GEODESIC VECTORS AND FLAT TOTALLY GEODESIC SUBALGEBRAS OF SIX-DIMENSIONAL FILIFORM METRIC LIE ALGEBRAS

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Abstract. A nilpotent Lie algebra $\mathfrak{n}$ equipped with an Euclidean inner product is called nilpotent metric Lie algebra. In this paper we describe the sets of the geodesic vectors and the flat totally geodesic subalgebras of the six-dimensional filiform metric Lie algebras. In this class with the exception of the metric Lie algebras corresponding to the standard filiform Lie algebra the flat totally geodesic subalgebras of every metric Lie algebra has dimension at most two.

1. Introduction

Let $\mathfrak{n}$ be a real nilpotent Lie algebra and N be the connected and simply connected Lie group having Lie algebra $\mathfrak{n}$. We call $(\mathfrak{n}, \langle .,. \rangle)$ a metric nilpotent Lie algebra if it is given an Euclidean inner product $\langle .,. \rangle$ on $\mathfrak{n}$. An inner product $\langle .,. \rangle$ on $\mathfrak{n}$ determines a left-invariant metric $\langle .,. \rangle_N$ on N and conversely. The corresponding nilpotent Lie group N endowed with the left-invariant metric $\langle .,. \rangle_N$ arising from $\langle .,. \rangle$ is a Riemannian nilmanifold.

Riemannian nilmanifolds have been studied extensively in the last decades. The first general studies for 2-step nilmanifolds were done by P. Eberlein (see [4], [5]).

We call a subalgebra $\mathfrak{h}$ of a metric Lie algebra $(\mathfrak{n}, \langle .,. \rangle)$ flat, respectively totally geodesic if its exponential image H in the Lie group N with the left invariant Riemann metric $\langle .,. \rangle_N$ is flat, respectively totally geodesic submanifold. A subalgebra $\mathfrak{h}$ of a metric Lie algebra $(\mathfrak{n}, \langle .,. \rangle)$ is totally geodesic if and only if it satisfies

$$\langle [X,Y], Z \rangle + \langle [X,Z], Y \rangle = 0, \tag{1.1}$$

for all $Y, Z \in \mathfrak{h}$ and X in the orthogonal complement $\mathfrak{h}^{\perp}$ of $\mathfrak{h}$ (cf. Lemma 1.2 in ([1])). Moreover, a non-zero vector $Y \in (\mathfrak{n}, \langle .,. \rangle)$ is geodesic precisely if for all $X \in (\mathfrak{n}, \langle .,. \rangle)$ one has

$$\langle [X,Y], Y \rangle = 0. \tag{1.2}$$

Cairns, Hinić Galić and Nikolayevsky presented a comprehensive study of totally geodesic subgroups of Riemannian nilmanifolds and the corresponding subalgebras of nilpotent metric Lie algebras. In particular the authors gave several results on the possible dimensions of totally geodesic subalgebras. They also found examples, where the obtained bounds on the dimensions of totally geodesic subalgebras are attained. Furthermore they gave an example of a 6-dimensional filiform nilpotent Lie algebra that has no totally geodesic subalgebra of dimension > 2, for any choice of inner product (see [1], [2]).

Nagy and Homolya in [8] studied geodesic vectors and flat totally geodesic subalgebras in 2-step nilpotent metric Lie algebras and proved that for isomorphic Lie algebras $\mathfrak{n}$ and $\mathfrak{n}^*$ there

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exists a bijective linear map $\mathfrak{n} \to \mathfrak{n}^*$ preserving the geodesic, respectively flat totally geodesic property of vectors, respectively subalgebras. Moreover they determined the sets of the geodesic vectors and the flat totally geodesic subalgebras in the 2-step nilpotent metric Lie algebras of dimension ≤ 6.

In [6] Figula and Nagy classified the isometry equivalence classes and determined the isometry groups of connected and simply connected Riemannian nilmanifolds on filiform Lie groups of arbitrary dimension and on five dimensional nilpotent Lie groups of nilpotency class > 2. Applying their approach Abbas and Figula proved that up to isometric isomorphism any six-dimensional filiform metric Lie algebra is of the form $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$, $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$, $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$, $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$ and $\mathfrak{n}_{6,18}(\alpha_i, \beta_j)$. These metric Lie algebras are defined by the following non-vanishing commutators (see [7]):

(1) $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$

$$
\begin{aligned}
&[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, && [E_1, E_3] = \alpha_2 E_4 - \frac{\alpha_5 \beta_1 + \alpha_2 \beta_7}{\alpha_4} E_5 + \beta_4 E_6, \\
&[E_1, E_4] = \frac{\alpha_1 \alpha_5}{\alpha_4} E_5 + \beta_5 E_6, && [E_2, E_3] = \alpha_3 E_5 + \beta_6 E_6, \\
&[E_2, E_4] = \beta_7 E_6, && [E_2, E_5] = \alpha_4 E_6, \\
&[E_4, E_3] = \alpha_5 E_6,
\end{aligned}
\tag{1.3}
$$

such that $\alpha_i > 0, i = 1, \ldots, 5$, $\beta_j \in \mathbb{R}, j = 1, \ldots, 7$, and if the set $J = \{j \in \{1, 4, 7\} : \beta_j \neq 0\} \neq \emptyset$, then $\beta_{j_\circ} > 0$ for the minimal element $j_\circ \in J$,

(2) $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$

$$
\begin{aligned}
&[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, && [E_1, E_3] = \alpha_2 E_4 + \beta_4 E_5 + \beta_5 E_6, \\
&[E_1, E_4] = \alpha_3 E_5 + \beta_6 E_6, && [E_2, E_3] = \frac{\alpha_2 \alpha_5}{\alpha_4} E_5 + \beta_7 E_6, \\
&[E_1, E_5] = \alpha_4 E_6, && [E_2, E_4] = \alpha_5 E_6,
\end{aligned}
\tag{1.4}
$$

such that $\alpha_i > 0, i = 1, \ldots, 5$, $\beta_j \in \mathbb{R}, j = 1, \ldots, 7$, and if the set $J = \{j \in \{1, 3, 4, 6, 7\} : \beta_j \neq 0\} \neq \emptyset$, then $\beta_{j_\circ} > 0$ for the minimal element $j_\circ \in J$,

(3) $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$

$$
\begin{aligned}
&[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, && [E_1, E_3] = \alpha_2 E_4 - \left(\frac{\alpha_3 \beta_1}{\alpha_1} + \frac{\alpha_2 \beta_8}{\alpha_4} \right) E_5 + \beta_4 E_6, \\
&[E_1, E_4] = \alpha_3 E_5 + \beta_5 E_6, && [E_1, E_5] = \beta_6 E_6, \\
&[E_2, E_3] = \beta_7 E_6, && [E_2, E_4] = \beta_8 E_6, \\
&[E_2, E_5] = \alpha_4 E_6, && [E_4, E_3] = \frac{\alpha_3 \alpha_4}{\alpha_1} E_6,
\end{aligned}
\tag{1.5}
$$

such that $\alpha_i > 0, i = 1, \ldots, 4$, $\beta_j \in \mathbb{R}, j = 1, \ldots, 8$, and one of the following cases is satisfied:

(a) $\beta_1 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_8 = 0$,
(b) $\beta_3 > 0$ or $\beta_5 > 0$, $\beta_1 = \beta_4 = \beta_6 = \beta_8 = 0$,
(c) $\beta_6 > 0$ or $\beta_4 > 0$, $\beta_1 = \beta_3 = \beta_5 = \beta_8 = 0$,
(d) $\beta_1 > 0$ or $\beta_8 > 0$, $\beta_3 = \beta_4 = \beta_5 = \beta_6 = 0$,
(e) at least two elements of the set $\{\beta_1, \beta_3, \beta_4, \beta_5, \beta_6, \beta_8\}$ are positive with the exceptions $(\beta_1 > 0, \beta_8 > 0)$, $(\beta_3 > 0, \beta_5 > 0)$, $(\beta_4 > 0, \beta_6 > 0)$,

(4) $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$

$$\begin{aligned} &[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, && [E_1, E_3] = \alpha_2 E_4 + \beta_4 E_5 + \beta_5 E_6, \\ &[E_1, E_4] = \alpha_3 E_5 + \beta_6 E_6, && [E_1, E_5] = \alpha_4 E_6, \\ &[E_2, E_3] = \alpha_5 E_6, \end{aligned} \tag{1.6}$$

such that $\alpha_i > 0, i = 1, \ldots, 5$, $\beta_j \in \mathbb{R}, j = 1, \ldots, 6$ and if the set $J = \{j \in \{1, 3, 4, 6\} : \beta_j \neq 0\} \neq \emptyset$, then $\beta_{j_\circ} > 0$ for the minimal element $j_\circ \in J$. Moreover, the Lie algebra $\mathfrak{n}_{6,18}(\alpha_i, \beta_j)$ is defined by the same commutators with $\alpha_5 = 0$.

Our aim in this work is to determine the sets of the geodesic vectors and the flat totally geodesic subalgebras in the class $\mathcal{C}$ of the six-dimensional filiform metric Lie algebras. Our investigation shows that in $\mathcal{C}$ only metric Lie algebras corresponding to the standard filiform Lie algebra allow a flat totally geodesic subalgebra of dimension > 2 (cf. Theorems 3.2, 3.4, 3.8, 4.2, 4.3).

2. Preliminaries

For any Lie algebra $\mathfrak{n}$, the lower central series of $\mathfrak{n}$ is defined by $\mathfrak{n}^{(0)} := \mathfrak{n}, \mathfrak{n}^{(1)} = [\mathfrak{n}, \mathfrak{n}]$ and $\mathfrak{n}^{(i+1)} = [\mathfrak{n}, \mathfrak{n}^{(i)}]$, $i \geq 1$. If $\mathfrak{n}^{(k)}$ is trivial for some integer k, then the Lie algebra $\mathfrak{n}$ is called nilpotent. Let k be the smallest integer so that $\mathfrak{n}^{(k)}$ is trivial, then $\mathfrak{n}$ is said to be k-step nilpotent. An n-dimensional nilpotent Lie algebra is called filiform if it is $n-1$-step nilpotent. We say an n-dimensional nilpotent Lie algebra $\mathfrak{n}$ is $\mathbb{N}$-graded filiform, if it can be decomposed in a direct sum of one-dimensional subspaces $\mathfrak{n} = \oplus_{i=1}^n V_i$ with $[V_1, V_i] = V_{i+1}$ for all $i > 1$ and $[V_i, V_j] \subset V_{i+j}$ for all $i, j \in \mathbb{N}$, where for convenience we set $V_i = 0$ for $i > n$.

Let $\mathbb{E}_6$ be a 6-dimensional Euclidean vector space with a distinguished orthonormal basis $\{E_1, E_2, E_3, E_4, E_5, E_6\}$. Let $(\mathfrak{n}, \langle ., . \rangle)$ be a filiform metric Lie algebra of dimension 6 defined on $\mathbb{E}_6$ by the commutators and relations described in cases (1.3), (1.4), (1.5), (1.6). The lower central series of the 6-dimensional filiform Lie algebra $\mathfrak{n}$ is given by $\mathfrak{n}^{(1)} = \mathrm{span}(E_3, E_4, E_5, E_6)$, $\mathfrak{n}^{(2)} = \mathrm{span}(E_4, E_5, E_6)$, $\mathfrak{n}^{(3)} = \mathrm{span}(E_5, E_6)$, $\mathfrak{n}^{(4)} = \mathrm{span}(E_6)$ which is the center ζ of $\mathfrak{n}$, and $\mathfrak{n}^{(5)} = \{0\}$. Denote by $\mathfrak{a}_i$ the orthogonal complement of $\mathfrak{n}^{(i)}$ in $\mathfrak{n}^{(i-1)}$, $i = 1, 2, 3, 4$. Hence we have $\mathfrak{a}_1 = \mathrm{span}(E_1, E_2)$, $\mathfrak{a}_2 = \mathrm{span}(E_3)$, $\mathfrak{a}_3 = \mathrm{span}(E_4)$, $\mathfrak{a}_4 = \mathrm{span}(E_5)$, and $\mathfrak{n} = \oplus_{i=1}^4 \mathfrak{a}_i \oplus \zeta$.

