Effective finiteness results for Diophantine equations over finitely generated domains

(*Survey and new results with J.-H. Evertse*)

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Diophantine equations

Ineffective finiteness results over **number fields** and more generally over **finitely generated domains**

$A = \mathbb{Z}[z_1, \ldots, z_r]$ finitely generated domain (**FGD**)
$A \supset \mathbb{Z}$, $z_1, \ldots, z_r$ *algebraic or transcendental* / $\mathbb{Q}$
Examples: $A = \mathbb{Z}$, $\mathcal{O}_K$ (K number field), $\mathcal{O}_S$ (S finite set of places), $\mathbb{Z}[X_1, \ldots, X_r], \ldots$

method: *Thue–Siegel–Roth–Schmidt* method

Effective finiteness results over **number fields**

$A = \mathbb{Z}$, $\mathcal{O}_K$, $\mathcal{O}_S$

method: *Baker's method*

Effective results over **function fields** (no finiteness)

method: *Mason,...*

Extension of the **effective theory** to the case of **finitely generated domains**

- **reduction** to the **number field case** and **function field case** by **effective specializations**, **use** of **effective results** over **number fields and function fields**, Gy (1983) $\Rightarrow$ **Thue equations, decomposable form equations, discriminant equations** over a *restricted* class of FGD's, Gy (1983)
- **combining Gy's method** with a result of Aschenbrenner (2004) $\Rightarrow$ **general method** for *arbitrary* FGD's; Evertse, Gy (2013) $\Rightarrow$ *unit equations*

further **applications** of the **general method** to:

- *Thue equations*: Bérczes, Evertse, Gy (2014)
- *superelliptic equations*, *Schinzel–Tijdeman equation*: Bérczes, Evertse, Gy (2014)

- *generalized unit equations:* Bérczes (2015)
- *Catalan equation*: Koymans (2017)
- *discriminant equations*: Evertse–Gy (2017)
- *decomposable form equations*: Evertse–Gy (202?)

$\Rightarrow$ a great number of **applications**

In my *talk:*

I Brief historical overview

II New general effective results on decomposable form equations over finitely generated domains and their applications (joint results with J.-H. Evertse)

I. Brief historical overview

UNIT EQUATIONS: Let $a, b, c \in A \setminus \{0\}$

$$ax + by = c \quad \text{in } x, y \in A^* \qquad \text{(U)}$$

Ineffective finiteness results:

Siegel (1921): $A = \mathcal{O}_K$, K number field, implicit

Mahler (1933): $A = \mathbb{Z}[(p_1 \cdot \ldots \cdot p_s)^{-1}]$, $p_1, \ldots, p_s$ primes

Parry (1950): $A = \mathcal{O}_S$, S-integers in K

Lang (1960): A arbitrary finitely generated over $\mathbb{Z}$

Effective results over number fields

First general effective finitenetss results, explicit bounds for the solutions:

Győry (1973, 1974): $A = \mathcal{O}_K$, K number field

Győry (1979): $A = \mathcal{O}_S$, S-integers in K

$$ax + by = c \quad \text{in } x, y \in \mathcal{O}_S^*, \; S\textit{-unit equation} \qquad (U_S)$$

Several **improvements** of the bounds, e.g. Bugeaud–Győry, Bugeaud, Győry–Yu, Le Fourn; the **best known bound** in terms of S : Győry (2019)

A great number of **applications**

method of proof: Baker's method; recent **alternative effective methods:** Bombieri, Bombieri–Cohen $A = \mathcal{O}_S$, over number fields, Murty–Pasten, von Känel, Matschke, Siksek, Bennett,. . . ,modular method over $\mathbb{Z}$

