ON THE DIOPHANTINE EQUATION $1 + 2^a + x^b = y^n$

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Abstract. Recently, mixed polynomial-exponential equations similar to the one in the title have been considered by many authors. In these results a certain non-coprimality condition plays an important role.

In this paper we completely solve the title equation for odd positive integers x with $x < 50$. Since we avoid the mentioned non-coprimality condition, this can be considered as a partial completion of the above mentioned results.

It seems that the deep effective tools (such as Baker's method) alone are not capable to handle the problem. We combine local arguments and Baker's method to prove our results.

1. Introduction

Mixed polynomial-exponential equations are of classical and recent interest. One of the most famous equation of this type is the so-called Ramanujan-Nagell equation

$$x^2 + 7 = 2^n$$

in positive integers x, n (see Ramanujan [20] and Nagell [19]). Later this equation has been generalized to

$$x^2 + D = y^n$$

in positive integers x, y, n with $n \geq 3$. Here the integer D is either fixed, or is of the form $D = \pm p_1^{a_1} \cdots p_s^{a_s}$ where $p_1, \ldots, p_s$ are given primes and $a_1, \ldots, a_s$ are unknown positive integers. For related results we only refer to the papers of Schinzel and Tijdeman [21], Cohn [12], Bugeaud, Mignotte and Siksek [10], Luca [18], Bugeaud and Muriefah

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2010 *Mathematics Subject Classification.* 11D41, 11D61.

Key words and phrases. Polynomial-exponential diophantine equations, local method, Baker's method.

Research supported in part by the OTKA grants K100339 and NK101680, and by the TÁMOP-4.2.2.C-11/1/KONV-2012-0001 project. The project has been supported by the European Union, co-financed by the European Social Fund. This work was partially supported by the European Union and the European Social Fund through project Supercomputer, the national virtual lab (grant no.: TÁMOP-4.2.2.C-11/1/KONV-2012-0010).

[11], Bérczes and Pink [6], Le and Zhu [16] and Xiaowei [23], and the references given there.

Recently the problem of perfect powers having few digits has been investigated in several papers. We only mention the results of Bennett, Bugeaud and Mignotte [4, 5], Corvaja and Zannier [13] and Bennett and Bugeaud [3] (see also the references in these papers). Beside deriving certain finiteness results, the authors solve several diophantine equations of the type

$$1 + x_1^a + x_2^b = y^n. \tag{1}$$

Here we also mention a nice result of Szalay [22], where the above equation is solved for $x_1 = x_2 = n = 2$.

Further, under certain restrictions even equations of type

$$1 + x_1^a + x_2^b + x_3^c = y^n \tag{2}$$

are considered; see e.g. [5] and the references there. However, in these results it is always necessary to assume that $\gcd(x_1, x_2)$ (or in case of equation (2) $\gcd(x_1, x_2, x_3)$) is greater than 1. The cases $x_1 = x_2$ and $x_1 = x_2 = x_3$ are of particular interest.

Much less is known about the solution of equation (1) when x_1 and x_2 are coprime. In particular, even the special case $x_1 = 2$, $x_2 = 3$ and $n = 2$ could not be handled by the deep methods of Corvaja and Zannier (see e.g. [13] and the references there). This particular equation has been recently solved by local methods by Leitner [17].

In this paper we completely solve the equation

$$1 + 2^a + x^b = y^n \tag{3}$$

in positive integers a, b, y, n with $n \geq 2$, for all odd values of x with $0 < x < 50$. Note that in particular, we avoid the condition of non-coprimality of x_1 and x_2 in (1). Obviously, we may assume that n is a prime in equation (3), so from this point on we shall do so without any further mentioning.

We note that classical methods (e.g. Baker's method) alone are apparently not sufficient to handle (1) in its full generality. Our method to handle (3) is the following. First by a local argument we show that one of a and n must be small in (3). If n is small then we apply local arguments again to find all solutions. If a is small then for the resulting equations we apply Baker's method to bound n, and we solve the remaining equations using local methods, and through elliptic equations and Thue equations of small degrees. At this stage we make use of the program package Magma [8], as well.

