ON GENERALIZED CONICS' THEORY AND AVERAGED RIEMANNIAN METRICS IN FINSLER GEOMETRY

CS. VINCZE

Abstract. In what follows we want to illustrate how to use averages to solve problems in Finsler geometry. We have two central topics which are closely related to each other via the notion of generalized Berwald manifolds. The linear connection on the base manifold is said to be compatible to the Finslerian metric if the parallel transport preserves the Finslerian lenght of tangent vectors. The generalized Berwald manifolds are Finsler spaces admitting compatible linear connections. The most important special cases are Berwald manifolds with torsion free compatible connections, Wagner manifolds with semi-symmetric compatible connections etc. The common feature of the compatible linear connections is that they must be metrical with respect to the averaged Riemannian metric coming from integration of the Riemann-Finsler metric on the indicatrix hypersurfaces. This central result allows us to consider the compatible linear connections in a Riemannian environment, i.e. the holonomy groups at the points of the base manifold can be interpreted as subgroups in the Euclidean orthogonal group. Of course if a generalized Berwald manifold is not a Riemannian space then the holonomy groups can not be transitive on the unit spheres in the tangent spaces with respect to the associated Riemannian metric. A continuity-type argumentation shows that the same is true for the topological closures of the holonomy groups at the points of the base manifold. Using the theory of generalized conics it can be also proved that it is a sufficient condition for a metrical linear connection of a Riemannian space to be the compatible connection for a non-Riemannian generalized Berwald manifold. Moreover the indicatrices can be always considered as generalized conics in the tangent spaces with respect to the Euclidean structure coming from the associated Riemannian metric. One of the advantages of this new approach is that we can avoid the classical theory of locally symmetric Riemannian spaces fitting the case of the classical Berwald manifolds because of the condition for the torsion being identically zero.

1. Introduction

There are several reasons why to use average (mean, mode, median, ecpectable value etc.) in mathematics. An average is a measure of the middle or typical value of a data set. The concept can be extended to functions too. The average value of an integrable function on a set D of finite positive measure is

$$\hat{f} = \frac{1}{\mu(D)} \int_D f \, d\mu.$$

Supported by JP-8/2009, TÉT_10 − 1 − 2011 − 0065. The paper contains the material of the lecture presented at 47th Symopsium on Finsler Geometry, 22-25 Nov. 2012, Kagoshima, Japan.

The general aim is to accumulate the information or to substitute more complicated mathematical objects with relatively simple ones. The object of the generalized conics' theory in the Euclidean coordinate space $\mathbb{R}^n$ is the investigation of subsets in the space all of whose points have the same average distance from the set of focuses. The "average" can be realized in several ways from classical (discrete) means to integration over the set of foci. In a significant part of the applications the common feature of functions measuring the average distance is the convexity. They also satisfy a kind of growth condition

$$\lim_{r\to\infty} \inf \left\{ \frac{F(\mathbf{x})}{r} \mid \|\mathbf{x}\| = r \right\} > 0.$$

These properties imply that the (lower) level sets of the form $F(\mathbf{x}) \leq c$ are compact convex subsets in the space bounded by compact convex hypersurfaces. They are called generalized conics[1]. The most important discrete cases are polyellipses with the classical arithmetic mean to calculate the average Euclidean distance from the elements of a finite point-set and lemniscates (with the classical geometric mean to calculate the average Euclidean distance from the elements of a finite point-set). Lemniscates in the plane play a central role in the theory of approximation in the sense that polynomial approximations[2] of holomorphic functions can be interpreted as approximations of curves with lemniscates [6], see also [22]. In terms of algebra we speak about the roots of polynomials (in terms of geometry we speak about the focuses of lemniscates). The polyellipses as additive versions of lemniscates have no such kind of properties (the problem was posed by E. Vázsonyi/Weiszfeld).

Definition 1. *Let $\Gamma = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ be a finite point-set in the Euclidean coordinate space. If the average distance is measured as the arithmetic mean*

$$A(\mathbf{x}) := \frac{d(\mathbf{x}, \mathbf{x}_1) + \ldots + d(\mathbf{x}, \mathbf{x}_m)}{m}$$

of distances from the points of Γ then hypersurfaces of the form A($\mathbf{x}$)=const. *are called polyellipses/polyellipsoids with focal set Γ.*

The result due to P. Erdős and I. Vincze [6] says that regular triangles can not be approximated by polyellipses even if the number of focuses can be arbitrary large, see also [23].

[1] The idea of generalization of classical conics is a periodic phenomenon in the history of mathematics. There are lots of points of view of investigations. Generalized conics (especially polyellipses as the additive versions of lemniscates) were investigated from the viewpoint of approximation theory [6]. On the other hand these geometric objects appear in optimization problems in a natural way [9], see also [14] and [10]. Another point of view in the literature is the theory of equidistant sets [15].

[2] Apart from the classical results such as Mergelyan's and Vituskin's theorem related to the so-called generalized lemniscates [22] there are some new trends [7] and [8] in the literature. Another possibility to take more steps forward is to give their geometric interpretations.

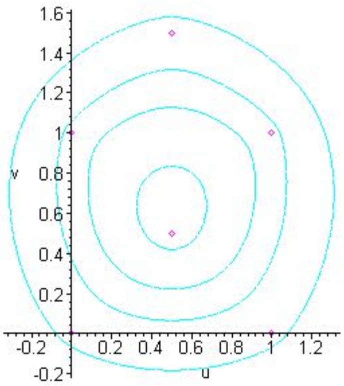

FIGURE 1. Confocal ellipses with six foci in the plane.