Generalization for finitely generated A

Let again $A = \mathbb{Z}[z_1, \dots, z_r]$, K quotient field, $a, b, c \in A \setminus \{0\}$,

$$ax + by = c \quad \text{in}\, x, y \in A^* \tag{U}$$

Gy (1983): for $q \leq r$, $\{z_1, \dots, z_q\} \subseteq \{z_1, \dots, z_r\}$, maximal algebraically independent, $A_0 = \mathbb{Z}[z_1, \dots, z_q]$, $K_0 = \mathbb{Q}(z_1, \dots, z_q)$; $\exists g \in A_0 \setminus \{0\}$ and $w \in K^*$ integral over A_0 such that

$$A \subseteq B := A_0 \left[\frac{1}{g}, w\right] \ \ (\subset K).$$

A effectively given if q and the minimal polynomials of $z_{q+1}, \dots, z_r$ over K_0 are given $\Rightarrow$ g, w and hence B can be determined

It follows from my results:

Theorem A (Győry, 1983)

The **unit equation**

$$ax + by = c \quad \text{in}\, x, y \in B^* \qquad (U_B)$$

has only finitely many solutions in B^* (and hence in A^* as well). Further, if q, g, w and a, b, c are effectively given, the solutions of (U_B) can be effectively determined.

Quantitative version: effective bound for the "size" of the solutions

basic idea of the **method of proof**, detailed description about 15 pages

reduction to the *function field* and *number field* case: in the number field case sufficiently many **effective** *ring homomorphisms* (*specializations*): any $\boldsymbol{u} = (u_1, \ldots, u_q) \in \mathbb{Z}^q$ yields a ring homomorphism $A_0 \to \mathbb{Z}$ by substituting u_i for z_i for $i = 1, \ldots, q$. This map can be extended to a ring homomorphism $B \to \overline{\mathbb{Q}}$ which sends (U_B) to an S-unit equation in a number field depending on $\boldsymbol{u}$.

use of *effective results* over *number fields* and *function fields* $\Rightarrow$ **algorithm** for solving (U_B)

The method works for B, $Z[X_1, \ldots X_r]$ and a class of other finitely generated domains of the form $A = \mathbb{Z}[z_1, \ldots, z_r]$. In general it was a

problem: in 1983, no general algorithm was known to select those solutions $x, y \in B^*$ of (U_B) for which $x, y \in A^*$.

Generalization for arbitrary finitely generated A (with J.-H. Evertse)

In what follows, **another representation** for $A = \mathbb{Z}[z_1, \dots, z_r]$.

Put

$$R = \mathbb{Z}[X_1, \dots, X_r],\ I = \{f \in R : f(z_1, \dots, z_r) = 0\}$$

$\Rightarrow$ $$A \cong R/I$$

I finitely generated ideal

Definitions

- *A effectively given* if a set of generators of I is given, say $I = (f_1, \dots, f_t)$
- for $\alpha \in A$, $\tilde{\alpha} \in R$ *representative* of α if $\alpha = \tilde{\alpha}(z_1, \dots, z_r)$
- $\alpha \in A$ is *effectively given* if a representative of α is given

Consider again the **unit equation**

$$ax + by = c \quad \text{in}\, x, y \in A^* \quad (a, b, c \in A \setminus \{0\}) \qquad \text{(U)}$$

Theorem B (Evertse–Győry, 2013)

If A and $a, b, c \in A$ are effectively given, the solutions x, y of (U) *can be effectively determined.*

method of proof: *refinement and combination* of *Győry's method* with the following *theorem of Aschenbrenner* (2004)

Theorem (Aschenbrenner, 2004)

Let $g_1, \ldots, g_m, g \in R := \mathbb{Z}[X_1, \ldots, X_r]$
Assume that

$$g_1 x_1 + \cdots + g_m x_m = g \tag{A}$$

is solvable in $x_1, \ldots, x_m \in R$. If $g_1, \ldots, g_m, g$ are given then (A) *has an effectively computable solution $x_1, \ldots, x_m \in R$.*

Remark

Theorem $\Rightarrow$ *algorithm* for deciding whether $x, y \in B^*$ are contained in A^* or not

Quantitative version of Theorem B

Definition

for $\alpha \in R = \mathbb{Z}[X_1, \ldots, X_r]$, the *degreee* $\deg \alpha$ is the total degree of α, and the *logarithmic height* $h(\alpha)$ of α is the logarithm of the maximum absolute value of its coeffiients. The *size* of α is defined by

$$s(\alpha) := \max\{1, \deg \alpha, h(\alpha)\}.$$

There are only **finitely many** $\alpha \in R = \mathbb{Z}[X_1, \ldots, X_r]$ of **bounded** size, and all of them can be determined effectively.