According to [8, Lemma 1, 2], we have the following:

Lemma 2.1. *A subalgebra $\mathfrak{h}$ in a nilpotent metric Lie algebra $(\mathfrak{n}, \langle ., . \rangle)$ is flat if and only if it is abelian. A subalgebra $\mathfrak{h}$ in $(\mathfrak{n}, \langle ., . \rangle)$ is flat totally geodesic precisely if each non-zero element of $\mathfrak{h}$ is geodesic.*

Lemma 2.2. *Let $(\mathfrak{n}, \langle ., . \rangle)$ be a 6-dimensional filiform metric Lie algebra. Each non-zero vector in $\cup_{i=1}^4 \mathfrak{a}_i \cup \zeta$ is geodesic. Any subalgebra contained in $\cup_{i=1}^4 \mathfrak{a}_i \cup \zeta$ is flat totally geodesic.*

Proof. If $Y \in \zeta$, then for every $X \in \mathfrak{n}$ we have $[X, Y] = 0$ and hence $\langle [X, Y], Y \rangle = 0$. Let $Y \in \mathfrak{a}_i$, $i = 1, 2, 3, 4$. Since $[X, Y]$ lies in $\mathfrak{n}^{(i)}$ for any $X \in \mathfrak{n}$ we obtain $\langle [X, Y], Y \rangle = 0$ which proves the first assertion. According to Lemma 2.1 the second claim follows from the first assertion. $\square$

We often use the following Proposition (see [1, Proposition 1.13, Theorems 1.17, 1.18]):

Proposition 2.3. *a) Filiform nilpotent metric Lie algebras do not possess any totally geodesic subalgebra of codimension one.*

b) A filiform nilpotent metric Lie algebra $(\mathfrak{n}, \langle ., . \rangle)$ possesses a totally geodesic subalgebra of codimension two if and only if $\mathfrak{n}$ is isomorphic to the standard filiform Lie algebra of dimension $n \geq 3$.

3. Geodesic vectors and flat totally geodesic subalgebras of metric Lie algebras $\mathfrak{n}_{6,14}(\alpha_i,\beta_j)$, $\mathfrak{n}_{6,15}(\alpha_i,\beta_j)$ and $\mathfrak{n}_{6,16}(\alpha_i,\beta_j)$

In this section we consider the metric Lie algebras $(\mathfrak{n}_{6,14}(\alpha_i,\beta_j),\langle .,.\rangle)$, $(\mathfrak{n}_{6,15}(\alpha_i,\beta_j),\langle .,.\rangle)$, and $(\mathfrak{n}_{6,16}(\alpha_i,\beta_j),\langle .,.\rangle)$. Our goal is to describe the sets of geodesic vectors and to compute the flat totally geodesic subalgebras.

Let us start with the metric Lie algebra $(\mathfrak{n}_{6,14}(\alpha_i,\beta_j),\langle .,.\rangle)$.

Theorem 3.1. *Let $(\mathfrak{n}_{6,14}(\alpha_i,\beta_j),\langle .,.\rangle)$ be the metric Lie algebra defined on $\mathbb{E}^6$ by non-vanishing commutators given by (1.3). The geodesic vectors of $(\mathfrak{n}_{6,14}(\alpha_i,\beta_j),\langle .,.\rangle)$ not belonging to $\mathfrak{a}_1\cup\mathfrak{a}_2\cup\mathfrak{a}_3\cup\mathfrak{a}_4\cup\zeta$ are the non-zero elements of the set $C_1\cup C_2\cup C_4$ in the case $\alpha_5\beta_1+\alpha_2\beta_7=0=\beta_4$, for $\alpha_5\beta_1+\alpha_2\beta_7\neq 0$ these are the non-zero elements of the set $C_1\cup C_3\cup C_4$, where*

$C_1 := \{bE_2+cE_3+dE_4 : b(\alpha_1 c+\beta_1 d)+c\alpha_2 d=0\}$, *at least two of the numbers b,c,d are non-zero with exception of the cases:*

1. $b=0$, 2. $d=0$, 3. $c=0$ with $\beta_1\neq 0$,

$$C_2 :=\Big\{a(E_1-\frac{\alpha_2}{\alpha_5}E_6)+cE_3+dE_4+eE_5 : a\neq 0,\ \alpha_5\beta_1+\alpha_2\beta_7=0=\beta_4,$$
$$e=(\beta_5 a-\alpha_5 c)\frac{\alpha_2\alpha_4}{\alpha_1\alpha_5^2},\ a(\alpha_1 c+\beta_1 d+\beta_2 e+\beta_3 f)-c(\alpha_3 e+\beta_6 f)-f(\beta_7 d+\alpha_4 e)=0\Big\},$$

$$C_3 :=\Big\{a\left(E_1-\frac{\alpha_2}{\alpha_5}E_6+(\frac{\beta_5}{\alpha_5}+\frac{\alpha_1\beta_4}{\alpha_5\beta_1+\alpha_2\beta_7})E_3-\frac{\alpha_2\alpha_4\beta_4}{\alpha_5(\alpha_5\beta_1+\alpha_2\beta_7)}E_5\right)+dE_4 : a\neq 0,$$
$$a(\alpha_1 c+\beta_1 d+\beta_2 e+\beta_3 f)-c(\alpha_3 e+\beta_6 f)-f(\beta_7 d+\alpha_4 e)=0\Big\},$$

$$C_4 :=\Big\{aE_1+cE_3+dE_4+eE_5+fE_6 : a\neq 0,\ f\neq -a\frac{\alpha_2}{\alpha_5},\ f\neq 0,\ e=(c\alpha_5-a\beta_5)\frac{f\alpha_4}{a\alpha_1\alpha_5},$$
$$d=\frac{a}{\alpha_5 f+\alpha_2 a}(\frac{\alpha_5\beta_1+\alpha_2\beta_7}{\alpha_4}e-\beta_4 f),\ a(\alpha_1 c+\beta_1 d+\beta_2 e+\beta_3 f)-c(\alpha_3 e+\beta_6 f)-$$
$$f(\beta_7 d+\alpha_4 e)=0\Big\}.$$

Proof. According to (1.2) a non-zero vector $Y=aE_1+bE_2+cE_3+dE_4+eE_5+fE_6\in\mathfrak{n}_{6,14}(\alpha_i,\beta_j)$ is geodesic if and only if one has $\langle [X,Y],Y\rangle=0$ for all $X=\sum_{i=1}^5 x_iE_i\in\mathfrak{n}_{6,14}(\alpha_i,\beta_j)$ or equivalently if and only if the system of equations

$$\begin{cases} b\alpha_4 f=0,\\ a(\frac{\alpha_1\alpha_5}{\alpha_4}e+\beta_5 f)-c\alpha_5 f=0,\\ a(\alpha_2 d-\frac{\alpha_5\beta_1+\alpha_2\beta_7}{\alpha_4}e+\beta_4 f)+b\alpha_3 e+d\alpha_5 f=0,\\ a(\alpha_1 c+\beta_1 d+\beta_2 e+\beta_3 f)-c(\alpha_3 e+\beta_6 f)-f(\beta_7 d+\alpha_4 e)=0,\\ b(\alpha_1 c+\beta_1 d+\beta_2 e)+c(\alpha_2 d-\frac{\alpha_5\beta_1+\alpha_2\beta_7}{\alpha_4}e+\beta_4 f)+d(\frac{\alpha_1\alpha_5}{\alpha_4}e+\beta_5 f)=0\end{cases}\tag{3.1}$$

is satisfied. Taking into account that $\alpha_i\neq 0$, $i=\{1,\dots,5\}$ and assume that $f=0$ the system of equations (3.1) reduces to

$$\begin{cases} ae=0,\\ a\alpha_2 d+b\alpha_3 e=0,\\ a(\alpha_1 c+\beta_1 d)-c\alpha_3 e=0,\\ b(\alpha_1 c+\beta_1 d+\beta_2 e)+c(\alpha_2 d-\frac{\alpha_5\beta_1+\alpha_2\beta_7}{\alpha_4}e)+d\frac{\alpha_1\alpha_5}{\alpha_4}e=0.\end{cases}\tag{3.2}$$

If $a=0$, then from (3.2) we obtain

$$\begin{cases} be = 0 = ce, \\ b\,(\alpha_1 c + \beta_1 d) + c\alpha_2 d + d\frac{\alpha_1\alpha_5}{\alpha_4} e = 0. \end{cases}$$

In the case $e=0=a=f$ the geodesic vectors of $(\mathfrak{n}_{6,14}(\alpha_i,\beta_j),\langle .,.\rangle)$ are the non-zero elements of the set $\overline{C_1} := \{bE_2 + cE_3 + dE_4 : b\,(\alpha_1 c + \beta_1 d) + c\alpha_2 d = 0\}$. For $b=0$ the element $Y = cE_3 + dE_4 \in \overline{C_1}$, $c,d \in \mathbb{R}$, satisfies the condition $\alpha_2 cd = 0$. Since $\alpha_2 \neq 0$ we obtain that either $c=0$ and the element $Y = dE_4$ is in $\mathfrak{a}_3$, or $d=0$ and the element $Y = cE_3$ is in $\mathfrak{a}_2$. If $d=0$, then for the element $Y = bE_2 + cE_3 \in \overline{C_1}$, $b,c \in \mathbb{R}$, the condition $\alpha_1 bc = 0$ holds. Since $\alpha_1 \neq 0$ we receive that either $c=0$ and the element $Y = bE_2$ is in $\mathfrak{a}_1$, or $b=0$ and the element $Y = cE_3$ is in $\mathfrak{a}_2$. If $c=0$ and $\beta_1 \neq 0$, then for the element $Y = bE_2 + dE_4 \in \overline{C_1}$, $b,d \in \mathbb{R}$ the condition $bd=0$ is satisfied. Hence we get either $b=0$ and the element $Y = dE_4$ is in $\mathfrak{a}_3$, or $d=0$ and the element $Y = bE_2$ is in $\mathfrak{a}_1$. Therefore the conditions for the set C_1 are proved.

In the case $b=c=0=a=f$ the vector $Y = dE_4 + eE_5$ is geodesic if and only if either $d=0$ or $e=0$, and hence it lies either in $\mathfrak{a}_4$ or $\mathfrak{a}_3$.

If $e=0$, then from (3.2) we get

$$\begin{cases} ad = 0 = ac, \\ b\,(\alpha_1 c + \beta_1 d) + c\alpha_2 d = 0. \end{cases}$$

The case $a=0=e=f$ is discussed above. In the case $d=c=0=e=f$ we receive that any non-zero vector $Y = aE_1 + bE_2$ is geodesic since it lies in the set $\mathfrak{a}_1$.