2. Alternative geometries

In what follows we present the solution of the problem how to find alternatives of the Euclidean geometry for subgroups in the Euclidean orthogonal group O(n). This is closely related to the theory of generalized Berwald manifolds (Berwald-, locally Minkowskian-, Wagner manifolds etc.). The idea is prepared by the following theorem due to L. Bieberbach.

Theorem 1. [4] *The holonomy group of any flat compact Riemannian manifold is finite.*

Let M be a flat compact Riemannian manifold and choose a point $\mathbf{x}$ of M. In the sense of Bieberbach's theorem we can find a finite system Γ of elements in the tangent space $T_{\mathbf{x}}M$ which is invariant under the holonomy group $H_{\mathbf{x}}$ of the Lévi-Civita connection. Therefore polyellipsoids with Γ as the focal set are also invariant. By parallel transports we can construct a smoothly varying family of compact convex bodies to provide a Finslerian environment for the Lévi-Civita connection: the Minkowski functionals induced by the generalized conics in the tangent spaces form a Finslerian fundamental function such that the parallel transport with respect to the Lévi-Civita connection preserves the Finslerian lenght of tangent vectors. This is just the notion of Berwald manifolds. Especially we have a flat connection and the process results in a locally Minkowski manifold. In general the holonomy group of a metrical linear connection is not finite. To adopt the previous method to the general situation we should develop the theory of conics with infinitely many focal points.

Definition 2. *Let $\Gamma \subset \mathbb{R}^n$ be a bounded orientable submanifold with finite positive measure with respect to the induced Riemannian volume form. The average distance is measured as the integral*

$$A(\mathbf{x}) := \frac{1}{\text{vol } \Gamma} \int_{\Gamma} \gamma \mapsto d(\mathbf{x}, \gamma)\, d\gamma. \tag{1}$$

Hypersurfaces of the form A($\mathbf{x}$)=const. *are called generalized conics with* Γ *as the set of focuses.*

According to the philosophy of integration (partitions, integral sums) conics defined by the equations of type A($\mathbf{x}$)=constant are just limits of polyellipsoids. Actually this is a purely topological way of the generalization – cf. Weiszfeld's problem of the topological closure of the set of polyellipses in the plane. We summarize some further possibilities of the generalization:

1. Consider curves in the Grassmannian (the set of k-dimensional linear subspaces) or the flag manifold (the collection of ordered sets of linear subspaces) to admit higher dimensional linear subspaces as focal objects (see parabolas).
2. Substitute the Euclidean distance in the integrand with another one: using the distance coming from the taxicab norm we have applications in geometric tomography, see [22].
3. Integral of type (1) can be considered as the expectable value of the random variable $d_2(\mathbf{x}, \xi)$ for any uniformly distributed random (vector) variable ξ on Γ. For different distributions, see [19].

In the preamble to his fourth problem presented at the International Mathematical Congress in Paris (1900) Hilbert suggested the examination of geometries standing next to Euclidean one in the sense that they satisfy much of Euclidean's axioms except some (tipically one) of them. In the classical non-Euclidean geometry the axiom taking to fail is the fameous parallel postulate. Another type of geometry standing next to Euclidean one is the geometry of normed spaces or, in a more general context, the geometry of Minkowski spaces. The crucial test is not the parallelism but the congruence via the group of linear isometries. Consider the standard n-dimensional real coordinate space $\mathbb{R}^n$ as an Euclidean space equipped with the canonical inner product and let G be a subgroup in the Euclidean orthogonal group O(n). We present a general method to construct a compact convex body K containing the origin in its interior such that

(A1) K is not a unit ball with respect to any inner product (ellipsoid problem),
(A2) K is invariant under the subgroup G,
(A3) its boundary ∂K is a smooth hypersurface (regularity condition).

Definition 3. *The Minkowski functional associated to K is defined as*

$$L(v) := \inf \{t > 0 \mid \frac{1}{t} v \in K\}.$$

The vector space equipped with such a functional is called a Minkowski space with unit ball K; the boundary of K is formed by the unit vectors with respect to L.

By the first condition (A1) the Minkowski space induced by the functional L associated to K is not Euclidean. The second condition (A2) says that G is a subgroup of the linear isometry group with respect to L. The third condition (A3) allows us to introduce the standard differential geometric objects in the space such as, for example, the metric with components formed by the second order partial derivatives of the energy function $E = (1/2)L^2$. In other words the Minkowski space $\mathbb{R}^n$ (with the functional L) is an alternative of the Euclidean geometry for the subgroup G. In case of differentiable manifolds with Riemannian structures the subgroup G will be interpreted as the holonomy group of a metrical linear connection at some point of the base manifold and the alternative geometry will be called Finsler geometry: instead of the Euclidean spheres in the tangent spaces, the unit vectors form the boundary of general convex sets containing the origin in their interiors (M. Berger). One of the main applications of generalized conics' theory is to present convex bodies satisfying conditions (A1) - (A3).