Because of the relatively large number of cases, in our local considerations we need to find efficiently moduli which witness that the appropriate equations have no solutions. (For finding appropriate moduli in special circumstances, see e.g. the papers of Brenner and Foster [9], Alex and Foster [1, 2] and Leitner [17], and the references there.) In doing so we shall use moduli composed of primes p such that $p-1$ has only small prime factors (such as $2, 3, 5$). The efficiency of such moduli is implied by work of Erdős, Pomerance and Schmutz [14], and have been successfully applied in case of purely exponential equations (see [7]). See also [5], where a similar sieve is applied.

Finally, we mention that we believe that our method is capable to solve equation (3) for other values of x, as well.

2. The main result

Our main result is the following.

Theorem 2.1. *The only solutions to equation* (3) *with* $0 < x < 50$ *odd and* $y^n > 100$ *are given by*

$$(a, x, b, y, n) = (4, 43, 3, 282, 2),\ (7, 15, 1, 12, 2),\ (1, 5, 3, 2, 7).$$

Note that the solutions (a, x, b, y, n) to equation (3) with $y^n \le 100$ are very easy to enumerate. The reason that we do not list them is that there are many such solutions.

3. Some lemmas

To prove our theorem we need the following lemmas. The first is a state-of-the-art lower bound for linear forms in the logarithms of two algebraic numbers, due to Laurent (Theorem 2 of [15]). For an algebraic number α of degree d over $\mathbb{Q}$, we define as usual the *absolute logarithmic height* of α by the formula

$$h(\alpha) = \frac{1}{d}\left(\log|a_0| + \sum_{i=1}^{d} \log\max\left(1, |\alpha^{(i)}|\right)\right),$$

where a_0 is the leading coefficient of the minimal polynomial of α over $\mathbb{Z}$ and the $\alpha^{(i)}$-s are the conjugates of α in the field of complex numbers.

Lemma 3.1. *Let* α_1 *and* α_2 *be multiplicatively independent algebraic numbers with* $|\alpha_1| \ge 1, |\alpha_2| \ge 1$ *and let* h, ρ *and* μ *be real numbers with* $\rho > 1$ *and* $1/3 \le \mu \le 1$. *Set*

$$\sigma = \frac{1+2\mu-\mu^2}{2}, \quad \lambda = \sigma\log\rho, \quad H = \frac{h}{\lambda} + \frac{1}{\sigma},$$

$$\omega = 2\left(1+\sqrt{1+\frac{1}{4H^2}}\right), \quad \theta = \sqrt{1+\frac{1}{4H^2}}+\frac{1}{2H}.$$

Consider the linear form $\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1$, *where* b_1 *and* b_2 *are positive integers. Put*

$$D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}] \,/\, [\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}]$$

and assume that

$$h \geq \max\left\{D\left(\log\left(\frac{b_1}{a_2}+\frac{b_2}{a_1}\right)+\log\lambda+1.75\right)+0.06, \lambda, \frac{D\log 2}{2}\right\},$$

$$a_i \geq \max\{1, \rho|\log\alpha_i| - \log|\alpha_i| + 2Dh(\alpha_i)\} \qquad (i=1,2),$$

and

$$a_1 a_2 \geq \lambda^2.$$

Then

$$\log|\Lambda| \geq$$

$$\geq -C\left(h+\frac{\lambda}{\sigma}\right)^2 a_1 a_2 - \sqrt{\omega\theta}\left(h+\frac{\lambda}{\sigma}\right) - \log\left(C'\left(h+\frac{\lambda}{\sigma}\right)^2 a_1 a_2\right)$$

with

$$C = C_0 \frac{\mu}{\lambda^3 \sigma}, \quad C' = \sqrt{\frac{C\sigma\omega\theta}{\lambda^3\mu}}$$

where

$$C_0 = \left(\frac{\omega}{6}+\frac{1}{2}\sqrt{\frac{\omega^2}{9}+\frac{8\lambda\omega^{5/4}\theta^{1/4}}{3\sqrt{a_1a_2}H^{1/2}}+\frac{4}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\frac{\lambda\omega}{H}}\right)^2.$$

We need the following corollary of Lemma 3.1 (see Corollary 2 of [15]). For a non-zero rational number $\alpha = u/v$ (given in reduced form) let $H(\alpha) = \max\{\log|u|, \log|v|, 1\}$.