Now, we suppose that $b=0$. In this case the system of equations (3.1) reduces to the following

$$\begin{cases} a\frac{\alpha_1\alpha_5}{\alpha_4} e = f(c\alpha_5 - a\beta_5), \\ d\alpha_5 f = -a(\alpha_2 d - \frac{\alpha_5\beta_1+\alpha_2\beta_7}{\alpha_4} e + \beta_4 f), \\ a\,(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) - c\,(\alpha_3 e + \beta_6 f) - f(\beta_7 d + e\alpha_4) = 0, \\ c(\alpha_2 d - \frac{\alpha_5\beta_1+\alpha_2\beta_7}{\alpha_4} e + \beta_4 f) + d(\frac{\alpha_1\alpha_5}{\alpha_4} e + \beta_5 f) = 0. \end{cases} \tag{3.3}$$

Assume that $a=0$. The first and second equations of (3.3) give $cf = 0 = df$. In the case $f=0=a$ we obtain from the third equation of (3.3) that $ce=0$. For $c=0=f=a=b$ the vector $Y = dE_4 + eE_5$ is geodesic if and only if it lies in $\mathfrak{a}_4$ or in $\mathfrak{a}_3$. For $e=0=f=a$ the vector $Y = cE_3 + dE_4$ is geodesic if and only if it is either in $\mathfrak{a}_3$ or in $\mathfrak{a}_2$.

In the case $c=d=0=a$ the third equation of (3.3) gives $fe=0$. In this case the geodesic vector Y has either the form $eE_5 \in \mathfrak{a}_4$ or the form $fE_6 \in \zeta$.

If $a \neq 0$, then the system of equations (3.3) is equivalent to the following system:

$$\begin{cases} \frac{\alpha_1\alpha_5}{\alpha_4} e = \frac{f}{a}(c\alpha_5 - a\beta_5), \\ d(\alpha_5 f + \alpha_2 a) = a(\frac{\alpha_5\beta_1+\alpha_2\beta_7}{\alpha_4} e - \beta_4 f), \\ a(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) - c\,(\alpha_3 e + \beta_6 f) - f(\beta_7 d + e\alpha_4) = 0. \end{cases} \tag{3.4}$$

Let $f = -a\frac{\alpha_2}{\alpha_5}$. Putting this expression into the first and second equations of (3.4) we receive

$$e = (a\beta_5 - c\alpha_5)\frac{\alpha_2\alpha_4}{\alpha_1\alpha_5^2}, \tag{3.5}$$

$$\frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4} e + \frac{\alpha_2}{\alpha_5} a\beta_4 = 0. \tag{3.6}$$

The substitution of (3.5) into (3.6) yields that

$$c(\alpha_5\beta_1+\alpha_2\beta_7)=\frac{a}{\alpha_5}\Big((\alpha_5\beta_1+\alpha_2\beta_7)\beta_5+\alpha_1\alpha_5\beta_4\Big). \tag{3.7}$$

If $\alpha_5\beta_1+\alpha_2\beta_7=0$, then the equation (3.7) gives $\beta_4=0$. In this case the geodesic vectors are the non-zero elements of the set C_2.

If $\alpha_5\beta_1+\alpha_2\beta_7\neq 0$, then from equation (3.7) we obtain:

$$c=a\Big(\frac{\beta_5}{\alpha_5}+\frac{\alpha_1\beta_4}{\alpha_5\beta_1+\alpha_2\beta_7}\Big). \tag{3.8}$$

Putting (3.8) into (3.5) we receive

$$e=-\frac{\alpha_2\alpha_4\beta_4}{\alpha_5(\alpha_5\beta_1+\alpha_2\beta_7)}a.$$

Hence the vector $Y=aE_1+cE_3+dE_4+eE_5+fE_6$ is a geodesic vector if and only if it lies in the set C_3.

If $f\neq -a\frac{\alpha_2}{\alpha_5}$, then from the first and second equations of (3.4) we have

$$e=\Big(c\alpha_5-a\beta_5\Big)\frac{f\alpha_4}{a\alpha_1\alpha_5},\quad d=\frac{a}{\alpha_5 f+\alpha_2 a}\Big(\frac{\alpha_5\beta_1+\alpha_2\beta_7}{\alpha_4}e-\beta_4 f\Big).$$

In this case the geodesic vectors of $(\mathfrak{n}_{6,14}(\alpha_i,\beta_j),\langle .,.\rangle)$ are the non-zero vectors of the set C_4. For $f=0$ the elements in C_4 has the shape $Y=aE_1+cE_3$, $a\neq 0$, and they satisfy the condition $\alpha_1 ac=0$. Therefore we have $c=0$ and hence the element $Y=aE_1$ lies in $\mathfrak{a}_1$. The intersection $C_1\cap C_2\cap C_4$ is empty because for the elements of C_1 one has $a=0$ but for the elements in $C_2\cup C_4$ we have $a\neq 0$. Moreover, for the elements in C_2 one gets $f=-a\frac{\alpha_2}{\alpha_5}$, in contrast to this for the elements in C_4 we have $f\neq -a\frac{\alpha_2}{\alpha_5}$. Hence the sets C_2 and C_4 are disjoint. Similarly we can see that the sets C_1, C_3 and C_4 are disjoint. This completes the proof. □

Now we determine the flat totally geodesic subalgebras of dimension >1 in the metric Lie algebra $(\mathfrak{n}_{6,14}(\alpha_i,\beta_j),\langle .,.\rangle)$.

Theorem 3.2. *Let $(\mathfrak{n}_{6,14}(\alpha_i,\beta_j),\langle .,.\rangle)$ be the metric Lie algebra defined on $\mathbb{E}^6$ by non-vanishing commutators given by (1.3). The flat totally geodesic subalgebras of dimension >1 in the metric Lie algebra $(\mathfrak{n}_{6,14}(\alpha_i,\beta_j),\langle .,.\rangle)$ are the 2-dimensional subalgebras:*

(1) $\mathfrak{h}_2=span(E_1,E_6)$ in the case $\beta_3=\beta_4=\beta_5=0$,
(2) $\mathfrak{h}_3=span(E_2,E_4)$ in the case $\beta_1=\beta_7=0$.

Proof. In view of Proposition 2.3 any flat totally geodesic subalgebra of $(\mathfrak{n}_{6,14}(\alpha_i,\beta_j),\langle .,.\rangle)$ has dimension at most 3. Hence firstly we compute the 2- and 3-dimensional abelian subalgebras in the Lie algebra $\mathfrak{n}_{6,14}(\alpha_i,\beta_j)$.

The 2- and 3- dimensional abelian subalgebras of the Lie algebra $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$ are:

$$\mathfrak{h}_1 = \text{span}\left(E_1 + k_2E_3 + k_3E_4 + k_4E_6, \quad E_5 + l_1E_6\right),$$
$$\mathfrak{h}_2 = \text{span}\left(E_1 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_5, \quad E_6\right),$$
$$\mathfrak{h}_3 = \text{span}\left(E_2 + k_1E_3 + k_2E_5 + k_3E_6, \quad E_4 + l_1E_5 + l_2E_6\right) \text{ with } \beta_7 + l_1\alpha_4 - k_1\alpha_5 = 0,$$
$$\mathfrak{h}_4 = \text{span}\left(E_2 + k_1E_3 + k_2E_4 + k_3E_5, \quad E_6\right),$$
$$\mathfrak{h}_5 = \text{span}\left(E_3 + k_1E_4 + k_2E_6, \quad E_5 + l_1E_6\right),$$
$$\mathfrak{h}_6 = \text{span}\left(E_3 + k_1E_4 + k_2E_5, \quad E_6\right),$$
$$\mathfrak{h}_7 = \text{span}\left(E_4 + k_1E_6, \quad E_5 + l_1E_6\right),$$
$$\mathfrak{h}_8 = \text{span}\left(E_4 + l_1E_5, \quad E_6\right),$$
$$\mathfrak{h}_9 = \text{span}\left(E_5, \quad E_6\right),$$
$$\mathfrak{h}_{10} = \text{span}\left(E_1 + k_2E_3 + k_3E_4, \quad E_5, \quad E_6\right),$$
$$\mathfrak{h}_{11} = \text{span}\left(E_3 + k_1E_4, \quad E_5, \quad E_6\right),$$
$$\mathfrak{h}_{12} = \text{span}\left(E_4, \quad E_5, \quad E_6\right),$$
$$\mathfrak{h}_{13} = \text{span}\left(E_2 + k_1E_3 + k_2E_5, \quad E_4 + l_1E_5, \quad E_6\right) \text{ with } \beta_7 + l_1\alpha_4 - k_1\alpha_5 = 0,$$

where $k_1, k_2, k_3, k_4, l_1, l_2 \in \mathbb{R}$.

The subalgebras $\mathfrak{h}_9$, $\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$, and $\mathfrak{h}_{12}$ are not flat totally geodesic because for the vector $E_5 + E_6$ the fourth equation of (3.1) gives the contradiction $\alpha_4 = 0$. Hence is the vector $E_5 + E_6 \in \mathfrak{h}_9 \cap \mathfrak{h}_{10} \cap \mathfrak{h}_{11} \cap \mathfrak{h}_{12}$ is not geodesic. Therefore the subalgebras $\mathfrak{h}_9$, $\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$, $\mathfrak{h}_{12}$ are excluded. The subalgebras $\mathfrak{h}_8$ and $\mathfrak{h}_{13}$ are not flat totally geodesic since for the vector $E_4 + l_1E_5 + E_6 \in \mathfrak{h}_8 \cap \mathfrak{h}_{13}$ the third equation of the system (3.1) yields $\alpha_5 = 0$ which is a contradiction. Therefore the vector $E_4 + l_1E_5 + E_6$ is not geodesic and hence by Lemma 2.1 the subalgebras $\mathfrak{h}_8$, $\mathfrak{h}_{13}$ are excluded, too The non-zer vector $E_2 + k_1E_3 + k_2E_4 + k_3E_5 + E_6 \in \mathfrak{h}_4$ is geodesic if and only if for $a = 0, b = f = 1$, $c = k_1$, $d = k_2$, $e = k_3$ the system (3.1) of equations holds. From the first equation of (3.1) we get the contradiction $\alpha_4 = 0$. Hence the vector $E_2 + k_1E_3 + k_2E_4 + k_3E_5 + E_6 \in \mathfrak{h}_4$ is not geodesic. Thus the subalgebra $\mathfrak{h}_4$ is not flat totally geodesic. Therefore the case $\mathfrak{h}_4$ is excluded. The element $E_3 + k_1E_4 + k_2E_5 + E_6 \in \mathfrak{h}_6$ is geodesic if and only if for $a = b = 0, c = f = 1$, $d = k_1$, $e = k_2$, the system (3.1) of equations is satisfied. From the second equation of (3.1) yields the contradiction $\alpha_5 = 0$. Therefore the subalgebras $\mathfrak{h}_6$ is not flat totally geodesic and hence it is excluded.

The non-zer vector $E_5 + l_1E_6 \in \mathfrak{h}_1 \cap \mathfrak{h}_5 \cap \mathfrak{h}_7$ is geodesic if and only if for $a = b = c = d = 0, e = 1, f = l_1$, the system (3.1) of equations holds. The fourth equation of (3.1) gives $\alpha_4 l_1 = 0$. As $\alpha_4 \neq 0$ we receive that $l_1 = 0$.

