$$
$$+1176x^4z^4 - 1000x^2y^6 + 200x^2y^4 - 2744x^2z^6 + 980z^4y^2 - 1372z^6 +$$
$$+700z^2y^4 - 32x^6 + 150y^4 + 40x^2y^2 + 294z^4 - 20y^2 - 8x^2 + 56z^2x^2 + 140z^2y^2$$

shows, by the substitution $x^2 = \frac{1}{2}$ and $y^2 = \frac{7}{5}z^2$ that

$$0 = -294z^4 + \frac{14406}{25}z^8$$

and, consequently, $z^2 = \frac{5}{7}$ and $y^2 = 1$. □

Using the previous lemma it can be easily seen that the rank of M is great or equal than 6 or

$$\frac{\alpha_1}{\alpha_4} = \pm\frac{1}{\sqrt{2}}, \quad \frac{\alpha_2}{\alpha_4} = \pm 1, \quad \text{and} \quad \frac{\alpha_3}{\alpha_4} = \pm\frac{\sqrt{5}}{\sqrt{7}}$$

which means that the third example reduces to the special case of the first one. If the rank is maximal, then the only solution is the identically zero additive function. If the rank is 7 then we can use a similar argumentation as in the second example, see Theorem 4.3. So the only interesting case is rank M = 6. Then the kernel of the linear transformation represented by M is of dimension 2 which means that

$$\begin{pmatrix} A(x) \\ A(\sqrt{2}x) \\ A(\sqrt{5}x) \\ A(\sqrt{7}x) \\ A(\sqrt{2}\sqrt{5}x) \\ A(\sqrt{2}\sqrt{7}x) \\ A(\sqrt{5}\sqrt{7}x) \\ A(\sqrt{2}\sqrt{5}\sqrt{7}x) \end{pmatrix} = \lambda(x)\begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{pmatrix} + \mu(x)\begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \end{pmatrix}$$

for some additive functions λ and μ. If $v_0 = w_0 = 0$ then we have that A is identically zero. Otherwise if, for example $v_0 \neq 0$, then

$$\lambda(x) = \frac{1}{v_0}A(x) - \frac{w_0}{v_0}\mu(x)$$

and, consequently,

$$A(\sqrt{2}x) = \frac{v_1}{v_0}A(x) + \left(w_1 - \frac{w_0}{v_0}\right)\mu(x).$$

If $w_1 - \frac{w_0}{v_0} = 0$ then we get immediately the semihomogeneity property

$$A(\sqrt{2}x) = \frac{v_1}{v_0}A(x).$$

Otherwise

$$\mu(x) = \frac{1}{w_1 - \frac{w_0}{v_0}}A(\sqrt{2}x) - \frac{v_1}{v_0\left(w_1 - \frac{w_0}{v_0}\right)}A(x)$$

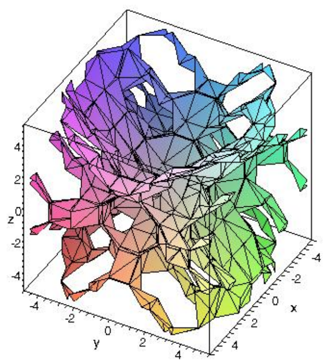
and, consequently,

$$A(\sqrt{5}x) = \frac{v_2}{v_0}A(x) - \frac{w_0 v_2}{v_0}\left(\frac{1}{w_1 - \frac{w_0}{v_0}}A(\sqrt{2}x) - \frac{v_1}{v_0\left(w_1 - \frac{w_0}{v_0}\right)}A(x)\right) +$$

$$w_2\left(\frac{1}{w_1 - \frac{w_0}{v_0}}A(\sqrt{2}x) - \frac{v_1}{v_0\left(w_1 - \frac{w_0}{v_0}\right)}A(x)\right)$$

wchich is discussed as the second example from the viewpoint of the semihomogeneity. The further arguments can be investigated in a similar way.

Remark 4.7. As an illustration the following figure shows the "irregularity surface" $\det M(x, y, z) = 0$.



References

1. Z. Daróczy, *Notwendige und hinreichende Bedingungen für die Existenz von nichtkonstanten Lösungen linearer Funktionalgleichungen*, Acta Sci. Math. Szeged, **22** (1961), 31–41.
2. Z. Daróczy and Gy. Maksa, *Functional equations on convex sets*, Acta Math. Hungar. **68** (1995), No. 3, 187-195.
3. Z. Daróczy, Gy. Maksa and Zs. Páles, *Functional equations involving means and their Gauss composition*, Proc. Amer. Math. Soc., **134** (2006), 521–530.
4. Zs. Páles, *Extension theorems for functional equations with bisymmetric operations*, Aequationes Math. **63** (2002), 266-291.
5. M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Prace Naukowe Universitetu Śląskiego w Katowicach Vol. CDLXXXIX, Państwowe Wydawnictwo Naukowe – Universitet Śląski, Warszawa–Kraków–Katowice, 1985.
6. L. Székelyhidi, *On a class of linear functional equations*, Publ. Math., **29** (1982), 19-28.
7. L. Székelyhidi, *Convolution type functional equations on topological Abelian groups*, World Scientific Publishing Co. Inc., Teaneck, NJ, 1991.
8. Z. Daróczy, K. Lajkó, R-L. Lovas, Gy. Maksa and Zs. Páles, *Functional equations involving means*, Acta Math. Hungar., submitted.
9. Gy. Maksa, *Functional equations involving means*, (in Hungarian), Talks at the Academy, Hungarian Academy of Sciences, Budapest (2006), under publication.
10. A. Varga, *On a functional equations containing four weighted arithmetic means*, BJMA, Vol. 2 (1), 2008, 21-32, www.math-analysis.org.
11. A. Varga, Cs. Vincze *On a functional equations containing weighted arithmetic means*, under publication in...

[1] [2]Institute of Mathematics, University of Debrecen, P. O. Box 12, Debrecen, Hungary.

E-mail address: `vargaa@math.klte.hu`

E-mail address: `csvincze@math.klte.hu`