Lemma 3.2. *Suppose that* α_1 *and* α_2 *are multiplicatively independent positive rational numbers with* $|\alpha_1| \geq 1, |\alpha_2| \geq 1$ *and assume that* $\alpha_1, \alpha_2, \log\alpha_1, \log\alpha_2$ *are positive real numbers. Consider the linear form*

$$\Lambda = b_2 \log\alpha_2 - b_1\log\alpha_1, \tag{4}$$

where b_1 *and* b_2 *are positive integers. Then*

$$\log|\Lambda| \geq -25.2\max\{\log h' + 0.38, 10\}^2 H(\alpha_1)H(\alpha_2), \tag{5}$$

where

$$h' = \frac{b_1}{H(\alpha_2)}+\frac{b_2}{H(\alpha_1)}.$$

In the course of the proof of our main result we have to solve completely several equations of the shape

$$x^b + t = y^n, \tag{6}$$

where x, t are given odd positive integers and b, y, n are unknown positive integers with $y \geq 2$ even and $n \geq 3$ an odd prime. By using Lemma 3.2 and Lemma 3.1 we derive relatively sharp explicit upper bounds for n in equation (6).

Lemma 3.3. *Let $3 \leq x \leq 47$ and $3 \leq t \leq 33$ be given odd, relatively prime integers. Consider equation (6) in unknown positive integers (b, y, n) with $n \geq 3$ an odd prime. Then for $y = 2$ we have $n < 14600$, for $y > 2$*

$$n < n_x = \begin{cases} 2800, & \textit{if} \quad x = 3, \\ 5000, & \textit{if} \quad 5 \leq x \leq 7, \\ 7000, & \textit{if} \quad 9 \leq x \leq 15, \\ 8600, & \textit{if} \quad 17 \leq x \leq 29, \\ 10000, & \textit{if} \quad 31 \leq x \leq 47 \end{cases}$$

holds, and for $y > 50000$ we have $n \leq 2003$.

Proof. First, by applying Lemma 3.2 we derive the indicated upper bounds for n in equation (6) valid for $y = 2$ and $y > 2$, respectively. Then, by using Lemma 3.1 we show that $n \leq 2003$, provided that $y > 50000$.

If in equation (6) we have $n \mid b$ then by writing $b = nb_1$ ($b_1 \geq 1$), we get from (6) that

$$t = y^n - (x^{b_1})^n = (y - x^{b_1})(y^{n-1} + y^{n-2}x^{b_1} \ldots + (x^{b_1})^{n-1}). \tag{7}$$

Since $t \geq 3 > 0$, equation (7) implies that $y - x^{b_1} \geq 1$, which together with $y \geq 2$ and (7) gives

$$t > 2^{n-1}.$$

Therefore by $t \leq 33$ we obtain $n < 1 + \log 33/\log 2 < 7$, which is a much better bound for n than stated. So, in what follows, we may write b occurring in (6) in the form $b = nB + r$, where B and r are integers for which $B \geq 0$ and $0 < r \leq n - 1$. Thus, equation (6) yields

$$\left| x^r \left(\frac{x^B}{y} \right)^n - 1 \right| = \frac{t}{y^n}. \tag{8}$$

Set

$$\Lambda := r \log x - n \log(y/x^B). \tag{9}$$

Since for every $z \in \mathbb{C}$ for which $|z-1| \leq 1/3$ we have $|\log z| < 2|z-1|$, we get by (8) and (9) that

$$|\Lambda| < \frac{2t}{y^n}. \tag{10}$$

Now, we are going to derive a lower bound for $|\Lambda|$ occurring in (9). Since x and t are relatively prime integers, it follows that in equation (6) we have $\gcd(x, y) = 1$. Thus the rational numbers y/x^B and x are multiplicatively independent. Hence we may apply Lemma 3.2 on taking

$$\alpha_1 = y/x^B, \alpha_2 = x, b_1 = n, b_2 = r.$$

By (6) we obviously have $y > x^B$, thus we may choose

$$H(\alpha_1) = H(y/x^B) = \begin{cases} 1, & \text{if} \quad y = 2, \\ \log y, & \text{if} \quad y > 2 \end{cases} \quad \text{and} \quad H(\alpha_2) = H(x) = \log x.$$