We treat the subalgebra $\mathfrak{h}_1$. The non-zero vector $E_1 + k_2E_3 + k_3E_4 + k_4E_6 \in \mathfrak{h}_1$ is geodesic if and only if for $b = e = 0, a = 1, c = k_2, d = k_3, f = k_4$ the system (3.1) of equations holds. From the second equation of (3.1) we receive

$$k_4(\beta_5 - \alpha_5 k_2) = 0. \tag{3.9}$$

Furthermore, The element $E_1 + k_2E_3 + k_3E_4 + k_4E_6 + E_5 \in \mathfrak{h}_1$ is geodesic if and only if for $b = 0, a = e = 1, c = k_2, d = k_3, f = k_4$ the system (3.1) of equations is satisfied. The second equation of (3.1) gives

$$\frac{\alpha_1\alpha_5}{\alpha_4} + k_4(\beta_5 - \alpha_5 k_2) = 0. \tag{3.10}$$

Taking into account (3.9), equation (3.10) yields the contradiction $\frac{\alpha_1\alpha_5}{\alpha_4} = 0$. Hence the subalgebra $\mathfrak{h}_1$ is not flat totally geodesic and therefore it is excluded.

Next we consider the subalgebra $\mathfrak{h}_5$. According to the system (3.1) of equations the non-zero vector $E_3 + k_1E_4 + k_2E_6 \in \mathfrak{h}_5$ is geodesic if and only if for $a = b = e = 0, c = 1, d = k_1, f = k_2$, the system (3.1) of equations holds. From the second equation of (3.1) one obtains $\alpha_5 k_2 = 0$. As $\alpha_5 \neq 0$ we have $k_2 = f = 0$. Using this the fifth equation of gives $\alpha_2 k_1$. Since $\alpha_2 \neq 0$ we receive $k_1 = 0$. But the element vector $E_3 + E_5 \in \mathfrak{h}_5$ is not geodesic because the fourth equation of the system (3.1) gives the contradiction $\alpha_3 = 0$. Therefore the subalgebra $\mathfrak{h}_5$ is excluded, too.

Here we deal with subalgebra $\mathfrak{h}_7$. The element $E_4 + k_1E_6 \in \mathfrak{h}_7$ is geodesic if and only if for $a = b = c = e = 0, d = 1, f = k_1$, the system (3.1) of equations is satisfied. From the third equation of (3.1) on obtains $\alpha_5 k_1 = 0$. Since $\alpha_5 \neq 0$ we get $k_1 = 0$. But the vector $E_4 + E_5 \in \mathfrak{h}_7$ is not geodesic because from the fifth equation of the system (3.1) we receive the contradiction $\frac{\alpha_1\alpha_5}{\alpha_4} = 0$. Hence the subalgebra $\mathfrak{h}_7$ is excluded.

Now we treat the subalgebra $\mathfrak{h}_2$. The non-zero vector $E_1 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_5 \in \mathfrak{h}_2$ is geodesic if and only if for $f = 0, a = 1, b = k_1, c = k_2, d = k_3, e = k_4$ the system (3.1) of equations is satisfied. The second equation of (3.1) gives $\alpha_1\alpha_5 k_4 = 0$. As $\alpha_1\alpha_5 \neq 0$ we get $k_4 = e = 0$. Using this from the third equation of (3.1) we receive $\alpha_2 k_3 = 0$. Since $\alpha_2 \neq 0$ we have $k_3 = d = 0$. Applying this the fourth equation of (3.1) yields $\alpha_1 k_2 = 0$. As $\alpha_1 \neq 0$ we receive $k_2 = c = 0$. The element $E_1 + k_1E_2 + E_6 \in \mathfrak{h}_2$ is geodesic if and only if for $e = c = d = 0, a = f = 1, b = k_1$ the system (3.1) of equations holds. From the first equation of (3.1) one has $\alpha_4 k_1 = 0$. Since $\alpha_4 \neq 0$ implies that $k_1 = 0$. Additionally, The non-zero vector $E_1 + E_6 \in \mathfrak{h}_2$ is geodesic if and only if for $b = c = d = e = 0, a = f = 1$ the system (3.1) of equations is satisfied. From the second, third, and fourth equation we get $\beta_5 = 0$, $\beta_4 = 0$, and $\beta_3 = 0$. Hence the assertion (1) follows.

Finally we consider the subalgebra $\mathfrak{h}_3$. The non-zero vector $E_2 + k_1E_3 + k_2E_5 + k_3E_6 \in \mathfrak{h}_3$ is geodesic if and only if for $a = d = 0, b = 1, c = k_1, e = k_2, f = k_3$ the system (3.1) of equations holds. The first equation of (3.1) gives $\alpha_4 k_3 = 0$. As $\alpha_4 \neq 0$ one has $k_3 = f = 0$. Using this from the third equation of (3.1) we obtain $\alpha_3 k_2 = 0$. Since $\alpha_3 \neq 0$ we receive $k_2 = e = 0$. Applying this the fifth equation of (3.1) one gets $\alpha_1 k_1 = 0$. Since $\alpha_1 \neq 0$ yields that $k_1 = c = 0$. The element $E_4 + l_1E_5 + l_2E_6 \in \mathfrak{h}_3$ is geodesic if and only if for $a = b = c = 0, d = 1, e = l_1, l = l_2$ the system (3.1) of equations is satisfied. From the third equation of (3.1) we obtain $\alpha_5 l_2 = 0$ which implies that $l_2 = f = 0$. Using this the fifth equation of (3.1) gives $\alpha_1\alpha_5 l_1 = 0$. This yields $l_1 = 0$. In addition the non-zero vector $E_2 + E_4 \in \mathfrak{h}_3$ is geodesic if and only if for $a = b = c = f = 0, b = d = 1$ the system (3.1) of equations holds. The fifth equation of (3.1) gives $\beta_1 = 0$. Taking into account the abelian condition when $l_1 = k_1 = 0$ we receive that $\beta_7 = 0$. Therefore the case (2) is proved. This proves Theorem 3.2. □

Next we deal with the $\mathbb{N}$-graded filiform metric Lie algebra $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ defined by non-vanishing commutators given by (1.4). Firstly we give the set of its geodesic vectors.

Theorem 3.3. *Let $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ be the metric Lie algebra defined on $\mathbb{E}^6$ by non-vanishing commutators given by (1.4). The geodesic vectors of $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ not belonging to $\mathfrak{a}_1 \cup \mathfrak{a}_2 \cup$*

$\mathfrak{a}_3 \cup \mathfrak{a}_4 \cup \zeta$ *are the non-zero elements of the set* $C_1 \cup C_2$, *where*

$$C_1 := \Big\{cE_3 + dE_4 + eE_5 + fE_6 : f \neq 0,\ c \neq 0,\ d = \frac{-c}{\alpha_5 f}\Big(\frac{\alpha_2\alpha_5}{\alpha_4}e + \beta_7 f\Big),$$
$$\frac{-c}{\alpha_5 f}\Big(\frac{\alpha_2\alpha_5}{\alpha_4}e + \beta_7 f\Big)(\alpha_3 e + \beta_6 f + c\alpha_2) + c(\beta_4 e + \beta_5 f) + e\alpha_4 f = 0\Big\},$$

$C_2 := \{bE_2 + cE_3 + dE_4 : b(\alpha_1 c + \beta_1 d) + c\alpha_2 d = 0\}$ *at least two of the numbers* b, c, d *are non-zero with exception of the cases:*

1. $b = 0$, 2. $d = 0$, 3. *for* $\beta_1 \neq 0,\ c = 0$.

Proof. Applying the commutators (1.4) and (1.2) we obtain that the non-zero element $Y = aE_1 + bE_2 + cE_3 + dE_4 + eE_5 + fE_6 \in \mathfrak{n}_{6,15}(\alpha_i, \beta_j)$ is geodesic if and only if for the real numbers a, b, c, d, e, f with respect to a basis $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ the system of equations

$$\text{(3.11)} \qquad \begin{cases} af = 0, \\ a\alpha_3 e + b\alpha_5 f = 0, \\ a(\alpha_2 d + \beta_4 e) + b(\frac{\alpha_2\alpha_5}{\alpha_4}e + \beta_7 f) = 0, \\ a(\alpha_1 c + \beta_1 d + \beta_2 e) - c(\frac{\alpha_2\alpha_5}{\alpha_4}e + \beta_7 f) - d\alpha_5 f = 0, \\ b(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) + c(\alpha_2 d + \beta_4 e + \beta_5 f) + d(\alpha_3 e + \beta_6 f) + e\alpha_4 f = 0 \end{cases}$$

is satisfied. If $a = 0$, then the second and the third equations of (3.11) gives $bf = 0 = be$. In the case $b = 0 = a$ the system (3.11) reduces to:

$$\text{(3.12)} \qquad \begin{cases} c(\frac{\alpha_2\alpha_5}{\alpha_4}e + \beta_7 f) + d\alpha_5 f = 0, \\ c(\alpha_2 d + \beta_4 e + \beta_5 f) + d(\alpha_3 e + \beta_6 f) + e\alpha_4 f = 0. \end{cases}$$

Suppose that $f = 0$. From the first equation of (3.12) we receive either $c = 0$ or $e = 0$. In the case $c = 0$ the vector $Y = dE_4 + eE_5$ is geodesic of $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle .,. \rangle)$ if and only if it lies either in $\mathfrak{a}_4$ or in $\mathfrak{a}_3$, whereas if $e = 0$ the vector $Y = cE_3 + dE_4$ is geodesic of $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle .,. \rangle)$ precisely if it lies either in $\mathfrak{a}_2$ or in $\mathfrak{a}_3$.

Assume that $f \neq 0$. From the first equation of (3.12) we obtain

$$\text{(3.13)} \qquad d = \frac{-c}{\alpha_5 f}\Big(\frac{\alpha_2\alpha_5}{\alpha_4}e + \beta_7 f\Big).$$

Substituting (3.13) into the second equation of (3.12) we recieve that

$$\frac{-c}{\alpha_5 f}\Big(\frac{\alpha_2\alpha_5}{\alpha_4}e + \beta_7 f\Big)(c\alpha_2 + \alpha_3 e + \beta_6 f) + c(\beta_4 e + \beta_5 f) + e\alpha_4 f = 0.$$

In this case the vector $Y = cE_3 + dE_4 + eE_5 + fE_6,\ f \neq 0$ is geodesic of $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle .,. \rangle)$ if and only if it lies in the set

$$\overline{C_1} := \Big\{cE_3 + dE_4 + eE_5 + fE_6 : f \neq 0,\ d = \frac{-c}{\alpha_5 f}\Big(\frac{\alpha_2\alpha_5}{\alpha_4}e + \beta_7 f\Big),$$
$$\frac{-c}{\alpha_5 f}\Big(\frac{\alpha_2\alpha_5}{\alpha_4}e + \beta_7 f\Big)(\alpha_3 e + \beta_6 f + c\alpha_2) + c(\beta_4 e + \beta_5 f) + e\alpha_4 f = 0\Big\}.$$

If $c = 0$, then any vector $Y = eE_5 + fE_6 \in \overline{C_1}$ lies either in $\mathfrak{a}_4$ or in ζ. Hence we get the set C_1 in the assertion.

In the case $f = e = 0 = a$, the vector $Y = bE_2 + cE_3 + dE_4$ is geodesic of $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle .,. \rangle)$ if and only if it lies in the set C_2. The set C_2 coincides with the set C_1 in the filiform metric Lie algebra $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle .,. \rangle)$. Therefore the validity of the condition for the numbers b, c, d in the case of the set C_2 in the assertion can be proved in the same manner as in the proof of Theorem 3.1 in the case C_1.