2.1. **The case of reducible subgroups – a survey.** If the group is reducible then there exists a non-trivial invariant linear subspace of dimension k+1 in the coordinate space under the elements of the subgroup. This subspace cuts a k - dimensional sphere from the unit sphere in the embedding space. In this case one of the spheres $S_1 \subset S_2 \subset \ldots \subset S_{n-2}$ plays the role of the set Γ of foci [20], see also [21]. Generalized conics with focal set S_{n-1} are spheres because the focal set is invariant under the whole orthogonal group. So do its levels.

Example 1. [20] see also [21]. As a basic example consider the space of dimension three (circular conics). Let

$$w\colon [0,2\pi] \to \mathbf{E}^3,\ w(t) := (\cos t, \sin t, 0) \tag{2}$$

be the unit circle S_1 in the (x, y)-coordinate plane and

$$A_1(x,y,z) := \frac{1}{2\pi}\int_0^{2\pi} \sqrt{\left(x-\cos t\right)^2 + \left(y-\sin t\right)^2 + z^2}\, dt.$$

The surface of the form

$$A_1(x,y,z) = \frac{8}{2\pi} \tag{3}$$

is a generalized conic.

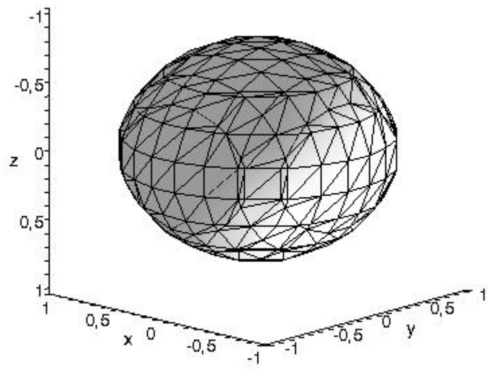

Figure 2. The generalized conic surface (3).

According to the invariance of the focal set under the rotation around the z-axis it is a revolution surface. To solve the ellipsoid problem by showing that (3) is not an ellipsoid it is enough to prove that the generatrix

$$\int_0^{2\pi} \sqrt{\cos^2 t + \left(y-\sin t\right)^2 + z^2}\, dt = 8 \tag{4}$$

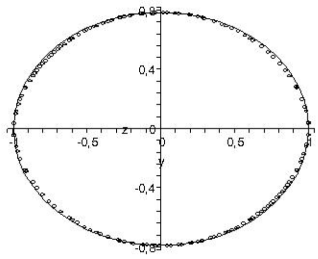

FIGURE 3. The generatrix (pointstyle) and its approximating ellipse.

is not an ellipse. If y=0 then we have that

$$z = \pm\sqrt{\left(\frac{8}{2\pi}\right)^2 - 1}.$$

On the other hand, if z=0 then the solutions of the equation

$$\int_0^{2\pi} \sqrt{\cos^2 t + \left(y - \sin t\right)^2}\, dt = 8$$

are just y=+1 or - 1. Therefore the only possible ellipse has the parametric form

$$y(s) = \cos s \quad \text{and} \quad z(s) = \sqrt{\left(\frac{8}{2\pi}\right)^2 - 1}\, \sin s. \tag{5}$$

The auxiliary function

$$v(s) := \int_0^{2\pi} \sqrt{\cos^2 t + \left(y(s) - \sin t\right)^2 + z^2(s)}\, dt$$

measures the difference between the generatrix and its approximating ellipse. We have that

$$v(0) = v\left(\frac{\pi}{2}\right) = 8 \quad \text{but} \quad v\left(\frac{\pi}{3}\right) = \frac{2}{\pi}\sqrt{2}\sqrt{3}\sqrt{8+\pi^2}\; E\left(\frac{2\sqrt{3}\pi}{3\sqrt{8+\pi^2}}\right),$$

where the symbol E refers to the elliptic integral

$$E(r) := \int_0^{\frac{\pi}{2}} \sqrt{1 - r^2 \sin^2 t}\, dt.$$

Although explicite calculations are impossible we can succesfully apply some recent results on elliptic integrals and the Gaussian hypergeometric function presented by Alzer - Qui [3] and Richards [16] to distinguish the generatrix and the approximating ellipse. The method is similar in case of higher dimensional spaces[3]. Finally we slightly modify the rate of the level in the following way

$$A_1(x, y, z) = c > \frac{8}{2\pi}. \tag{6}$$

A continuity-type argumentation shows that the generalized conic (6) also satisfies conditions (A1) and (A2). On the other hand the focal set is contained in the interior of the conic. This means that the regularity condition (A3) is also satisfied.

2.2. **The case of irreducible subgroups.**

Definition 4. *The group $G \subset O(n)$ is called transitive on the Euclidean unit sphere if any two elements of the unit sphere can be transported into each other by a transformation from G. G has dense orbits if there is at least one unit vector v such that its orbit under G is a dense subset of the Euclidean unit sphere.*

If the subgroup G is transitive on the Euclidean unit sphere then there are no alternatives. Using a continuity-type argumentation it can be easily seen that if G has dense orbits, i.e. the closure of G is transitive then the Euclidean geometry is the only possible one for G. In what follows we suppose that G has no dense orbits.