Since $r < n$ and $H(\alpha_1) \geq 1$ we obtain

$$h' < \frac{n}{\log x} + n,$$

whence we obtain the lower bound

$$|\Lambda| > -25.2 \max\left\{\log\left(\frac{n}{\log x} + n\right) + 0.38, 10\right\}^2 \log x H(\alpha_1). \tag{11}$$

On comparing (10) and (11) we infer that

$$n < 25.2 \max\left\{\log\left(\frac{n}{\log x} + n\right) + 0.38, 10\right\}^2 \log x \frac{H(\alpha_1)}{\log y} + \frac{\log 2t}{\log y}, \tag{12}$$

which, by $y \geq 2$, implies that either

$$n < 2520 \log x \frac{H(\alpha_1)}{\log y} + \frac{\log 2t}{\log 2}, \tag{13}$$

or

$$n < 25.2\left(\log\left(\frac{n}{\log x} + n\right) + 0.38\right)^2 \log x \frac{H(\alpha_1)}{\log y} + \frac{\log 2t}{\log 2}. \tag{14}$$

Since

$$\frac{H(\alpha_1)}{\log y} = \begin{cases} \frac{1}{\log 2}, & \text{if} \quad y = 2, \\ 1, & \text{if} \quad y > 2 \end{cases}$$

then by (13) and (14) a simple calculation gives an upper bound for n valid for every $t \leq 33$ and $y \geq 2$. Namely, we may write $n < 14600$ if $y = 2$, and $n < n_x$ if $y > 2$, where

$$n_x = \begin{cases} 2800, & \text{if } x = 3, \\ 5000, & \text{if } 5 \le x \le 7, \\ 7000, & \text{if } 9 \le x \le 15, \\ 8600, & \text{if } 17 \le x \le 29, \\ 10000, & \text{if } 31 \le x \le 47. \end{cases} \tag{15}$$

In what follows we will show that in equation (6) we have $n \le 2003$, provided that $y > 50000$. We may write b in equation (6) in the form $b = nB + r$ as above, but now with $r \in \mathbb{Z}$ for which $0 < |r| \le \frac{n-1}{2}$. Since $y > 50000$, we may assume that $B \ge 1$. Set

$$\Lambda_r = \begin{cases} r \log x - n \log(y/x^B), & \text{if } r > 0 \\ |r| \log x - n \log(x^B/y), & \text{if } r < 0. \end{cases} \tag{16}$$

By applying the same argument as above we obtain by (8) and (16) that

$$|\Lambda_r| < \frac{2t}{y^n}. \tag{17}$$

Now, we are going to derive a lower bound for $|\Lambda_r|$ using Lemma 3.1. Choose $\alpha_1 = y/x^B, \alpha_2 = x, b_1 = n, b_2 = r$ if $r > 0$ and set $\alpha_1 = x^B/y, \alpha_2 = x, b_1 = n, b_2 = |r|$ if $r < 0$. Then we have $D = 1$. In the case when $r < 0$ using $n \ge 5, B \ge 1, t \le 33$ and $1 \le |r| \le \frac{n-1}{2}$ we obtain that in equation (6) we have $x^B > y$. Thus, we may write

$$h(\alpha_1) = h(x^B/y) = \log(x^B).$$

Further, by some routine calculus we find $\log(x^B/y) < (n-1)\log x/2n$, whence

$$\rho|\log(x^B/y)| - \log|x^B/y| + 2h(x^B/y) < \frac{\rho+2}{2}\log x + 2\log y.$$

Therefore we may choose

$$a_1 = \frac{\rho+2}{2}\log x + 2\log y \quad \text{and} \quad a_2 = (\rho+1)\log x. \tag{18}$$

By using the same argument as above we see that the values occurring in (18) are convenient also for the case $r > 0$. In what follows, we suppose that $n > 2003$ and by fixing the parameters (μ, ρ) occurring in Lemma 3.1, we derive a lower bound for Λ_r defined in (16). We give the details only in the case $x = 47$, since the proof for the values $3 \le x \le 45$ is similar.