Now, if $f = 0$, the system (3.11) of equations reduces to

$$\begin{cases} ae = 0, \\ ad + \frac{\alpha_5}{\alpha_4} be = 0, \\ a(\alpha_1 c + \beta_1 d) - \frac{\alpha_2 \alpha_5}{\alpha_4} ce = 0, \\ b(\alpha_1 c + \beta_1 d + \beta_2 e) + c(\alpha_2 d + \beta_4 e) + d\alpha_3 e = 0. \end{cases} \tag{3.14}$$

We discussed the case $a = 0$ above. In the case $e = 0$, from the second and the third equation of (3.14) one has $ad = 0 = ac$. The case $f = e = a = 0$ is discussed above. In the case $d = c = 0 = e$ any vector $Y = aE_1 + bE_2$ is geodesic because it lies in $\mathfrak{a}_1$. The intersection $C_1 \cap C_2$ is empty since for the elements of C_2 one has $f = 0$ but for the elements in C_1 we have $f \neq 0$. This proves Theorem 3.3. □

According to [2, Theorem 5.17], the metric Lie algebra $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ is isomorphic to the $\mathbb{N}$-graded filiform Lie algebra $(\mathfrak{m}_2(6), \langle ., . \rangle)$.

Theorem 3.4. *Let $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ be the metric Lie algebra defined on $\mathbb{E}^6$ by non-vanishing commutators given by (1.4). Then the unique flat totally geodesic subalgebra of dimension > 1 in $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ is $\mathfrak{h}_6 = span(E_3, E_6)$ in the case $\beta_5 = \beta_7 = 0$.*

Proof. In view of Proposition 2.3 and [2, Theorem 4.2], the dimension of flat totally geodesic subalgebras of $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ is at most 3. Hence firstly we determine the 2- and 3-dimensional abelian subalgebras in the Lie algebra $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$.

The 3- and 2-dimensional abelian subalgebras of $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$ are:

$$\begin{aligned}
\mathfrak{h}_1 &= \mathrm{span}\,(E_2 + k_1 E_3 + k_2 E_4, \quad E_5, \quad E_6)\,, \\
\mathfrak{h}_2 &= \mathrm{span}\,(E_3 + k_1 E_4, \quad E_5, \quad E_6)\,, \\
\mathfrak{h}_3 &= \mathrm{span}\,(E_4, \quad E_5, \quad E_6)\,, \\
\mathfrak{h}_4 &= \mathrm{span}\,(E_3 + k_1 E_6, \quad E_4 + l_2 E_6, \quad E_5 + s_1 E_6)\,, \\
\mathfrak{h}_5 &= \mathrm{span}\,(E_3 + k_1 E_5, \quad E_4 + l_1 E_5, \quad E_6)\,, \\
\mathfrak{h}_6 &= \mathrm{span}\,(E_3 + k_1 E_4 + k_2 E_5, \quad E_6)\,, \\
\mathfrak{h}_7 &= \mathrm{span}\,(E_1 + k_1 E_2 + k_2 E_3 + k_3 E_4 + k_4 E_5, \quad E_6)\,, \\
\mathfrak{h}_8 &= \mathrm{span}\,(E_2 + k_1 E_3 + k_2 E_4 + k_3 E_6, \quad E_5 + s_1 E_6)\,, \\
\mathfrak{h}_9 &= \mathrm{span}\,(E_2 + k_1 E_3 + k_2 E_4 + k_3 E_5, \quad E_6)\,, \\
\mathfrak{h}_{10} &= \mathrm{span}\,(E_3 + k_1 E_5 + k_2 E_6, \quad E_4 + l_1 E_5 + l_2 E_6)\,, \\
\mathfrak{h}_{11} &= \mathrm{span}\,(E_3 + k_1 E_4 + k_2 E_6, \quad E_5 + s_1 E_6)\,, \\
\mathfrak{h}_{12} &= \mathrm{span}\,(E_4 + l_2 E_6, \quad E_5 + s_1 E_6)\,, \\
\mathfrak{h}_{13} &= \mathrm{span}\,(E_4 + l_1 E_5, \quad E_6)\,, \\
\mathfrak{h}_{14} &= \mathrm{span}\,(E_5, E_6)\,,
\end{aligned}$$

where $k_1, k_2, k_3, k_4, l_1, l_2, s_1 \in \mathbb{R}$.

The subalgebras $\mathfrak{h}_1$, $\mathfrak{h}_2$, $\mathfrak{h}_3$, and $\mathfrak{h}_{14}$ are not flat totally geodesic because for the vector $E_5 + E_6$ the fifth equation of (3.11) gives the contradiction $\alpha_4 = 0$. Hence is the vector $E_5 + E_6 \in \mathfrak{h}_1 \cap \mathfrak{h}_2 \cap \mathfrak{h}_3 \cap \mathfrak{h}_{14}$ is not geodesic. Therefore the subalgebras $\mathfrak{h}_1$, $\mathfrak{h}_2$, $\mathfrak{h}_3$, $\mathfrak{h}_{14}$ are excluded. By the system (3.11) of equations the vector $E_4+l_2E_6 \in \mathfrak{h}_4\cap\mathfrak{h}_{12}$, respectively $E_5+s_1E_6 \in \mathfrak{h}_4\cap\mathfrak{h}_8\cap\mathfrak{h}_{11}\cap\mathfrak{h}_{12}$, respectively $E_4+l_1E_5 \in \mathfrak{h}_5\cap\mathfrak{h}_{13}$ is totally geodesic precisely if and only if one has $\alpha_5l_2 = 0 = \beta_6l_2$, respectively $\alpha_4s_1 = 0$, respectively $\alpha_3l_1 = 0$. Since $\alpha_5\alpha_4\alpha_3 \neq 0$ we receive that $l_2 = l_1 = s_1 = 0$. Using this for the vector $E_4+E_5 \in \mathfrak{h}_4\cap\mathfrak{h}_{12}$ the fifty equation of (3.11) leads to the contradiction $\alpha_3 = 0$, whereas for the vector $E_4+E_6 \in \mathfrak{h}_5\cap\mathfrak{h}_{13}$ the fourth equation (3.11) gives the contradiction $\alpha_5 = 0$. Therefore the vectors $E_4+E_5 \in \mathfrak{h}_4\cap\mathfrak{h}_{12}$, $E_4+E_6 \in \mathfrak{h}_5\cap\mathfrak{h}_{13}$ are not geodesic. Hence the subalgebras $\mathfrak{h}_4$, $\mathfrak{h}_5$, $\mathfrak{h}_{12}$, $\mathfrak{h}_{13}$ are excluded, too.

The non-zero vector $E_1+k_1E_2+k_2E_3+k_3E_4+k_4E_5+E_6 \in \mathfrak{h}_7$ is not geodesic bacause the first equation of (3.11) leads to the contradiction $1 = 0$. Therefore the subalgebra $\mathfrak{h}_7$ is not flat totally geodesic and hence it is excluded.

The element $E_2+k_1E_3+k_2E_4+k_3E_6+E_5 \in \mathfrak{h}_8$ is geodesic if and only if for $b = e = 1$, $c = k_1$, $d = k_2$, $f = k_3$ the system (3.11) of equations is satisfied. From the second equation of (3.11) we get $\alpha_5k_3 = 0$. As $\alpha_5 \neq 0$ we have $k_3 = 0$. Using this the third equation of (3.11) gives the contradiction $\frac{\alpha_2\alpha_5}{\alpha_4} = 0$. Hence the vector $E_1+k_1E_3+k_2E_4+k_3E_6+E_5 \in \mathfrak{h}_8$ is not geodesic. Therefore the subalgebra $\mathfrak{h}_8$ is excluded.

The non-zero vector $E_2+k_1E_3+k_2E_4+k_3E_5+E_6 \in \mathfrak{h}_9$ is geodesic if and only if for $b = f = 1$, $c = k_1$, $d = k_2$, $e = k_3$ the system (3.11) of equations holds. From the second equation of (3.11) we get the contradiction $\alpha_5 = 0$. Hence the vector $E_2+k_1E_3+k_2E_4+k_3E_5+E_6 \in \mathfrak{h}_8$ is not geodesic. So the subalgebra $\mathfrak{h}_9$ is excluded.

The element $E_3+k_1E_4+k_2E_6 \in \mathfrak{h}_{11}$ is geodesic if and only if for $a = b = e = 0$, $c = 1$, $d = k_1$, $f = k_2$ the system (3.11) of equations is satisfied. The fourth equation gives

$$\beta_7k_2+\alpha_5k_1k_2 = 0. \tag{3.15}$$

In addition, The non-zero vector $E_3+k_1E_4+k_2E_6+E_5 \in \mathfrak{h}_{11}$ is geodesic if and only if for $c = e = 1$, $d = k_1$, $f = k_2$ the system (3.11) of equations holds. The fourth equation of (3.11) yields

$$\frac{\alpha_2\alpha_5}{\alpha_4}+\beta_7k_2+\alpha_5k_1k_2 = 0. \tag{3.16}$$

Comparing (3.15) with (3.16) we obtain the contradiction $\frac{\alpha_2\alpha_5}{\alpha_4} = 0$. Hence the vector $E_3+k_1E_4+k_2E_6+E_5 \in \mathfrak{h}_{11}$ is not geodesic and the subalgebra $\mathfrak{h}_{11}$ is excluded.

The vector $E_4+l_1E_5+l_2E_6 \in \mathfrak{h}_{10}$ is geodesic if for $a = b = c = 0$, $d = 1$, $e = l_1$, $f = l_2$ the system (3.11) of equations holds. The fourth and the fifth equations of (3.11) yield $\alpha_5l_2 = 0$, $\alpha_3l_1+\beta_6l_2+\alpha_4l_1l_2 = 0$. As $\alpha_5\alpha_3 \neq 0$ we obtain that $l_1 = l_2 = 0$. The element $E_3+k_1E_5+k_2E_6 \in \mathfrak{h}_{10}$ is geodesic if for $a = b = d = 0$, $c = 1$, $e = l_1$, $f = l_2$ the system (3.11) of equations is satisfied. From the fourth and fifth equations of (3.11) we get

$$\frac{\alpha_2\alpha_5}{\alpha_4}k_1+\beta_7k_2 = 0,\ \beta_4k_1+\beta_5k_2+\alpha_4k_1k_2 = 0. \tag{3.17}$$

The element $E_3+E_4+k_1E_5+k_2E_6 \in \mathfrak{h}_{10}$ is geodesic if for $a = b = 0$, $c = d = 1$, $e = l_1$, $f = l_2$ the system (3.11) of equations is valid. From the fourth of (3.11) one has

$$\frac{\alpha_2\alpha_5}{\alpha_4}k_1+\beta_7k_2+\alpha_5k_2 = 0. \tag{3.18}$$

Comparing it with the first equation of (3.17) we obtain $\alpha_5k_2 = 0$. Since $\alpha_5 \neq 0$ one gets $k_2 = 0$. Using this from (3.18) we receive that also $k_2 = 0$. But the vector $E_3+E_4 \in \mathfrak{h}_{10}$ is not geodesic since for $a = b = e = f = 0$, $c = d = 1$ the fifth equation of (3.11) gives the contradiction $\alpha_2 = 0$. Therefore the subalgebra $\mathfrak{h}_{10}$ is excluded.