Theorem 2. (H. C. Wang) *If G has no dense orbits then its topological closure is of dimension less or equal than $\frac{(n-1)(n-2)}{2}$.*

Proof. Using the closed subgroup theorem from the classical Lie-theory the topological closure H is considered as a compact Lie-subgroup in O(n). The proof of the Wang's theorem is a simple induction. In case of $n = 2$ the statement is true because dim H=1 implies the existence of a one-parameter subgroup which obviously runs through the whole circle by its orbits. The inductive step is based on the mapping $\Omega\colon g \in H \mapsto g(v)$, where v is an arbitrarily fixed unit vector. Differentiating at the identitiy we have that

$$\dim H = \dim T_e H = \dim \mathrm{Ker}\ \Omega'(e) + \mathrm{Rank}\ \Omega'(e) = \dim H_v + \mathrm{Rank}\ \Omega'(e),$$

---

[3]In case of the unit sphere S_k in the coordinate (k+1) - plane $(x^1, \ldots, x^{k+1}, 0, \ldots, 0)$ the function measuring the average distance is

$$A_k(\mathbf{x}) := \frac{1}{\mathrm{Vol}\ S_k} \int_{S_k} \gamma \mapsto d(\mathbf{x}, \gamma)\, d\gamma =$$

$$\frac{1}{\mathrm{Vol}\ S_k} \int_{S_{k-1}} \gamma \mapsto \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{D(\mathbf{x}, \gamma, v)} \cos^{k-1}(v)\, dv \right) d\gamma, \quad \text{where}$$

$$D(\mathbf{x}, \gamma, v) := \sum_{i=1}^{k} (x^i - \gamma^i \cos(v))^2 + (x^{k+1} - \sin(v))^2 + (x^{k+2})^2 + \ldots + (x^n)^2.$$

The intersections of conics of the form $A_k(\mathbf{x}) = \text{const.}$ with the plane $x^1 = \ldots = x^k = 0$ and $x^{k+3} = \ldots = x^n = 0$ are niveau's of the function

$$f_k(y, z) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 + y^2 + z^2 - 2y \sin t} \cos^{k-1} t\, dt$$

with variables $y := x^{k+1}$ and $z := x^{k+2}$, respectively. It can be considered as a correction of the variables in [21], p. 820.

where $H_v := \{g \in H \mid g(v) = v\}$ is the stabilizer of the element v. Since Ω obviously commutates with the left translations it has a constant rank. Especially the rank is less than $n-1$ because the group H is not transitive on the unit sphere. By the same reason we can choose the element v in such a way that the stabilizer is not transitive on the sphere S_{n-2} in the orthogonal complement to v. Therefore (using the inductive hypothesis for the stabilizer group)

$$\dim H \leq \frac{(n-2)(n-3)}{2} + (n-2) = \frac{(n-1)(n-2)}{2}$$

as was to be proved. □

Remark 1. *The bound is just the same as that in Wang's theorem* [28] *without translation parameters. The mapping Ω plays a central role in Simon's fundamental work* [17] *on the transitivity of holonomy systems too.*

Corollary 1. *If $n = 2, 3$ then for any subgroup $G \subset O(n)$ having no dense orbits there is a finite system of elements in $\mathbb{R}^n$ which is invariant under G.*

Proof. Wang's theorem states that the dimension of the topological closure H is zero or 1 (in O(3)). If H is of dimension zero then the unit component is trivial (it contains only the identity). Because of its compactness H has at most finitely many connected components. In other words it is a finite subgroup. In case of dimension 1 the Lie algebra of H is generated by the anti-symmetric matrix A. Its action can be given as $Aw = v \times w$ for some uniquely determined element v. Therefore Av=**0** which means that v is invariant under the elements of the unit component. The compactness implies that H has only finitely many connected components and thus the system consisiting of the image of v under elements from different components is finite as was to be proved. □

Corollary 2. *The alternative geometry of dimension two or three always can be realized by Minkowski functionals induced by polyellipses/polyellipsoids.*

In case of higher dimensional spaces the convex hulls of the orbits will play the role of the focal set. Let G be a subgroup in O(n) having no dense orbits and consider its topological closure H. Taking a point **z** on the unit sphere the orbit P(**z**) under H is obviously invariant under any subgroup of H; especially G. Like the case of reducible subgroups the explicite calculation of integrals of the form

$$A(\mathbf{x}) := \int\limits_{\text{conv } P(\mathbf{z})} d_2(\mathbf{x}, \gamma)\, d\gamma \tag{7}$$

seems to be impossible in general. Therefore we follow another way to solve the ellipsoid problem (A1) by the help of a theorem of alternatives, see Theorem 3. Consider the function $f_z\colon w \mapsto \sup_{g \in G}\ d(w, g(\mathbf{z}))$. Since f_z is convex it follows that it is a continuous function and its infimum on the unit sphere is attained at some point z*. Therefore we can formulate the following definition.

Definition 5. *The minimax point of* **z** *is such a point* $\mathbf{z}^*$ *where the infimum* $a_z := \inf_{\|w\|=1}\ f_z(w)$ *is attained at.*

Corollary 3. *To find the minimax point is just the solution of the optimization problem subject to an equality constrain: minimize* $\sup_{g\in G}||w-g(\mathbf{z})||^2$ *subject to* $\|w\|=1$.