Suppose that

$$n > 2003, \tag{19}$$

and set $x = 47$. Further, choose $\mu = 0.58$ and $\rho = 7$. Then we get that $\sigma = 0.9118$ and $\lambda = 0.9118 \log 7 < 1.7743$. Further, by (18) we obtain

$$a_1 = 4.5 \log 47 + 2 \log y \quad \text{and} \quad a_2 = 8 \log 47, \tag{20}$$

so, by $y > 50000$ we easily check that $a_1 a_2 > \lambda^2$ holds. Now, we are going to derive an upper bound for the quantity

$$h_1 = D\left(\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log\lambda + 1.75\right) + 0.06.$$

Since $D = 1, |r| < n/2$ and $y > 50000$ we get by (20) that

$$h_1 < \log\left(\frac{n}{8\log 47} + \frac{n/2}{4.5\log 47 + 2\log 50000}\right) + \log(0.9118\log 7) + 1.81.$$

Hence we find that

$$h_1 < \log n - 0.71. \tag{21}$$

Set

$$h := \log n - 0.71. \tag{22}$$

By (19) and (22) we see that

$$h > \log(2003) - 0.71 > 6.892, \tag{23}$$

which implies that $h > 1.7743 > \lambda$. Thus, the assumptions of Lemma 3.1 concerning the parameter h are satisfied. By (23) we may write for the quantity H occurring in Lemma 3.1 the lower bound

$$H = \frac{h}{\lambda} + \frac{1}{\sigma} > \frac{6.892}{0.9118\log 7} + \frac{1}{0.9118} > 4.98.$$

Thus, we check that $\omega < 4.01006$ and by $\lambda = 0.9118 \log 7 < 1.7743$, $H > 4.98$ and $y > 50000$ we get $C_0 < 1.96$. Further, by (9) of Section 3.1 and by (24) of Section 3.3 of [15] we have

$$\sqrt{\omega\theta} \leq (5 + \sqrt{17})/4 < 2.3 \tag{24}$$

and

$$C' < 4C. \tag{25}$$

We check that in our case

$$\mu/\lambda^3\sigma = 0.58/((\log 7)^3 \cdot 0.9118^4) < 0.114 \tag{26}$$

and

$$h + \lambda/\sigma = \log n - 0.71 + \log 7 < \log n + 1.24. \tag{27}$$

By combining (25), (26), (27) with $a_1 = 4.5 \log 47 + 2 \log y, a_2 = 8 \log 47, C_0 < 1.96, n < n_{47} = 10000$ and using $y > 50000$ we obtain that

$$C' \left(h + \frac{\lambda}{\sigma} \right)^2 a_1 a_2 < 2 \log y. \tag{28}$$

By setting

$$A := \log n + 1.24 \tag{29}$$

and combining $C_0 < 1.96$ with (20), (24), (26) and (28) we may write that

$$\log |\Lambda_r| \geq -0.114 \cdot 1.96 (4.5 \log 47 + 2 \log y)(8 \log 47) A^2 - 2.3A - 2 \log y. \tag{30}$$

On comparing (30) with (17) we obtain by (29), $t \leq 33$ and $y > 50000$ that

$$n < 24.8(\log n + 1.24)^2 + 0.23(\log n + 1.24) + 3.$$

This by a simple calculation yields that $n \leq 2003$, which contradicts (19). So we find that in equation (6) we have $n \leq 2003$, provided that $y > 50000$. By applying the same approach as above for the values $3 \leq x \leq 45$ we conclude that in each case under consideration we always have that $n \leq 2003$. Thus the lemma is proved. □

4. Proof of Theorem 2.1

In this section we give the proof of our main result.

Proof of Theorem 2.1. Observe first that since a is assumed to be positive, the left hand side of (3) is always even. Hence y must always be even. Moreover, considering equation (3) modulo an appropriate power of 2, we get that either a or n must be small. For example, if $x = 1$ then modulo 4 we see that $a \geq 2$ would yield that $2 \mid y^n$ but $4 \nmid y^n$, a contradiction. Thus $a = 1$ must hold in this case (yielding the solution $(a, x, b, y, n) = (1, 1, b, 2, 2)$). By a similar argument, for $1 \leq x \leq 49$ odd we get Table 1.

x	modulus	conclusion
$1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49$	4	$a = 1$
$3, 11, 19, 27, 35, 43$	8	$a \leq 2$ or $n = 2$
$7, 23, 39$	16	$a \leq 3$ or $n = 3$
$15, 47$	32	$a \leq 4$ or $n = 2$
31	64	$a \leq 5$ or $n = 5$

TABLE 1. Bounding a or n in equation (3).