Finally we treat th subalgebra $\mathfrak{h}_6$. The element $E_3 + k_1 E_4 + k_2 E_5 \in \mathfrak{h}_6$ is geodesic if and only if the system (3.11) of equations are satisfied for $a = b = f = 0$, $c = 1$, $d = k_1$, $e = k_2$. From the fourth and fifth equations of (3.11) we obtain $\frac{\alpha_2\alpha_5}{\alpha_4} k_2 = 0$ $\alpha_2 k_1 + \beta_4 k_2 + k_1 \alpha_3 k_2 = 0$. As $\alpha_2\alpha_5 \neq 0$ we get $k_2 = k_1 = 0$. Using this the vector $E_3 + E_6$ lies in $\in \mathfrak{h}_6$, which is geodesic if for $a = b = d = e = 0$, $c = f = 1$ the system (3.11) is valid. From the fourth and fifth equations of (3.11) we receive $\beta_7 = \beta_5 = 0$. Therefore the subalgebra $\mathfrak{h}_6 = \mathrm{span}(E_3, E_6)$ is flat totally geodesic in the case $\beta_5 = \beta_7 = 0$. Hence Theorem 3.4 is proved. □

Now we treat the filiform Lie algebra $\ell_{6,16}$ defined by the non-vanishing Lie brackets

$$[G_1, G_2] = G_3,\ [G_1, G_3] = G_4,\ [G_1, G_4] = G_5,\ [G_2, G_5] = G_6,\ [G_4, G_3] = G_6.$$

The following theorem describes the isometric isomorphism classes of the metric Lie algebras $(\ell_{6,16}, \langle .,. \rangle)$ and the group of orthogonal automorphisms of $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle)$.

Theorem 3.5. *Let $\langle .,. \rangle$ be an inner product on the 6-dimensional filiform Lie algebra $\ell_{6,16}$.*

(1) There is a unique metric Lie algebra $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle)$ which is isometrically isomorphic to the metric Lie algebra $(\ell_{6,16}, \langle .,. \rangle)$ with $\alpha_i > 0, i = 1, \dots, 4$ and one of the following cases is satisfied:
- *(i) $\beta_1 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_8$,*
- *(ii) $\beta_3 > 0$ or $\beta_5 > 0$, $\beta_1 = \beta_4 = \beta_6 = \beta_8 = 0$,*
- *(iii) $\beta_6 > 0$ or $\beta_4 > 0$, $\beta_1 = \beta_3 = \beta_5 = \beta_8 = 0$,*
- *(iv) $\beta_1 > 0$ or $\beta_8 > 0$, $\beta_3 = \beta_4 = \beta_5 = \beta_6 = 0$,*
- *(v) at least two of the elements of the set $\{\beta_1, \beta_3, \beta_4, \beta_5, \beta_6, \beta_8\}$ are positive with the exception of the cases $(\beta_1 > 0, \beta_8 > 0)$, $(\beta_3 > 0, \beta_5 > 0)$, $(\beta_4 > 0, \beta_6 > 0)$.*

(2) The group $\mathcal{OA}(\mathfrak{n}_{6,16}(\alpha_i, \beta_j))$ of orthogonal automorphisms of the metric Lie algebra $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle)$ is the following group:
- *(a) in case (1i) one has $\mathcal{OA}(\mathfrak{n}_{6,16}(\alpha_i, \beta_j)) = \{TE_i = \varepsilon_1 E_i, i = 1, 6, TE_i = \varepsilon_2 E_i, i = 2, 4, TE_i = \varepsilon_1\varepsilon_2 E_i, i = 3, 5,\ \varepsilon_1, \varepsilon_2 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$,*
- *(b) in case (1ii) one has $\mathcal{OA}(\mathfrak{n}_{6,16}(\alpha_i, \beta_j)) = \{TE_2 = E_2, TE_4 = E_4, TE_i = \varepsilon E_i, i = 1, 3, 5, 6,\ \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,*
- *(c) in case (1iii) one has $\mathcal{OA}(\mathfrak{n}_{6,16}(\alpha_i, \beta_j)) = \{TE_3 = E_3, TE_5 = E_5, TE_i = \varepsilon E_i, i = 1, 2, 4, 6,\ \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,*
- *(d) in case (1iv) one has $\mathcal{OA}(\mathfrak{n}_{6,16}(\alpha_i, \beta_j)) = \{TE_1 = E_1, TE_6 = E_6, TE_i = \varepsilon E_i, i = 2, 3, 4, 5,\ \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,*
- *(e) in case (1v) the group $\mathcal{OA}(\mathfrak{n}_{6,16}(\alpha_i, \beta_j))$ is trivial.*

Remark 3.6. *The proof of Theorem 3.5 can be found in [7, Theorem 3.8, p. 8.] with the exception of the case (1ii) and its group of orthogonal automorphism given by (2b) which are missing from there.*

Firstly we determine the geodesic vectors of the filiform metric Lie algebra $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle)$.

Theorem 3.7. *Let $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle)$ be the metric Lie algebra defined on $\mathbb{E}^6$ by non-vanishing commutators given by (1.5). The geodesic vectors of $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle)$ not belonging to $\mathfrak{a}_1 \cup \mathfrak{a}_2 \cup$*

$\mathfrak{a}_3 \cup \mathfrak{a}_4 \cup \zeta$ are the non-zero elements of the set $C_1 \cup C_2$, where

$$C_1 := \Big\{ bE_2 + cE_3 + dE_4 + eE_5 : b(\alpha_1 c + \beta_1 d + \beta_2 e) + c\Big(\alpha_2 d - (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})e\Big) + d\alpha_3 e = 0 \Big\},$$

at least two of the numbers b, c, d, e are non-zero with exception of the cases:

1. $b = c = 0$, 2. $b = e = 0$, 3. $d = e = 0$,
4. $b = d = 0$ *with* $\alpha_3\alpha_4\beta_1 + \alpha_1\alpha_2\beta_8 \neq 0$, 5. $c = d = 0$ *with* $\beta_2 \neq 0$,
6. $c = e = 0$ *with* $\beta_1 \neq 0$,

$$C_2 := \Big\{ a\Big(E_1 - \frac{\beta_6}{\alpha_4}E_2\Big) + cE_3 + dE_4 + eE_5 + fE_6 : af \neq 0,\ ae = \frac{f}{\alpha_3}\Big(\frac{a\beta_6\beta_8}{\alpha_4} + c\frac{\alpha_3\alpha_4}{\alpha_1} - a\beta_5\Big),$$

$$ac = \frac{f}{\alpha_1}(c\beta_7 + d\beta_8 + e\alpha_4) - \frac{a}{\alpha_1}(\beta_1 d + \beta_2 e + \beta_3 f),$$

$$ad = \frac{a}{\alpha_2}\Big((\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})e - \beta_4 f\Big) + \frac{f}{\alpha_2}\Big(\frac{a\beta_6\beta_7}{\alpha_4} - d\frac{\alpha_3\alpha_4}{\alpha_1}\Big),$$

$$d(\alpha_2 c + \alpha_3 e) = a\frac{\beta_6}{\alpha_4}(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) + ce\Big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\Big) - f(c\beta_4 + d\beta_5 + e\beta_6)\Big\}.$$

Proof. According to (1.2) the vector $Y = aE_1 + bE_2 + cE_3 + dE_4 + eE_5 + fE_6$ is geodesic of $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle)$ if and only if the following system of equations

(3.19)
$$\begin{cases} f(a\beta_6 + b\alpha_4) = 0, \\ a(\alpha_3 e + \beta_5 f) + f(b\beta_8 - c\frac{\alpha_3\alpha_4}{\alpha_1}) = 0, \\ a\Big(\alpha_2 d - (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})e + \beta_4 f\Big) + f(b\beta_7 + d\frac{\alpha_3\alpha_4}{\alpha_1}) = 0, \\ a(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) - f(c\beta_7 + d\beta_8 + e\alpha_4) = 0, \\ b(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) + c\Big(\alpha_2 d - (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})e + \beta_4 f\Big) + d(\alpha_3 e + \beta_5 f) + e\beta_6 f = 0 \end{cases}$$

is satisfied for $a, b, c, d, e, f \in \mathbb{R}$.

Firstly, we suppose that $f = 0$. Take into account that $\alpha_i > 0$, $i = 1, \dots, 4$, the system (3.19) of equations gives $ae = 0 = ad = ac$.

For $a = 0 = f$ the vector $Y = bE_2 + cE_3 + dE_4 + eE_5$ is geodesic of $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), .\langle .,. \rangle)$ if and only if it lies in set $\overline{C_1} = \{bE_2 + cE_3 + dE_4 + eE_5 : b(\alpha_1 c + \beta_1 d + \beta_2 e) + c(\alpha_2 d - (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})e) + d\alpha_3 e = 0\}$. If $b = c = 0$, then for the element $Y = dE_4 + eE_5 \in \overline{C_1}$, $d, e \in \mathbb{R}$ one has $d\alpha_3 e = 0$. As $\alpha_3 \neq 0$ we obtain that Y lies either in $\mathfrak{a}_4$ or in $\mathfrak{a}_3$. If $b = e = 0$, then for the element $Y = cE_3 + dE_4 \in \overline{C_1}$, $c, d \in \mathbb{R}$ we have $c\alpha_2 d = 0$. As $\alpha_2 \neq 0$ we obtain that Y lies either in $\mathfrak{a}_3$ or in $\mathfrak{a}_2$. In the case $d = e = 0$ for the element $Y = bE_2 + cE_3 \in \overline{C_1}$, $b, c \in \mathbb{R}$ one gets $b\alpha_1 c = 0$. Since $\alpha_1 \neq 0$ we receive that Y lies either in $\mathfrak{a}_3$ or in $\mathfrak{a}_2$. In the case $b = c = 0$ for the element $Y = cE_3 + eE_5 \in \overline{C_1}$, $c, e \in \mathbb{R}$ one obtains $ce(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}) = 0$. If $\alpha_3\alpha_4\beta_1 + \alpha_1\alpha_2\beta_8 \neq 0$, then we have $ce = 0$, or equivalently the element Y is either in $\mathfrak{a}_3$ or in $\mathfrak{a}_2$. In the case $c = d = 0$ for the element $Y = bE_2 + eE_5 \in \overline{C_1}$, $b, e \in \mathbb{R}$ we receive $b\beta_2 e = 0$. If $\beta_2 \neq 0$, then the element Y lies either in $\mathfrak{a}_4$ or in $\mathfrak{a}_1$. In the case $c = e = 0$ for the element $Y = bE_2 + dE_4 \in \overline{C_1}$, $b, d \in \mathbb{R}$ we receive $b\beta_1 d = 0$. If $\beta_1 \neq 0$, then the element Y lies either in $\mathfrak{a}_3$ or in $\mathfrak{a}_1$. This proves the conditions for the set C_1 in the theorem. I

For $e = d = c = 0 = f$ any vector $Y = aE_1 + bE_2$ is geodesic of $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), .\langle .,. \rangle)$ because it lies in $\mathfrak{a}_1$.

Secondly, assume that $a\beta_6 + b\alpha_4 = 0$. Hence we receive $b = -\frac{a\beta_6}{\alpha_4}$. Putting this expression into (3.19) one obtains that

(3.20)
$$\begin{cases} ae = \frac{f}{\alpha_3}(\frac{a\beta_6\beta_8}{\alpha_4} + c\frac{\alpha_3\alpha_4}{\alpha_1} - a\beta_5), \\ ad = \frac{a}{\alpha_2}\Big((\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})e - \beta_4 f\Big) + \frac{f}{\alpha_2}(\frac{a\beta_6\beta_7}{\alpha_4} - d\frac{\alpha_3\alpha_4}{\alpha_1}), \\ ac = \frac{f}{\alpha_1}\left(c\beta_7 + d\beta_8 + e\alpha_4\right) - \frac{a}{\alpha_1}\left(\beta_1 d + \beta_2 e + \beta_3 f\right), \\ a\frac{\beta_6}{\alpha_4}\left(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f\right) + ce(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}) - f(c\beta_4 + d\beta_5 + e\beta_6) = d(\alpha_2 c + \alpha_3 e). \end{cases}$$

Suppose that $a = 0$. Hence one has $b = 0$. From system (3.20) of equations we receive $fc = 0 = fd = fe$. The case $f = 0 = a = b$ is discussed above. If $c = d = e = 0 = a = b$, then any vector $Y = fE_6$, $f \in \mathbb{R}$ is geodesic since it lies in ζ.