Lemma 1. *The function* f_z *is constant on the sphere if and only if* G *has dense orbits.*

Proof. The condition says that $f_z(w)=a_z$ for any unit vector w. If f_z attains its infimum at w, then the antipodal point is just the position where $\sup_{\|w\|=1}\inf_{g\in G}\ d(w,g(\mathbf{z}))$ is attained at. Therefore $w\mapsto\inf_{g\in G}d(w,g(\mathbf{z}))$ is also constant – especially it is identically zero which means that the orbit of z under G is dense. □

Consider the function

$$s\colon\mathbf{R}\to\mathbf{R},\quad s(t):=\begin{cases}0 & \text{if } t\leq a_z\\ (t-a_z)e^{-\frac{1}{t-a_z}} & \text{if } t>a_z.\end{cases}$$

By the help of the standard calculus [12] it can be seen that it is a smooth convex function on the real line. Define

$$m(t):=t+s(t);$$

as we can see nothing happens as far as $t\leq a_z$. If $t>a_z$ then the function m increases its value relative to the argument t. Therefore

$$A(\mathbf{x}):=\int\limits_{\text{conv }P(\mathbf{z})}\gamma\mapsto d(\mathbf{x},\gamma)\,d\gamma\quad\text{and}\quad A^*(\mathbf{x}):=\int\limits_{\text{conv }P(\mathbf{z})}\gamma\mapsto m(d(\mathbf{x},\gamma))\,d\gamma$$

agree at the minimax point: $c:=A(\mathbf{z}^*)=A^*(\mathbf{z}^*)$ but one of the hypersurfaces A($\mathbf{x}$)=c or A*($\mathbf{x}$)=c must be different from the sphere unless the mapping f_z is constant. It is impossible because G has no dense orbits. Therefore the ellipsoid problem (A1) is solved for irreducible subgroups because invariant ellipsoids under an irreducible subgroup in O(n) must be Euclidean spheres.

Theorem 3. (Theorem of the alternatives [21]) *If* G *has no dense orbits,* $\mathbf{z}$ *is a point on the Euclidean sphere and* c *is the common value of the functions* A *and* A^* *at the minimax point* $\mathbf{z}^*$ *then at least one of the hypersurfaces* A($\mathbf{x}$)=c *or* A*($\mathbf{x}$)=c *induces a non-Euclidean Minkowski functional* L *such that* G *is a subgroup of the linear isometries with respect to* L.

The theorem of the alternatives motivates the following definition.

Definition 6. *Let* $\Gamma\subset\mathbb{R}^n$ *be a bounded orientable submanifold of finite positive measure with respect to the induced Riemannian volume form. If* m *is a strictly monotone increasing convex function on the non-negative real numbers with initial value* m(0)=0 *and*

$$A_m(\mathbf{x}):=\frac{1}{\text{vol }\Gamma}\int_\Gamma\gamma\mapsto m(d_2(\mathbf{x},\gamma))\,d\gamma \tag{8}$$

then hypersurfaces of the form

$$A_m(\mathbf{x})=c \tag{9}$$

are called generalized conics with distorsion m.

Remark 2. *We have just realized the second possibility of generalizations: substitute the Euclidean distance function in the integrand with another one.*

Corollary 4. *If G is reducible or it is irreducible without dense orbits then an alternative geometry for G can be given by generalized conics of type* (9) *in the space.*

2.3. **Alternatives of Riemannain geometry.** Let M be a differentiable manifold with local coordinates u^1, ..., u^n on $U \subset M$. The induced coordinate system on the tangent manifold consists of the functions

$$x^1 := u^1 \circ \pi, \dots, x^n = u^n \circ \pi \quad \text{and} \quad y^1 := du^1, \dots, y^n = du^n,$$

where $\pi\colon TM \to M$ is the canonical projection.

Definition 7. *A Finsler structure on a differentiable manifold M is a smoothly varying family $F\colon TM \to \mathbb{R}$ of Finsler-Minkowski functionals in the tangent spaces satisfying the following conditions:*

- *the function F is of class at least $\mathcal{C}^4$ in all of its variables x^1, ..., x^n and y^1, ..., y^n for any non-zero element $v \in TM$ (i.e. it has an open neighbourhood such that the restricted function is of class at least $\mathcal{C}^4$ in all of its variables x^1, ..., x^n and y^1, ..., y^n),*
- *the Hessian matrix*

$$g_{ij} = \frac{\partial^2 E}{\partial y^j \partial y^i}$$

of the energy function $E := (1/2)F^2$ with respect to the variables $y^1, \dots, y^n$ is positive definite at any non-zero element $v \in TM$.

Theorem 4. [21] *Suppose that M is a connected Riemannian manifold and ∇ is a metrical linear connection on M. If $\mathbf{x} \in M$ and the holonomy group $G_{\mathbf{x}}$ of ∇ has no dense orbits in the tangent space $T_{\mathbf{x}}M$ then there is a non-Riemannian Finsler manifold equipped with the fundamental function $F\colon TM \to \mathbb{R}$ such that the parallel transports with respect to ∇ preserve the Finslerian lenght of tangent vectors and the unit spheres in the tangent spaces are generalized conics of type* (9).

Proof. In the sense of Corollary 4 there is a $G_{\mathbf{x}}$ - invariant generalized conic of type (9) in $T_{\mathbf{x}}M$. Using parallel transports (with respect to ∇) a smoothly varying family of compact convex bodies can be constructed to provide a Finslerian environment for ∇: the Minkowski functionals induced by the translated conics in the tangent spaces form a Finslerian fundamental function such that the parallel transports with respect to ∇ preserve the Finslerian lenght of tangent vectors. □

3. Averaged Riemannian metrics

Definition 8. *A linear connection on the base manifold is compatible to the Finslerian metric structure if and only if the parallel transport preserves the Finslerian lenght of tangent vectors. Finsler spaces admitting such a compatible linear connection are called generalized Berwald manifolds.*

In the previous section we investigated the problem how to find alternative geometries for a subgroup G in the orthogonal group. Such a subgroup was also interpreted as the holonomy group of a metrical linear connection at some point of a Riemannian manifold. The question how to build a non-Euclidean/non-Riemannian metric structure from the Euclidean/Riemannian one was answered by using group-invariant generalized conics and parallel transports. In what follows we are interested in the converse problem: suppose that we have a generalized Berwald manifold with a compatible linear connection ∇ on the base manifold. The converse problem is how to find a Riemannian structure for ∇ to be metrical. This question will be answered in terms of the averaged Riemannian metric [26], see also [1] and [5].