Theorem B' (Evertse–Győry, 2013)

Assume that in $A = \mathbb{Z}[z_1, \dots, z_r]$, $r \geq 1$. *Let* $\tilde{a}, \tilde{b}, \tilde{c}$ *be representatives for* $a, b, c \in A$ *in* $R = \mathbb{Z}[X_1, \dots, X_r]$. *Assume that* $f_1, \dots, f_t \in R$ *and* $\tilde{a}, \tilde{b}, \tilde{c}$ *all have degree at most* d *and logarithmic height at most* h, *where* $d \geq 1, h \geq 1$. *Then for each solution* (x, y) *of* (U) $ax + by = c$ *in* $x, y \in A^*$, *there are representatives* $\tilde{x}, \tilde{x'}, \tilde{y}, \tilde{y'}$ *of* x, x^{-1}, y, y^{-1} *such that*

$$s(\tilde{x}), s(\tilde{x'}), s(\tilde{y}), s(\tilde{y'}) \leq \exp\{(2d)^{c_1^r}(h+1)\},$$

where c_1 *is an effectively computable absolute constant* > 1.

Theorem B' $\Rightarrow$ Theorem B, easy

Thue equations

Let $A = \mathbb{Z}[z_1, \ldots, z_r], K$ quotient field of A, and

$$F(X, Y) = a_0 X^n + a_1 X^{n-1} Y + \cdots + a_n Y^n \in A[X, Y],$$

$b \in A \setminus \{0\}, n \geq 3, F$ has no multiple factor.

$$F(x, y) = b \quad \text{in}\, x, y \in A \tag{T}$$

Ineffective finiteness results

Thue (1909): $A = \mathbb{Z}$

⋮

Lang (1960): A arbitrary finitely generated domain

Generalization:

Theorem C (Siegel, K number field, 1929; Lang, A finitely generated, 1960)

Let $F \in K[X, Y]$ be a polynomial irreducible over $\overline{K}$ such that the affine curve $F(x, y) = 0$ is of genus ≥ 1. Then this curve has only finitely many points with coordinates in A.

Effective finiteness results for (T)

Baker (1968): $A = \mathbb{Z}$, bound for x, y

Coates (1969): $A = \mathbb{Z}[(p_1 \cdot \ldots \cdot p_s)^{-1}]$

Kotov–Sprindžuk (1973): $A = \mathcal{O}_S$, ring of S-integers in a number field K

Improvements of the bounds for x, y:

Feldman (1971),...

method of proof: Baker's method

Gy (1983): for a restricted class of finitely generated domains A

General case: recall $A = \mathbb{Z}[z_1, \ldots, z_r]$, K quotient field, $R = \mathbb{Z}[X_1, \ldots, X_r]$, $I = \{f \in R : f(z_1, \ldots, z_r) = 0\}$ finitely generated *ideal* in R; for $\alpha \in A, \tilde{\alpha} \in R$ *representative* of α if $\alpha = \tilde{\alpha}(z_1, \ldots, z_r)$

Theorem D (Bérczes, Evertse, Gy, 2014)

Given generators $f_1, \ldots, f_t$ of I and representatives of $a_0, a_1, \ldots, a_n, b$, the solutions $x, y \in A$ of (T) *can be effectively determined*

+ quantitative version

method of proof: E–Gy's method

major open problems: make effective the Siegel–Lang Theorem C (first over $\mathbb{Z}$ and then over A)