In the case $a \neq 0$ the geodesic vectors of $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle ., . \rangle)$ are in the set

(3.21)
$$\begin{aligned} \overline{C_2} =& \Big\{ a(E_1 - \frac{\beta_6}{\alpha_4}E_2) + cE_3 + dE_4 + eE_5 + fE_6 : a \neq 0,\ ae = \frac{f}{\alpha_3}\Big(\frac{a\beta_6\beta_8}{\alpha_4} + c\frac{\alpha_3\alpha_4}{\alpha_1} - a\beta_5\Big), \\ & ac = \frac{f}{\alpha_1}\left(c\beta_7 + d\beta_8 + e\alpha_4\right) - \frac{a}{\alpha_1}\left(\beta_1 d + \beta_2 e + \beta_3 f\right), \\ & ad = \frac{a}{\alpha_2}\Big((\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})e - \beta_4 f\Big) + \frac{f}{\alpha_2}\Big(\frac{a\beta_6\beta_7}{\alpha_4} - d\frac{\alpha_3\alpha_4}{\alpha_1}\Big), \\ & d(\alpha_2 c + \alpha_3 e) = a\frac{\beta_6}{\alpha_4}\left(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f\right) + ce\Big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\Big) - f\left(c\beta_4 + d\beta_5 + e\beta_6\right) \Big\}. \end{aligned}$$

If $f = 0$, then from (3.21) it follows that $e = d = c = 0$. Hence any element $Y = a(E_1 - \frac{\beta_6}{\alpha_4}E_2)$ lies in $\mathfrak{a}_1$. Therefore we may assume that $f \neq 0$. The intersection $C_1 \cap C_2$ is empty since for the elements of C_1 one has $a = 0$ and $f = 0$ but for the elements in C_2 we have $af \neq 0$. This completes the proof. $\square$

Theorem 3.8. *Let $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle ., . \rangle)$ be a metric Lie algebra defined on $\mathbb{E}^6$ by non-vanishing commutators given by (1.5). The flat totally geodesic subalgebras of dimension > 1 in $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle ., . \rangle)$ are the 2-dimensional subalgebras:*

(1) $\mathfrak{h}_{10} = span(E_3, E_5)$ in the case $\alpha_3\alpha_4\beta_1 + \alpha_2\alpha_1\beta_8 = 0$,

(2) $\mathfrak{h}_7 = span(E_1 - \frac{\beta_6}{\alpha_4}E_2, \quad E_6)$ in the case $\beta_3 = 0$, $\beta_4 = \frac{\beta_6\beta_7}{\alpha_4}$, $\beta_5 = \frac{\beta_6\beta_8}{\alpha_4}$,

(3) $\mathfrak{h}_8 = span(E_2 + k_1E_4 + k_2E_5, \quad E_3 + l_1E_4 + l_2E_5)$ if and only if the equations

$$\beta_7 + l_1\beta_8 + l_2\alpha_4 + k_1\frac{\alpha_3\alpha_4}{\alpha_1} = 0, \quad \beta_1 k_1 + \beta_2 k_2 + k_1\alpha_3 k_2 = 0,$$

$$\alpha_2 l_1 - (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})l_2 + l_1\alpha_3 l_2 = 0, \tag{3.22}$$

$$\alpha_1 + \beta_1 l_1 + \beta_2 l_2 + \alpha_2 k_1 - (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})k_2 + \alpha_3(k_1 l_2 + l_1 k_2) = 0$$

are satisfied,

(4) $\mathfrak{h}_9 = span(E_2 + \frac{\beta_8\alpha_1}{\alpha_3\alpha_4}E_3 - \frac{1}{\alpha_3}(\beta_1 + \frac{\beta_8\alpha_1}{\alpha_3\alpha_4})E_5, \quad E_4)$ if and only if for α_i, β_j, $i = 1, 3, 4$, $j = 1, 9$ the equation

$$\frac{(\alpha_1)^2\beta_8}{\alpha_3\alpha_4} + \Big(\beta_1 + \frac{\beta_8\alpha_1}{\alpha_3\alpha_4}\Big)\Big(-\frac{\beta_2}{\alpha_3} + \frac{\beta_8\alpha_1}{(\alpha_3)^2\alpha_4}(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})\Big) = 0$$

holds.

Proof. According to Proposition 2.3 any totally geodesic subalgebra $\mathfrak{h}$ of $(\mathfrak{n}_{6,16}(\alpha_i,\beta_j),\langle .,.\rangle)$ has dimension less or equal to 3. Firstly we determine the 2- and the 3-dimensional abelian subalgebras of the Lie algebra $\mathfrak{n}_{6,16}(\alpha_i,\beta_j)$.

The 3- and the 2-dimensional abelian subalgebras of $(\mathfrak{n}_{6,16}(\alpha_i,\beta_j),\langle .,.\rangle)$ are:

$$\mathfrak{h}_1 = \operatorname{span}\left(E_1 + k_1E_2 + k_2E_3 + k_3E_4, \quad E_5, \quad E_6\right) \text{ with } \beta_6 + k_1\alpha_4 = 0,$$
$$\mathfrak{h}_2 = \operatorname{span}\left(E_3 + k_1E_4, \quad E_5, \quad E_6\right),$$
$$\mathfrak{h}_3 = \operatorname{span}\left(E_4, \quad E_5, \quad E_6\right),$$
$$\mathfrak{h}_4 = \operatorname{span}\left(E_2 + k_1E_4 + k_2E_5, \quad E_3 + l_1E_4 + l_2E_5, \quad E_6\right) \text{ with } \beta_7 + l_1\beta_8 + l_2\alpha_4 + k_1\frac{\alpha_3\alpha_4}{\alpha_1} = 0,$$
$$\mathfrak{h}_5 = \operatorname{span}\left(E_2 + k_1E_3 + k_2E_5, \quad E_4 + l_1E_5, \quad E_6\right) \text{ with } \beta_8 + l_1\alpha_4 - k_1\frac{\alpha_3\alpha_4}{\alpha_1} = 0,$$
$$\mathfrak{h}_6 = \operatorname{span}\left(E_1 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_6, \quad E_5 + s_1E_6\right) \text{ with } k_1 = -\frac{\beta_6}{\alpha_4},$$
$$\mathfrak{h}_7 = \operatorname{span}\left(E_1 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_5, \quad E_6\right),$$
$$\mathfrak{h}_8 = \operatorname{span}\left(E_2 + k_1E_4 + k_2E_5 + k_3E_6, \quad E_3 + l_1E_4 + l_2E_5 + l_3E_6\right)$$
$$\text{with } \beta_7 + l_1\beta_8 + l_2\alpha_4 + k_1\frac{\alpha_3\alpha_4}{\alpha_1} = 0,$$
$$\mathfrak{h}_9 = \operatorname{span}\left(E_2 + k_1E_3 + k_2E_5 + k_3E_6, \quad E_4 + l_1E_5 + l_2E_6\right) \text{ with } \beta_8 + l_1\alpha_4 - k_1\frac{\alpha_3\alpha_4}{\alpha_1} = 0,$$
$$\mathfrak{h}_{10} = \operatorname{span}\left(E_3 + k_1E_4 + k_2E_6, \quad E_5 + s_1E_6\right),$$
$$\mathfrak{h}_{11} = \operatorname{span}\left(E_2 + k_1E_3 + k_2E_4 + k_3E_5, \quad E_6\right),$$
$$\mathfrak{h}_{12} = \operatorname{span}\left(E_3 + l_1E_4 + l_2E_5, \quad E_6\right),$$
$$\mathfrak{h}_{13} = \operatorname{span}\left(E_4 + l_2E_6, \quad E_5 + s_1E_6\right),$$
$$\mathfrak{h}_{14} = \operatorname{span}\left(E_4 + l_1E_5, \quad E_6\right),$$
$$\mathfrak{h}_{15} = \operatorname{span}\left(E_5, \quad E_6\right),$$

where $k_1, k_2, k_3, l_1, l_2, l_3, s_1 \in \mathbb{R}$.

The vector $E_5 + E_6 \in \mathfrak{h}_1 \cap \mathfrak{h}_2 \cap \mathfrak{h}_3 \cap \mathfrak{h}_{15}$ is not geodesic, because putting $a = b = c = d = 0$, $e = f = 1$ into the fourth equation of the system (3.19) we obtain the contradiction $\alpha_4 = 0$. Therefore the subalgebras $\mathfrak{h}_1$, $\mathfrak{h}_2$, $\mathfrak{h}_3$ and $\mathfrak{h}_{15}$ are not flat totally geodesic. Hence they are excluded. The vector $E_3 + l_1E_4 + l_2E_5 + E_6 \in \mathfrak{h}_4 \cap \mathfrak{h}_{12}$ is not geodesic, since putting $a = b = 0$, $c = f = 1$, $d = l_1$, $e = l_2$ into the second equation of the system (3.19) we get the contradiction $\frac{\alpha_3\alpha_4}{\alpha_1} = 0$. Hence the subalgebras $\mathfrak{h}_4$ and $\mathfrak{h}_{12}$ are not flat totally geodesic. The vector $E_4 + l_1E_5 + E_6 \in \mathfrak{h}_5 \cap \mathfrak{h}_{14}$ is not geodesic, because substituting $a = b = c = 0$, $d = 1 = f$, $e = l_1$ into the third equation of the system (3.19) we receive the contradiction $\frac{\alpha_3\alpha_4}{\alpha_1} = 0$. Hence the subalgebras $\mathfrak{h}_5$ and $\mathfrak{h}_{14}$ are not flat totally geodesic. The vector $E_5 + s_1E_6 \in \mathfrak{h}_6 \cap \mathfrak{h}_{10} \cap \mathfrak{h}_{13}$ is geodesic if for $a = b = c = d = 0$, $e = 1$, $f = s_1$ the system (3.19) of equations holds. From the fourth and fifth equations of (3.19) we have $\alpha_4 s_1 = 0$, $\beta_6 s_1 = 0$. Since $\alpha_4 \neq 0$ we get $s_1 = 0$. The vector $E_4 + l_2E_6 \in \mathfrak{h}_{13}$ is geodesic if for $a = b = c = e = 0$, $d = 1$, $f = l_2$ the system (3.19) of equations is satisfied. From the third, fourth and fifth equations of (3.19) we have $\frac{\alpha_3\alpha_4}{\alpha_1} l_2 = 0$, $\beta_8 l_2 = 0$, $\beta_5 l_2 = 0$. Since $\frac{\alpha_3\alpha_4}{\alpha_1} \neq 0$ one has $l_2 = 0$. But the vector $E_4 + E_5 \in \mathfrak{h}_{13}$ is not geodesic because the substitution $a = b = c = f = 0$, $d = e = 1$ into the fifth equation of the system (3.19) leads to the contradiction $\alpha_3 = 0$. Therefore the subalgebra $\mathfrak{h}_{13}$ is not flat totally geodesic. The vector $E_2 + k_1E_3 + k_2E_4 + k_3E_5 + E_6 \in \mathfrak{h}_{11}$ is not geodesic, since putting $a = 0$, $b = f = 1$, $c = k_1$, $d = k_2$, $e = k_3$ into the first equation of the system (3.19) we obtain the contradiction $\alpha_4 = 0$. Hence the subalgebra $\mathfrak{h}_{11}$ is not flat totally geodesic. The vector $E_3 + k_1E_4 + k_2E_6 \in \mathfrak{h}_{10}$ is geodesic if for $a = b = e = 0$, $c = 1$, $d = k_1$, $f = k_2$ the system (3.19) of equations holds. The second, fourth and fifth equations yield

$$k_2\frac{\alpha_3\alpha_4}{\alpha_1} = 0, \quad k_2(\beta_7 + k_1\beta_8) = 0, \quad \alpha_2k_1 + \beta_4k_2 + k_1\beta_5k_2 = 0. \tag{3.23}$$

As $\frac{\alpha_3\alpha_4}{\alpha_1} \neq 0, \alpha_2 \neq 0$ we receive that $k_2 = k_1 = 0$. The vector $E_3 + E_5 \in \mathfrak{h}_{10}$ is geodesic if for $a = b = d = f = 0$, $c = e = 1$, the system (3.19) of equations is satisfied. The fifth equation gives $\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4} = 0$. Hence the subalgebra $\mathfrak{h}_{10} = \mathrm{span}(E_3, E_5)$ is flat totally geodesic if and only if $\alpha_3\alpha_4\beta_1 + \alpha_2\alpha_1\beta_8 = 0$. This proves the case (1).