Definition 9. *Let $f\colon TM \to \mathbb{R}$ be a zero homogeneous function and let us define the average-valued function*

$$A_f(\mathbf{x}) := \int_{\partial K_{\mathbf{x}}} f\, \mu_{\mathbf{x}},$$

where $\partial K_{\mathbf{x}}$ is the indicatrix hypersurface with respect to the Finsler-Minkowski functional of the tangent space $T_{\mathbf{x}}M$,

$$d\mu = \sqrt{\det g_{ij}}\; dy^1 \wedge \ldots \wedge dy^n \tag{10}$$

is the canonical volume form on the tangent space $T_{\mathbf{x}}M$ as an oriented Riemannian manifold equipped with the metric tensor g and

$$\mu = \sqrt{\det g_{ij}}\; \sum_{i=1}^{n} (-1)^{i-1} y^i dy^1 \wedge \ldots \wedge dy^{i-1} \wedge dy^{i+1} \ldots \wedge dy^n \tag{11}$$

denotes the induced volume form on the indicatrix hypersurface. Especially the averaged Riemannian metric is defined as

$$\gamma(v, w) := \int_{\partial K} g(v, w)\, \mu. \tag{12}$$

To clarify that the averaged Riemannian metric (the average-valued function) is well-defined, i.e. the definition is independent of the coordinate systems and local orientations we can refer to (10) and (11) as canonical Riemannian densities instead of volume forms. In case of orientable Riemannian manifolds like the tangent spaces (finite dimensional real vector spaces) and indicatrices (compact convex hypersurfaces) the integral of functions will be the same independently of using Riemannian volume forms or Riemannian densities [12].

Example 2. [24] Let (M, α) be a Riemannian manifold and consider a nonzero 1-form β on M such that its supremum norm is less than 1 at each point of the manifold. The *Randers space* constructed from (M, α) by perturbation with β is defined as a Finsler manifold, where the fundamental function is a simple sum $F(v) := \|v\| + \beta(v)$ of the Riemannian fundamental function (the norm coming from the inner product) and β. The associated Riemannian metric can be expressed as the combination

$$\gamma(X, Y) = f\alpha(X, Y) + g\beta(X)\beta(Y)$$

of the initial data, where

1. In case of dimension 2 we have elliptic integrals of the second kind

$$f := \int_0^{2\pi} \sqrt{1+\|\beta^\sharp\|\sin v}\left(1-\frac{\|\beta^\sharp\|}{1+\|\beta^\sharp\|\sin v}\cos^2 v\sin v\right) dv,$$

$$g := \int_0^{2\pi} \frac{1}{\|\beta^\sharp\|\sqrt{1+\|\beta^\sharp\|\sin v}}(1+2\cos^2 v\sin v) + \frac{1}{\sqrt{1+\|\beta^\sharp\|\sin v}}\, dv.$$

2. In case of higher dimensional Randers manifolds

$$f := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\text{vol } S_{n-2}}{(1+\|\beta^\sharp\|\sin v)^{\frac{n-3}{2}}}\cos^{n-2} v\, dv-$$

$$-\frac{\|\beta^\sharp\|}{n-1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\text{vol } S_{n-2}}{(1+\|\beta^\sharp\|\sin v)^{\frac{n-1}{2}}}\cos^{n} v\sin v\, dv,$$

$$g := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\text{vol } S_{n-2}}{\|\beta^\sharp\|(1+\|\beta^\sharp\|\sin v)^{\frac{n-1}{2}}}(1+\frac{n}{n-1}\cos^2 v)\sin v\cos^{n-2} v\, dv+$$

$$+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\text{vol } S_{n-2}}{(1+\|\beta^\sharp\|\sin v)^{\frac{n-1}{2}}}\cos^{n-2} v\ dv.$$

From the viewpoint of the generalized Berwald manifolds' theory the key result can be formulated as follows.

Theorem 5. [26] *If ∇ is a linear connection on the manifold M such that the parallel transport with respect to ∇ preserves the Finslerian length of tangent vectors then it must be metrical with respect to the averaged Riemannian metric.*

The averaged Riemannian metric allows us to detect linear connections on the base manifod which are compatible to the Finslerian metric structure in the following way:

1 Compute the averaged Riemannian metric of the Finslerian structure.
2. Find the group $H_{\mathbf{x}}$ of orthogonal transformations leaving the Finslerian indicatrix at the point $\mathbf{x}$ invariant – it is obviously a closed subgroup in O(n).
3. If $G_{\mathbf{x}}$ is the holonomy group of any compatible linear connection at the point $\mathbf{x}$ then $G_{\mathbf{x}} \subset H_{\mathbf{x}}$.