Superelliptic equations

Let

$$F(X) = a_0X^n + \cdots + a_n \in A[X], \; b \in A \setminus \{0\},$$

$m \geq 2, F$ has no multiple zero

$$F(x) = by^m \quad \text{in}\, x, y \in A \qquad \text{(HS)}$$

$n \geq 2$ if $m \geq 3$, superelliptic case

$n \geq 3$ if $m = 2$, hyperelliptic case

Ineffective finiteness results

Siegel (1926), LeVeque (1964): $A = \mathbb{Z}$ or $\mathcal{O}_K, K$ number field

Lang (1960), A arbitrary finitely generated domain

Effective results

Baker (1969): $A = \mathbb{Z}$

Schinzel–Tijdeman (1976): bound for m

Brindza (1984): $A = \mathcal{O}_S$, number field case

Brindza (1989): A domain considered by Gy (1983)

Theorem E (B, E, Gy, 2014)

If A and $a_0, \ldots, a_n, b$ are effectively given, then (HS) *has only finitely many solutions and all of them can be effectively determined*

+ effective bound for m

+ quantitative version

method of proof: E–Gy's method

Generalized unit equations

A finitely generated over $\mathbb{Z}$, K quotient field, $F \in A[X, Y], \Gamma \subset K^*$ finitely generated

$$(*) \begin{cases} F \text{ has no divisor of the form } X^m Y^n - \alpha \\ \text{or } X^m - \alpha Y^n, m, n \geq 0 \text{ integers, } m + n > 0 \end{cases}$$

$$F(x, y) = 0 \quad \text{in } x, y \in A^* \text{ or more generally in } \Gamma \qquad \text{(GU)}$$

Ineffective finiteness results: $(*)$ *necessary*

Lang (1960): finitely many solutions in A^* and in Γ

Lang's conjecture: the same in $x, y \in \overline{\Gamma}$, the division group of Γ

$$\overline{\Gamma} := \{u \in \overline{K}^* \ : \ \exists\, m > 0 \text{ integer}, u^m \in \Gamma\}$$

Liardet (1974,75): proof of Lang's conjecture

Effective finiteness results in number fields

Bombieri–Gubler (2006): (GU), in Γ

Bérczes, Evertse, Gy (2009): (U) in $\overline{\Gamma}$

Bérczes, Evertse, Gy, Pontreau (2009): (GU) in $\overline{\Gamma}$

Effective finiteness result over FGD's

Theorem F (Bérczes, 2015)

If A, Γ are finitely generated and A, Γ, F are effectively given, then (GU) *has only finitely many solutions +* **effective** *+ quantitative*

method of proof: Evertse–Gy (2013)

Catalan equation

Let A be a FGD

$$x^m - y^n = 1 \text{ in } x, y \in A \setminus \{0\}, \text{ not root of unity}, m, n > 1,\ mn > 4 \quad \text{(C)}$$

Catalan conjecture (1844): for $A = \mathbb{Z}$, $3^2 - 2^3 = 1$ is the only solution

Tijdeman (1976): $A = \mathbb{Z}$, effective finiteness result
Brindza, Gy, Tijdeman (1986): $A = \mathcal{O}_K$, effective finiteness result
Brindza (1987): $A = \mathcal{O}_S$, effective finiteness result
Brindza (1993): for a class of FGD's effective finiteness result

Baker's method

Mihailescu (2002): proof of Catalan conjecture

other method

Theorem G (Koymans, 2017)

If A is an effectively given FGD, then (C) *has only finitely many solutions*
$+$ **effective** $+$ *quantitative*

method of proof: Evertse–Győry (2013)

Discriminant equations

$A = \mathbb{Z}[z_1, \dots, z_r], K$ quotient field, L finite extension of $K, D \in A \setminus \{0\}$

many *diophantine problems* $\Rightarrow$ **discriminant equation**

$$D(F) = D \text{ in monic } F \in A[X] \text{ of given degree } n \geq 2 \text{ having its zeros in } L \qquad (D_1)$$

$F(X), F(X+a)$ $(a \in A)$ *A-equivalent* $\Rightarrow$ same discriminant.