Assume first that for some given odd x with $1 \leq x \leq 49$ we have that n is bounded, according to Table 1. Then the resulting equation is solved by a local argument. We explain our method by an example, the situation is similar for the other cases. So consider the case $x = 43$, and observe that then Table 1 gives $n = 2$. Take the modulus

$$m := 2^5 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 41 \cdot 61 \cdot 73 \cdot 109 \cdot 151 \cdot 181 \cdot 241 \cdot 401 \cdot 433 \cdot 1801 \cdot 2161.$$

Observe that for all the factors f_i of m we have that $\varphi(f_i)$ is composed exclusively of (at most) the primes $2, 3, 5$. In this way we may intersect the information obtained by considering (3) modulo f_i separately. In fact it turns out that if $a \geq 5$ then the system of linear congruences

$$1 + 2^a + 43^b \equiv y^2 \pmod{f_i} \quad (i = 1, \ldots, 16)$$

has no solutions. (Observe the occurrence of $f_1 = 2^5$ as a factor in m.) In case of $a < 5$, we write $b = 3u + t$ with $t = 0, 1, 2$, and consider the elliptic equations $43^t z^3 + c = y^2$ with $c = 3, 5, 9, 17$, where $z = 43^u$. Solving these equations with Magma [8], in this case we get the only solution

$$(a, x, b, y, n) = (4, 43, 3, 282, 2).$$

When instead of $n = 2$ we have $n = 3$ or $n = 5$, then in the remaining small cases equation (3) is reduced to Thue equations of degrees 3 or 5, respectively. These equations can also be handled by Magma. For example, when $x = 7$ then by Table 1 we have $n = 3$. Now using the modulus

$$m = 2^4 \cdot 3 \cdot 5 \cdot 7$$

we get that our equation has no solutions, provided that $a \geq 4$. In the remaining cases $a < 4$ we just need to solve Thue equations of the form $7^t z^3 + c = y^3$ with $t = 0, 1, 2$, $c = 3, 5, 9$, $z = 7^u$, $b = 3u + t$.

In all the cases considered, this approach works. We summarize the appropriate moduli in Table 2 (corresponding to the values of n coming from Table 1). Note that if the corresponding condition does not hold then (3) reduces to simple elliptic or Thue equations as above, which are all solved by Magma.

Now we turn to the case where n is arbitrary, but a is bounded according to Table 1. Some of the resulting equations

$$x^b + t = y^n \tag{31}$$

where we write t for the possible values $1 + 2^a$, can be handled easily. First of all, we do not need to consider the cases where x is a prime power, hence $x = 9, 25, 27, 49$ can be excluded. The case $x = 1$ is trivial and has already been discussed. We include all the other pairs (x, t) in Table 3, which can be excluded using appropriate moduli. Namely, it

x	n	modulus	condition
3	2	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	$a \geq 4$ and $b \geq 2$
11	2	$2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 97 \cdot 241$	$a \geq 5$
19	2	$2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 41 \cdot 43 \cdot 61 \cdot$ $\cdot 73 \cdot 109 \cdot 181 \cdot 211 \cdot 337 \cdot 421 \cdot 631$	$a \geq 5$
27	2	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	$a \geq 4$
35	2	$2^2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 61$	$a \geq 2$ and $b \geq 2$
43	2	$2^5 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 41 \cdot 61 \cdot 73 \cdot 109 \cdot 151 \cdot$ $\cdot 181 \cdot 241 \cdot 401 \cdot 433 \cdot 1801 \cdot 2161$	$a \geq 5$
7	3	$2^4 \cdot 3 \cdot 5 \cdot 7$	$a \geq 4$
23	3	$2^2 \cdot 3^2 \cdot 7 \cdot 13 \cdot 37$	$a \geq 2$
39	3	$2^2 \cdot 3^2 \cdot 7 \cdot 13$	$a \geq 2$ and $b \geq 2$
15	2	$2^2 \cdot 3^2 \cdot 5^2 \cdot 7$	$a \geq 2$ and $b \geq 2$
47	2	$2^6 \cdot 3 \cdot 5^2 \cdot 11 \cdot 13 \cdot 61$	$a \geq 6$
31	5	$2^2 \cdot 11 \cdot 31 \cdot 41 \cdot 61 \cdot 101 \cdot$ $\cdot 151 \cdot 181 \cdot 241 \cdot 601$	$a \geq 2$