We can formulate two important problems related to compatible connections: the problem of unicity and the problem of intrinsic characterization, i.e. how to express them in terms of the canonical data of the Finsler manifold. It is well-known that metrical linear connections are uniquely determined by the torsion tensor. Consider the decomposition

$$T(X,Y) := T_1(X,Y) + T_2(X,Y),$$

where

$$T_1 := T(X,Y) - T_2(X,Y) \text{ and } T_2 := \frac{1}{n-1}\left(\tilde{T}(X)Y - \tilde{T}(Y)X\right).$$

The traceless part T_1 is automatically zero in case of $n = 2$. In case of $n \geq 3$ the traceless part can be divided into two further components A_1 and S_1 by separating the axial (or totally antisymmetric) part A_1 which means that its lowered tensor with respect to the Riemannian metric is totally antisymmetric. Then we have eight classes of linear connections with torsion depending on that the terms A_1, S_1 and T_2 are surviving or not [2]. This results in eight classes of possible generalized Berwald manifolds. The most important special cases are summarized in the following.

1. Classical Berwald manifolds with $T = 0$.

From the viewpoint of Riemannian holonomies the case of the Berwald manifolds was systematically investigated by Z. Szabó [18]. He successfully used the results on the holonomy of Riemannian manifolds together with the foundations of symmetric Lie algebras, especially M. Berger's and J. Simon's [17] results.

2. Exact Wagner manifolds with vanishing traceless part and exact trace tensor
$$\tilde{T} = \frac{n-1}{2} d\alpha$$
in the torsion (we can speak about *closed Wagner manifolds* via the requirement of a closed trace tensor). Without extra conditions of exactness and closedness of the trace tensor this is the case of generalized Berwald manifolds admitting semi-symmetric compatible linear connections.

In case of Wagner manifolds the torsion involves the exterior derivative of a function. The geometric meaning is the global (local) conformal equivalence to a Berwald manifold via the exponent of the function α as Hashiguchi-Ichijyō's theorem [11] states. In 2001 M. Matsumoto [13] posed the problem of conformal equivalence of Berwald manifolds. According to Hashiguchi-Ichijyō's theorem the unicity problem of the compatible linear connection in a Wagner manifold is obviously related to the question whether how many essentially different ways there are for a Finsler manifold to be conformal to a Berwald manifold. In terms of Matsumoto's problem: Is there non-homothetic Berwald manifolds or not? All of these problems were solved by using the construction of averaged Riemannian metrics [26], see also [24] and [25]. In [27] we have completed the theory by solving the problems of the unicity and the intrinsic characterization for semi-symmetric compatible linear connections without extra conditions of exactness and closedness of the trace tensor. Since the traceless part T_1 is automatically zero in case of $n = 2$ this means the full solution of the basic problems for two-dimensional generalized Berwald manifolds.

3. If the torsion tensor has only the pure axial (anti-symmetric) component then the linear connection has the same geodesics as the Lévi-Civita connection and vice-verse.

Finally we present a lower dimensional unicity theorem related to the case of compatible linear connections with pure axial (anti-symmetric) components in the torsions.

Theorem 6. *Suppose that M is a three-dimensional non-Riemannian Finsler manifold, ∇_1 and ∇_2 are compatible linear connections with pure axial components in the torsions. Then $\nabla_1 = \nabla_2$.*

Proof. Taking the difference tensor $\nabla_2(X,Y) - \nabla_1(X,Y) = D(X,Y)$ of the connections we have that $\gamma(D(X,Y),Z) = -\gamma(D(X,Z),Y)$ because both ∇_1 and ∇_2 is metrical with respect to the averaged Riemannian metric γ. On the other hand they have the same geodesics as the Lévi-Civita connection which implies that $D(X,X) = 0$. Since the parallel transports (with respect to both ∇_1 and ∇_2) preserve the Finslerian lenght of tangent vectors the mapping $Y_{\mathbf{x}} \mapsto D_{\mathbf{x}}(X,Y)$ is an element of the Lie algebra of the group of orthogonal transformations leaving the Finslerian indicatrix at the point $\mathbf{x}$ invariant for any vector field X. In the sense of Wang's theorem this group is of dimension zero or 1. Therefore we can choose a basis X_1, X_2 and X_3 in $T_{\mathbf{x}}M$ in such a way that X_1 and X_2 belong to the kernel of the linear transformation $X_{\mathbf{x}} \mapsto D_{\mathbf{x}}(X, \cdot\ )$. We have that

$$D(X_3,X_1) = -D(X_1,X_3) = 0, \quad D(X_3,X_2) = -D(X_2,X_3) = 0$$

and, by the anti-symmetry again, $D(X_3,X_3) = 0$, i.e. the difference tensor is identically zero and the connections coincide. □

Example 3. The unicity of the compatible linear connection does not follow in general. Consider a compatible linear connection ∇ to the Finslerian metric structure on a connected manifold M and let $\tau_{\mathbf{xz}}$ be the set of parallel transport along curves joining the points $\mathbf{x}$ and $\mathbf{z}$. Since the indicatrix hypersurface is invariant under the holonomy group of ∇ at $\mathbf{x}$ the holonomy algebra acts trivially on the Finslerian fundamental function. By the Ambrose-Singer theorem it is spanned by the elements

$$R_\varphi(v,w) := \varphi^{-1} \circ R_{\mathbf{z}}(v,w) \circ \varphi$$

as $\mathbf{z}$ runs through the points of the manifold, $\varphi \in \tau_{\mathbf{xz}}$ and v, $w \in T_{\mathbf{z}}M$. Therefore if

$$D_{\mathbf{x}}(X,Y) \in \text{Linspan}\ \{R_\varphi(v,w) \mid \mathbf{z} \in M, \varphi \in \tau_{\mathbf{xz}} \text{ and } v,w \in T_{\mathbf{z}}M\}\ Y_{\mathbf{x}}$$

for any point $\mathbf{x}$ in M then the connection

$$\nabla_2(X,Y) = \nabla_1(X,Y) + D(X,Y) \tag{13}$$

is compatible to the Finslerian metric structure. Especially we put

$$D(X,Y) = \beta(X)R(V,W)Y$$

for some 1-form β and fixed vector fields V and W on the manifold.