Ineffective finiteness results on A-equivalence classes of solutions

Delone, Nagell (1930), independently: $A = \mathbb{Z}$, $n = 3$

Nagell (1967): $A = \mathbb{Z}$, $n = 4$, F irreducible

In full generality:

Gy (1982): assume that A is integrally closed (in K). Then (D_1) has only finitely many A-equivalence classes of solutions

Consequences:

L/K finite extension, A_L integral closure of A in L

$$D_{L/K}(\alpha) = D \quad \text{in}\, \alpha \in A_L \tag{D_2}$$

$\alpha, \alpha + a$ ($a \in A$) *A-equivalent* $\Rightarrow$ same discriminant

Gy (1982): *Up to A-equivalence, (D_2) has only finitely many solutions*

$$A_L = A[\alpha] \quad \text{for}\, \alpha \in A_L \tag{D_3}$$

$\Leftrightarrow \{1, \alpha, \ldots, \alpha^{d-1}\}$ power integral basis of A_L over A, $d = [L : K]$

Examples: $A = \mathbb{Z}, K = \mathbb{Q}, L$ quadratic or cyclotomic,
if α *solution* of $(D_3) \Rightarrow$ so is $\varepsilon\alpha + a, \varepsilon \in A^*, a \in A$

Gy (1982): *Up to multiplication by elements of A^* and translation by elements of A, there are only finitely many $\alpha \in A_L$ with (D_3).*

method of proof: reduction of (D_1) to unit equations; $(D_2) \Rightarrow (D_1)$; $(D_3) \Rightarrow (D_2)$

Effective finiteness results for equations (D_1), (D_2), (D_3)

Gy (1973–1976): $A = \mathbb{Z}$, in (D_1) L not fixed

Gy (1978–1981): $A = \mathcal{O}_K$, $\mathcal{O}_S$, number field case

method of proof: reduction to unit equations, Baker's method

Gy (1984): for a class of finitely generated A over $\mathbb{Z}$

general case

$A = \mathbb{Z}[z_1, \dots, z_r], K$ quotient field, L finite extension of K

L is *given effectively* if an irreducible $P \in K[X]$ is given such that $L \cong K[X]/(P)$

Theorem H (Evertse–Gy, 2017)

Assume that A is integrally closed. Then up to A-equivalence, equation (D_1) has only finitely many solutions. Further, if A, L and D are given, all solutions can be determined effectively.

- The condition that A is integrally closed can be weakened to

$$\left(\frac{1}{n}A^{+} \cap A_{K}^{+}\right)/A^{+} \text{ finite, decidable}$$

where A_K is the integral closure of A in K

- Similar results for equations (D_2), (D_3) under some additional conditions

method of proof: reduction to unit equations in L, use of general Theorem B on unit equations and some effective linear algebra

II. Decomposable form equations

(survey and some new general effective results with J.-H. Evertse)

basic importance in diophantine number theory

$A = \mathbb{Z}[z_1, \ldots, z_r]$, K quotient field, $\overline{K}$ an algebraic closure of K, $b \in K^*$

Definition

$F \in K[X_1, \ldots, X_m]$ *decomposable form*, if it factorizes into linear factors, say $\ell_1, \ldots, \ell_n$ over $\overline{K}$. Assume that at least three of $\ell_1, \ldots, \ell_n$ are pairwise linearly independent

Decomposable form equation:

$$F(x_1, \ldots, x_m) = b \quad \text{in } x_1, \ldots, x_m \in A \tag{DF_1}$$

$m = 2$, Thue equation

Further important classes of decomposable form equations with $m \geq 2$:

norm form equations, discriminant form equations, index form equations

Norm form equations and discriminant form equations

Norm form equation:

$$N(\alpha_1 x_1 + \cdots + \alpha_m x_m) = b \text{ in } x_1, \ldots, x_m \in A \qquad \text{(NF)}$$

where $\alpha_1 = 1, \alpha_2, \ldots, \alpha_m \in \overline{K}$, linearly independent over K,
$N(\alpha_1 X_1 + \cdots + \alpha_m X_m)$ norm form with coefficients in K.