TABLE 2. Moduli yielding that (3) has no solutions for x, under the indicated condition.

turns out that for these pairs either b or n is small. If b gets bounded, then to find all solutions for the corresponding equation is trivial. If n is bounded then we can reduce the problem to the solutions of elliptic or Thue equations just as previously. All these equations were solved by Magma.

(x,t)	excl. mod	(x,t)	excl. mod	(x,t)	excl. mod
$(3,3)$	3	$(21,3)$	3	$(39,3)$	3
$(7,3)$	8	$(23,3)$	8	$(39,5)$	16
$(7,5)$	8	$(23,5)$	8	$(39,9)$	8
$(11,3)$	8	$(31,3)$	8	$(41,3)$	8
$(15,3)$	3	$(31,5)$	8	$(43,3)$	8
$(15,5)$	8	$(31,9)$	16	$(45,3)$	3
$(15,9)$	16	$(31,17)$	32	$(47,3)$	8
$(17,3)$	8	$(33,3)$	3	$(47,5)$	8
$(19,3)$	8	$(35,3)$	8	$(47,9)$	16
$(19,5)$	9	$(35,5)$	8		

TABLE 3. The easily excludable pairs (x,t) in (31).

Thus we are left with the pairs $(x,t) \in H$ to solve equation (31) where

$$H = \{(3,5),(5,3),(7,9),(11,5),(13,3),(15,17),$$

$$(23,9),(29,3),(31,33),(37,3),(43,5),(47,17)\}.$$

By Lemma 3.3 in all these cases we get that $n \leq 2003$, provided that $y > 50000$.

Assume first that $n \leq 2003$. If $n \leq 7$ then we get elliptic or Thue equations as before, and these equations can be solved by Magma. So we may further suppose that $n \geq 11$. In case of $11 \leq n \leq 2003$, we can solve all the occurring equations locally, by finding appropriate moduli as earlier. However, this has to be done rather carefully, since for the prime factors p of m we also need that $n \mid p-1$. Here we applied the following strategy. For a given $(x,t) \in H$, we found the first 6 primes $p \nmid x$ such that $n \mid p-1$, and the order of x modulo p is smaller than $p/6$. Then testing the corresponding equation modulo the product of these primes, we always got that there are no solutions other than those given in the theorem. We could find appropriate primes in each case to handle the pairs (x,t) with $11 \leq n \leq 2003$.

Suppose next that $n > 2003$. Then we have $y \leq 50000$. Since $y \geq 2$ is even, working modulo 8 in equation (31) we may assume that b is odd for every $(x,t) \in H$. We will use a local argument to reduce the number of possibilities for y in equation (31). We explain our method by an example, the situation is similar for the other cases. So consider the case $(x,t) = (47,17)$. Then equation (31) becomes

$$47^b + 17 = y^n \tag{32}$$

where $y \geq 2$ is even. Further, Lemma 3.3 implies that $n < n_0$, where

$$n_0 = \begin{cases} 14600, & \text{if } y = 2, \\ 10000, & \text{if } y > 2. \end{cases}$$

By a simple calculation we obtain that

$$47^b + 17 \equiv \begin{cases} 2 & \pmod{3}, \\ 0,4 & \pmod{5}, \\ 9,12 & \pmod{13}, \\ 4,13 & \pmod{17}, \\ 18 & \pmod{23}. \end{cases} \tag{33}$$

Since n is an odd prime, (32) and (33) yields

$$y \equiv \begin{cases} 1 & \pmod{3}, \\ 0, 4 & \pmod{5}, \\ 3, 4, 9, 10, 12 & \pmod{13}, \\ 4, 13 & \pmod{17}, \\ 2, 3, 4, 6, 8, 9, 12, 13, 16, 18 & \pmod{23}. \end{cases} \tag{34}$$