References

[1] T. Aikou, Averaged Riemannian metrics and connections with application to locally conformal Berwald manifolds, Publ. Math. Debrecen **81**/1-2 (2012), 179-198.

[2] I. Agricola and T. Friedrich, On the holonomy of connections with skew-symmetric torsion, Math. Ann. 328 (2004), 711-748.

[3] H. Alzer and S. L. Qui, Monotonicity theorems and inequalities for the complete elliptic integrals, J. Comp. and Appl. Math. **172** (2004), 289-312.
[4] L. S. Charlap, Bieberbach groups and flat manifolds, Springer 1986.
[5] M. Crampin, On the construction of Riemannian metrics for Berwald spaces by averaging, accepted for publication in Houston J. Math.
[6] P. Erdős and I. Vincze, *On the approximation of closed convex plane curves*, Mat. Lapok 9 (1958), 19-36 (in Hungarian, summaries in Russian and German). MR 20 6070.
[7] A. A. Danielyan, On a polynomial approximation problem, J. of Approx. Theory, Vol. **162**, Issue 4, (April 2010), pp. 717-722.
[8] T. Gharibyan, W. Luh and J. Müller, A lacunary version of Mergelian's approximation theorem, J. of Approx. Theory, Vol. **162**, Issue 4, (April, 2010), pp. 709-716.
[9] C. Gross and T.-K. Strempel, On generalizations of conics and on a generalization of the Fermat-Toricelli problem, Amer. Math. Monthly **105**, no. 8 (1998) 732-743.
[10] S. Guernon and R. Tessler, The Fermat-Steiner problem, Amer. Math. Monthly **109**, Vol. 5 (2002), 443-451.
[11] M. Hashiguchi and Y. Ichijyō, On conformal transformations of Wagner spaces, Rep. Fac. Sci. Kagoshima Univ. (Math., Phys., Chem.) No. **10** (1977), 19-25.
[12] J. M. Lee, Introduction to Smooth Manifolds, Springer-Verlag New York, Inc. (2003)
[13] M. Matsumoto, Conformally Berwald and conformally flat Finsler spaces, Publ. Math. Debrecen, **58** (1-2) (2001), 275-285.
[14] Z. A. Melzak and J. S. Forsyth, Polyconics 1. Polyellipses and optimization, Quart. Appl. Math. **35**, Vol. 2 (1977), 239-255.
[15] M. Ponce and P. Santibanez, On equidistant sets and generalized conics: the old and the new, Amer. Math. Monthly, in appear.
[16] K. C. Richards, Sharp power mean bounds for the Gaussian hypergeometric function, J. Math. Anal. Appl. **308** (2005), 303-313.
[17] J. Simons, On the transitivity of holonomy systems, annals of Math., Vol. 76, No. 2, September, 1962, 213-234.
[18] Z. I. Szabó, Positive definite Berwald spaces, Tensor, N. S. Vol. **35** (1981), 25-39.
[19] Cs. Vincze, Á. Nagy, Cs. Noszály and M. Barczy, A stochastic algorithm for computing global minimizer of generalized conic functions, arXiv:1301.6112.
[20] Cs. Vincze and Á. Nagy, Examples and notes on generalized conics and their applications, AMAPN Vol. **26** (2010), 359-575.
[21] Cs. Vincze and Á. Nagy, An introduction to the theory of generalized conics and their applications, Journal of Geom. and Phys. Vol. **61** (2011), 815-828.
[22] Cs. Vincze and Á. Nagy, On the theory of generalized conics with applications in geometric tomography, J. of Approx. Theory **164** (2012), 371-390.
[23] Cs. Vincze and A. Varga, On a lower and upper bound for the curvature of ellipses with more than two foci, Expo. Math. **26** (2008), 55-77.
[24] Cs. Vincze, On a scale function for testing the conformality of Finsler manifolds to a Berwald manifold, Journal of Geometry and Physics. 54 (2005), 454-475.
[25] Cs. Vincze, On geometric vector fields of Minkowski spaces and their applications, Journal of Diff. Geom. and Its Appl. Vol. 24 (1), 2006, 1-20.
[26] Cs. Vincze, A new proof of Szabó's theorem on the Riemann-metrizability of Berwald manifolds, J. AMAPN, 21 (2005), 199-204.
[27] Cs. Vincze, On generalized Berwald manifolds with semi-symmetric linear connections, submitted to Publ. Math. Debrecen.
[28] H. C. Wang, Finsler spaces with completely integrable equations of Killing, J. London Math. Soc. **22** (1947), 5-9.

University of Debrecen, Hungary
E-mail address: `csvincze@science.unideb.hu`