Discriminant form equation

$$D(\alpha_1 x_1 + \cdots + \alpha_m x_m) = b \text{ in } x_1, \ldots, x_m \in A \qquad (DF_2)$$

where $1, \alpha_1, \ldots, \alpha_m \in \overline{K}$, linearly independent over K;
$D(\alpha_1 X_1 + \cdots + \alpha_m X_m)$ discriminant form with coefficients in K.

Ineffective finiteness results on equations (DF_1), (DF_2) and (NF)

over number fields:

Schmidt (1971): (NF), $A = \mathbb{Z}$, finiteness criterion, description of the set of solutions

Schlickewei (1977): (NF), $A = \mathbb{Z}_S$, finiteness result

method of proof: Subspace theorem

over finitely generated domains A:

Gy (1982): (DF_1), (DF_2), (NF) finiteness, under certain restrictions on (DF_1), (NF)

method of proof: reduction to unit equations, Lang's theorem

Laurent (1984): (NF), finiteness

Evertse–Gy (1988): (DF_1), (NF), finiteness criteria

Gy (1993): (DF_1), description of the structure of the set of solutions

method of proof: reduction to multivariate unit equations

Effective finiteness results

over number fields:

Gy (1976, 1981): (DF_2), $A = \mathbb{Z}, \mathcal{O}_K, \mathcal{O}_S$

Gy–Papp (1978), Gy (1981): (DF_1), (NF), $A = \mathbb{Z}, \mathcal{O}_K, \mathcal{O}_S$, under certain restrictions on F

method: Baker's method

over a restricted class of finitely generated domains

Gy (1983): (DF_1), (DF_2), (NF), under certain restrictions on (DF_1), (NF)

method of proof: effective specialization method

Effective finiteness results over arbitrary finitely generated domains

$A = \mathbb{Z}[z_1, \ldots, z_r]$, K quotient field, $F \in K[X_1, \ldots, X_m]$ decomposable form, i.e. factorizes into linear forms, say $\ell_1, \ldots, \ell_n$ over $\overline{K}$

Decomposable form equation

$$F(x_1, \ldots, x_m) = b \quad \text{in } x_1, \ldots, x_m \in A, \qquad (DF_1)$$

where $b \in K^*$

Let $\mathcal{L}_F = \{\ell_1, \ldots, \ell_n\}$, suppose $\mathcal{L}_F$ has at least 3 pairwise linearly independent linear forms. Further, to simplify the presentation, we assume that rank $\mathcal{L}_F = m$.

Definition (Győry and Papp, 1978)

$\mathcal{G}(\mathcal{L}_F)$ *graph* with *vertex* system $\mathcal{L}_F$ in which ℓ_i, ℓ_j $(i \neq j)$ connected by an *edge* if ℓ_i, ℓ_j linearly dependent or linearly independent and $\lambda_i\ell_i + \lambda_j\ell_j + \lambda_q\ell_q = 0$ for some $q \not\subset \{i, j\}$ with $\lambda_i, \lambda_j, \lambda_q \in L \setminus \{0\}$

$A \cong \mathbb{Z}[X_1, \ldots, X_r]/\mathcal{I}$, $\mathcal{I} = \{f \in \mathbb{Z}[X_1, \ldots, X_r] \,:\, f(z_1, \ldots, z_r) = 0\}$, $\mathcal{I} = (f_1, \ldots, f_t)$

To state quantitative result we generalize the size of elements $\alpha \in K$ to the case $\alpha \in \overline{K}$.