The assertion (34) reduces the number of possibilities for $y \geq 2$ to 67 cases. Finally, supposing $b \geq 5$ and working modulo 47^5 in equation (32) for the remaining 67 values of $y \geq 2$ and for every odd prime $n < n_0$, we always get a contradiction. This shows that we necessarily have $b \in \{1, 3\}$ in (32), whence the solutions of (32) can be easily listed. By repeating the above argument for every $(x, t) \in H$ we could determine all solutions of (32), and the theorem follows. □

References

[1] L. J. Alex and L. L. Foster, *On the diophantine equation* $1 + x + y = z$, Rocky Mountain J. Math. **22** (1992), 11–62.

[2] L. J. Alex and L. L. Foster, *On the diophantine equation* $w + x + y = z$*, with* $wxyz = 2^r 3^s 5^t$, Revista Mat. Univ. Comp. Madrid **8** (1995), 13–48.

[3] M. A. Bennett and Y. Bugeaud, *Perfect powers with three digits*, submitted.

[4] M. A. Bennett, Y. Bugeaud and M. Mignotte, *Perfect powers with few binary digits and related Diophantine problems*, Ann. Scuola Normale Pisa, to appear.

[5] M. A. Bennett, Y. Bugeaud and M. Mignotte, *Perfect powers with few binary digits and related Diophantine problems, II*, Math. Proc. Camb. Phil. Soc. **153** (2012), 525–540.

[6] A. Bérczes and I. Pink, *On the diophantine equation* $x^2 + p^{2k} = y^n$, Arch. Math. **91** (2008), 505–517.

[7] Cs. Bertók and L. Hajdu, *On a Hasse-type principle for exponential diophantine equations and its applications*, submitted.

[8] W. Bosma, J. Cannon and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235–265.

[9] J. L. Brenner and L. L. Foster, *Exponential diophantine equations*, Pacific J. Math. **101** (1982), 263–301.

[10] Y. Bugeaud, M. Mignotte and S. Siksek, *Classical and modular approaches to exponential and diophantine equations II. The Lebesque-Nagell equation*, Compos. Math. **142** (2006), 31–62.

[11] Y. Bugeaud and F. S. A. Muriefah, *The Diophantine equation* $x^2 + c = y^n$*: a brief overview*, Rev. Colomb. Mat. **40** (2006), 31–37.

[12] J. H. E. Cohn, *The diophantine equation* $x^2 + C = y^n$, Acta Arith. **65** (1993), 367–381.

[13] P. Corvaja and U. Zannier, *Finiteness of odd perfect powers with four nonzero binary digits*, Annales de l'institut Fourier **63** (2013), 715–731.

[14] P. Erdős, C. Pomerance and E. Schmutz, *Carmichael's lambda function*, Acta Arith. **58** (1991), 365–385.

[15] M. Laurent, *Linear forms in two logarithms and interpolation determinants II*, Acta Arith. **133** (2008), 325–348.

[16] M. H. Le and L. H. Zhu *On some generalized Lebesgue-Nagell equations*, J. Number Theory **131** (2011), 458–469.

[17] D. J. Leitner, *Two exponential Diophantine equations*, J. Théor. Nombres Bordeaux **23** (2011), 479–487.

[18] F. Luca, *On the equation* $x^2+2^a3^b=y^n$, Int. J. Math. Sci. **29** (2002), 239–244.

[19] T. Nagell, *Løsning till oppgave nr 2*, Norsk. Mat. Tidsskrift **30** (1948), 62–64.

[20] S. Ramanujan, *Question 446*, J. Indian Math. Soc. **5** (1913), 130.

[21] A. Schinzel and R. Tijdeman, *On the equation* $y^m = P(x)$, Acta Arith. **31** (1976), 199–204.

[22] L. Szalay, *The equation* $2^n + 2^m + 2^l = z^2$, Indag. Math. **13** (2002), 131–142.

[23] P. Xiaowei, *The Exponential Lebesgue-Nagell Equation* $x^2+p^{2m}=y^n$, Period. Math. Hungar. **67** (2013), 231–242.

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