Definition

For $\alpha \in \overline{K}$, let $\deg_K \alpha$ the degree of α over K. A tuple or representatives for α : $(g_0, \ldots, g_n)$, where $g_0, \ldots, g_n \in \mathbb{Z}[X_1, \ldots, X_r]$, $g_0 \notin \mathcal{I}$ and

$$X^n + \frac{g_1(z_1, \ldots, z_r)}{g_0(z_1, \ldots, z_r)} X^{n-1} + \cdots + \frac{g_n(z_1, \ldots, z_r)}{g_0(z_1, \ldots, z_r)}$$

monic minimal polynomial of α over K. We say that $\deg(g_0, \ldots, g_n) \leq d$, logarithmic height $h(g_0, \ldots, g_n) \leq h$ if $\deg g_i \leq d$, $h(g_i) \leq h$ for $i = 0, \ldots, n$.

Definition

Given $\boldsymbol{x} = (x_1, \ldots, x_m) \in A^m$, a representative for $\boldsymbol{x}$ is a tuple $\tilde{\boldsymbol{x}} = (\tilde{x}_1, \ldots, \tilde{x}_m)$ with $\tilde{x}_i \in \mathbb{Z}[X_1, \ldots, X_r]$, $x_i = \tilde{x}_i(z_1, \ldots, z_r)$ for $i = 1, \ldots, m$. The size of $\tilde{\boldsymbol{x}}$ is defined by

$$s(\tilde{\boldsymbol{x}}) := \max_i s(\tilde{x}_i) = \max_i \max(1, \deg \tilde{x}_i, h(\tilde{x}_i))$$

$$F(\boldsymbol{x}) = \ell_1(\boldsymbol{x}) \dots \ell_n(\boldsymbol{x}) \quad \text{in} \quad \boldsymbol{x} \in A^m \qquad (DF_1)$$

Theorem I (Evertse–Gy, 202?)

Suppose the following:

- $\mathcal{G}(\mathcal{L}_F)$ *is connected*;
- *the generators* $f_1, \dots, f_t$ *of* $\mathcal{I}$ *have degree* $\leq d$ *and logarithmic height* $\leq h$;
- b *and the coefficients of* $\ell_1, \dots, \ell_n$ *have tuples of representatives of degree* $\leq d$ *and logarithmic height* $\leq h$;
- *the coefficients of* $\ell_1, \dots, \ell_n$ *have degree* $\leq D$ *over* K.

Then every solution $\boldsymbol{x}$ *of* (DF_1) *is represented by* $\tilde{\boldsymbol{x}} \in \mathbb{Z}[X_1, \dots, X_r]^m$ *such that*

$$s(\tilde{\boldsymbol{x}}) \leq \exp((2mn \cdot D^{Dmn} d)^{\exp O(r)} h).$$

Theorem I has many consequences and applications.

$A \cong \mathbb{Z}[X_1, \ldots, X_r]/I$ where $I = \{f \in \mathbb{Z}[X_1, \ldots, X_r] : f(z_1, \ldots, z_r) = 0\}$ finitely generated ideal, $I = (f_1, \ldots, f_t)$; A *effectively given* if $f_1, \ldots, f_t$ effectively given

Definition

A finite extension L of K *effectively given* if it is given in the form $K[X]/(P)$, P effectively given monic, irreducible in $K[X]$; $L = K(\Theta)$, $\Theta := X \pmod{P} \Rightarrow$ any $\beta \in L, \beta = \sum_{i=0}^{d-1} a_i \Theta^i$ with $a_0, \ldots, a_{d-1} \in K$, $d = [L : K]$; $\beta \in L$ *given / can be determined effectively* if $a_0, \ldots, a_{d-1}$ are given / can be determined effectively.

$$F(\boldsymbol{x}) = \ell_1(\boldsymbol{x}) \dots \ell_n(\boldsymbol{x}) \quad \text{in} \quad \boldsymbol{x} \in A^m \qquad (DF_1)$$

Theorem I $\Rightarrow$

Theorem J (Evertse–Gy, 202?)

If $\mathcal{G}(\mathcal{L}_F)$ is connected, then equation (DF_1) has only finitely many solutions. Moreover, if the coefficients of $\ell_1, \dots, \ell_n$ belong to a finite extension L of K and if A, K, L, b and the coefficients of $\ell_1, \dots, \ell_n$ are given effectively, then all solutions can be effectively determined.

method of proof of Theorems I and J: following Gy–Papp (1978) over number fields, use the connectedness of $\mathcal{G}(\mathcal{L}_F)$, reduce (DF_1) to a finite system of unit equations over a finitely generated overring A' of A in L, apply the effective Theorems B resp. B' (Evertse–Gy, 2013) on unit equations, and utilize so-called 'degree-height estimates'.

Theorems I and J $\Rightarrow$

- For m = 2 $\Rightarrow$ **Theorem D** (Bérczes, Evertse, Gy, 2014) on Thue equations
- For m > 2, more general version (Evertse–Gy, 202?): $\mathcal{G}(\mathcal{L}_F)$ not necessarily connected, rank $\mathcal{L}_F \leq m$
- The first assertion of **Theorem J**: Gy (1982)
- The second assertion of **Theorem I** for a restricted class of A: Gy (1983)

alternative proof: Evertse–Gy (2013) method, i.e. reduction to the number field and function field case, effective specializations, use of effective results over number fields and function fields; see also Gy (1983)

Consequences of Theorem I:

Norm form equation

$$N(\alpha_1 x_1 + \cdots + \alpha_m x_m) = b \quad \text{in } x_1, \ldots, x_m \in A \qquad \text{(NF)}$$

more general version of **Therem J** $\Rightarrow$

Theorem K (Evertse–Gy, 202?)

Suppose that in (NF) α_m *is of degree* ≥ 3 *over* $K(\alpha_1, \ldots, \alpha_{m-1})$*. Then equation* (NF) *has only finitely many solutions with* $x_m \neq 0$*. Further, if* $A, K, L, \alpha_1, \ldots, \alpha_m$ *and* b *are effectively given, all solutions of* (NF) *with* $x_m \neq 0$ *can be effectively determined.* + *quantitative version*

- The first assertion of **Theorem K**: Gy (1982)
- The second assertion of **Theorem K** for a restricted class of A:
  Gy (1983)
- ≥ 3 and $x_m \neq 0$ *necessary*

Discriminant form equation

$$D(\alpha_1 x_1 + \cdots + \alpha_m x_m) = b \quad \text{in } x_1, \ldots, x_m \in A \qquad (DF_2)$$

Theorem J $\Rightarrow$

Theorem L (Evertse–Gy, 20?)

Under the above assumptions concerning (DF_2), equation (DF_2) has only finitely many solutions. Moreover, if $A, K, L, \alpha_1, \ldots, \alpha_m$ and b are effectively given, all solutions of (DF_2) can be effectively determined. + *quantitative version*

- The first assertion of **Theorem L**: Gy (1982)
- The second assertion of **Theorem L** for a restricted class of A: Gy (1983)

Applications to *index form equations* and *integral elements of given discriminant*

More general versions of equations (DF_1), (NF), (DF_2)

$$F(x_1, \ldots, x_m) \in bA^* \quad \text{in} \quad x_1, \ldots, x_m \in A \qquad (DF_1^*)$$

$\Rightarrow$ e.g. simple ring extensions of A

Many other **applications** of **Theorems B–L** and their more general versions

Our general effective method $\Rightarrow$ general program

Given a polynomial equation

$$P(\boldsymbol{x}) = 0 \quad \text{in } \boldsymbol{x} \in A^m \qquad (*)$$

If we have effective finiteness results for the ($S-$) integral solutions of the corresponding equation over number fields and effective results over function fields of char 0, our method gives an effective finiteness result for equation $(*)$

+quantitative versions

Thank you for your attention!