The non-zero vector $E_1 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_6 \in \mathfrak{h}_6$ is geodesic if and only if for $a = 1, e = 0$, $b = k_1 = -\frac{\beta_6}{\alpha_4}, c = k_2, d = k_3, f = k_4$ the equations (3.19) holds. The second equation of (3.19) gives

$$k_4(\beta_5 + \beta_8k_1 - \frac{\alpha_3\alpha_4}{\alpha_1}k_2) = 0 \tag{3.24}$$

Furthermore, the element $E_1 + E_5 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_6 \in \mathfrak{h}_6$ is geodesic if and only if for $a = 1, e = 1$, $b = k_1 = -\frac{\beta_6}{\alpha_4}, c = k_2, d = k_3, f = k_4$ the system (3.19) of equations is valid. From the second equation of (3.19) one has

$$\alpha_3 + k_4(\beta_5 + \beta_8k_1 - \frac{\alpha_3\alpha_4}{\alpha_1}k_2) = 0. \tag{3.25}$$

Comparing (3.25) with (3.24) we obtain the contradiction $\alpha_3 = 0$, which excludes the subalgebra $\mathfrak{h}_6$.

The non-zero vector $E_1 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_5 \in \mathfrak{h}_7$ is geodesic if for $a = 1, f = 0$, $b = k_1, c = k_2, d = k_3, e = k_4$ the system (3.19) of equations holds. From the second equation of (3.19) we get $\alpha_3k_4 = 0$. As $\alpha_3 \neq 0$ we have $e = k_4 = 0$. Using this from the third equation of (3.19) we obtain $\alpha_2k_3 = 0$. Since $\alpha_2 \neq 0$ we receive that $d = k_3 = 0$. Taking into account that $e = d = 0$ from the fourth equations of (3.19) we have $\alpha_1k_2 = 0$. As $\alpha_1 \neq 0$ we obtain $k_2 = 0$. Using this the vector $E_1 + k_1E_2 + E_6 \in \mathfrak{h}_7$ is geodesic if for $a = 1, b = k_1, c = d = e = 0, f = 1$ the system (3.19) is valid. From the first equation of (3.19) we receive that $k_1 = -\frac{\beta_6}{\alpha_4}$. Using this it follows from the second equation of (3.19) that $\beta_5 = \frac{\beta_6\beta_8}{\alpha_4}$, whereas from the third equation of (3.19) that $\beta_4 = \frac{\beta_6\beta_7}{\alpha_4}$. From the fourth equation of (3.19) we obtain that $\beta_3 = 0$. This proves case (2) of the Theorem.

Now, we consider the subalgebra $\mathfrak{h}_8$. The element $E_2 + k_1E_4 + k_2E_5 + k_3E_6 \in \mathfrak{h}_8$ is geodesic if and only if for $a = c = 0$, $b = 1, d = k_1, e = k_2, f = k_3$ the system (3.19) of equations holds. The first equation of (3.19) yields $\alpha_4k_3 = 0$. As $\alpha_4 \neq 0$ one gets $f = k_3 = 0$. Applying this the fifth equation of (3.19) gives

$$\beta_1k_1 + \beta_2k_2 + k_1\alpha_3k_2 = 0. \tag{3.26}$$

The non-zero vector $E_3 + l_1E_4 + l_2E_5 + l_3E_6 \in \mathfrak{h}_8$ is geodesic if for $a = b = 0$, $c = 1, d = l_1, e = l_2, f = l_3$ the system (3.19) of equations is satisfied. It follows from the second equation of (3.19) that $\frac{\alpha_3\alpha_4}{\alpha_1}l_3 = 0$. Since $\frac{\alpha_3\alpha_4}{\alpha_1} \neq 0$ we receive $f = l_3 = 0$. Using this the fifth equation of (3.19) gives

$$\alpha_2 l_1 - \Big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\Big) l_2 + l_1\alpha_3 l_2 = 0. \tag{3.27}$$

Let us consider the non-zero vector $E_2 + E_3 + (k_1 + l_1)E_4 + (k_2 + l_2)E_5 \in \mathfrak{h}_8$. It is geodesic if and only if and only if for $a = f = 0, b = c = 1, d = k_1 + l_1, e = k_2 + l_2$ the system (3.19) of equations is valid. From the fifth equation of (3.19) one has

$$\alpha_1+\beta_1(k_1+l_1)+\beta_2(k_2+l_2)+\alpha_2(k_1+l_1)-\Big(\frac{\alpha_3\beta_1}{\alpha_1}+\frac{\alpha_2\beta_8}{\alpha_4}\Big)(k_2+l_2)+(k_1+l_1)\alpha_3(k_2+l_2) = 0. \tag{3.28}$$

Taking into account (3.26) and (3.27) equation (3.28) reduces to

$$\alpha_1 + \beta_1 l_1 + \beta_2 l_2 + \alpha_2 k_1 - \Big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\Big) k_2 + \alpha_3(k_1 l_2 + l_1 k_2) = 0. \tag{3.29}$$

Therefore the subalgebra $\mathfrak{h}_8$ is flat totally geodesic if and only if it satisfies the conditions of the case (3.22) in the Theorem.

Finally we treat the subalgebra $\mathfrak{h}_9$. The non-zero vector $E_4 + l_1 E_5 + l_2 E_6 \in \mathfrak{h}_9$ is geodesic precisely if for $a = b = c = 0$, $d = 1$, $e = l_1$, $f = l_2$ the system (3.19) of equations is satisfied. From the third equation we obtain $\frac{\alpha_3\alpha_4}{\alpha_1} l_2 = 0$. As $\frac{\alpha_3\alpha_4}{\alpha_1} \neq 0$ we have $f = l_2 = 0$. Using this the fifth equation of (3.19) gives $\alpha_3 l_1 = 0$. Since $\alpha_3 \neq 0$ one gets $l_1 = 0$. Using this from the condition $\beta_8 + l_1\alpha_4 - k_1\frac{\alpha_3\alpha_4}{\alpha_1} = 0$ to be abelian the subalgebra $\mathfrak{h}_9$ we receive that $k_1 = \frac{\beta_8\alpha_1}{\alpha_3\alpha_4}$. The element $E_2 + k_1 E_3 + k_2 E_5 + k_3 E_6 \in \mathfrak{h}_9$ is geodesic if and only if for $a = d = 0$, $b = 1$, $c = k_1$, $e = k_2$, $f = k_3$ the system (3.19) of equations holds. From the first equation of (3.19) we have $k_3\alpha_4 = 0$. As $\alpha_4 \neq 0$ it follows that $f = k_3 = 0$. Using this the fifth equation of (3.19) yields

$$\alpha_1 k_1 + \beta_2 k_2 - k_1 k_2 (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}) = 0. \tag{3.30}$$

The element $E_2 + k_1 E_3 + k_2 E_5 + E_4 \in \mathfrak{h}_9$ is geodesic precisely if for $a = f = 0$, $b = d = 1$, $c = k_1$, $e = k_2$ the system (3.19) of equations is valid. From the fifth equation of (3.19) we obtain

$$\alpha_1 k_1 + \beta_1 + \beta_2 k_2 + k_1\Big(\alpha_2 - (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})k_2\Big) + \alpha_3 k_2 = 0. \tag{3.31}$$

Taking into account equation (3.30) equation (3.31) reduces to

$$\beta_1 + k_1\alpha_2 + \alpha_3 k_2 = 0. \tag{3.32}$$

Putting the expression for k_1 into (3.32) we obtain that $k_2 = -\frac{1}{\alpha_3}\Big(\beta_1 + \frac{\beta_8\alpha_1}{\alpha_3\alpha_4}\Big)$. Substituting the expressions of k_1 and k_2 into (3.30) for the parameters α_i, β_j, $i = 1, 3, 4$, $j = 1, 8$ of the filiform metric Lie algebra $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle ., . \rangle)$ we obtain equation (3.31). This proves case (4) of the Theorem. Thus Theorem 3.8 is shown. □

4. Geodesic vectors and flat totally geodesic subalgebras of Lie algebras $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$ and $\mathfrak{n}_{6,18}(\alpha_i, \beta_j)$

In this section we deal with the metric Lie algebras $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ and $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$. The next result describes the sets of geodesic vectors of the metric Lie algebras $(\mathfrak{n}_{6,j}(\alpha_i, \beta_j), \langle ., . \rangle), j \in \{17, 18\}$.

Theorem 4.1. *Let $(\mathfrak{n}_{6,17}(\alpha_i,\beta_j),\langle .,.\rangle)$ be the metric Lie algebra defined on $\mathbb{E}^6$ by non-vanishing commutators given by (1.6). The geodesic vectors of $(\mathfrak{n}_{6,17}(\alpha_i,\beta_j),\langle .,.\rangle)$ not belonging to $\mathfrak{a}_1\cup\mathfrak{a}_2\cup\mathfrak{a}_3\cup\mathfrak{a}_4\cup\zeta$ are the non-zero elements of the set $C_1\cup C_2$, where*

$C_1 := \{dE_4+eE_5+fE_6 : d(\alpha_3 e+\beta_6 f)+e\alpha_4 f=0, d,e,f\in\mathbb{R}\}$, *such that at least two of the numbers d,e,f are non-zero with exception of the cases:*

1. $f=0$, 2. $d=0$, 3. $e=0$ *with* $\beta_6\neq 0$,

$C_2 := \{bE_2+cE_3+dE_4+eE_5 : b(\alpha_1 c+\beta_1 d+\beta_2 e)+c(\alpha_2 d+\beta_4 e)+d\alpha_3 e=0, b,c,d,e\in\mathbb{R}\}$, *such that at least two of the numbers b,c,d,e are non-zero with exception of the cases:*

1. $b=c=0$, 2. $b=e=0$, 3. $d=e=0$, 4. $b=d=0$ *with* $\beta_4\neq 0$,
5. $c=d=0$ *with* $\beta_2\neq 0$, 6. $c=e=0$, *with* $\beta_1\neq 0$.

Let $(\mathfrak{n}_{6,18}(\alpha_i,\beta_j),\langle .,.\rangle)$ be the metric Lie algebra defined on $\mathbb{E}^6$ by the non-vanishing commutators given by (1.6) with $\alpha_5=0$. The geodesic vectors of $(\mathfrak{n}_{6,18}(\alpha_i,\beta_j),\langle .,.\rangle)$ not belonging to $\mathfrak{a}_1\cup\mathfrak{a}_2\cup\mathfrak{a}_3\cup\mathfrak{a}_4\cup\zeta$ are the non-zero elements of the set

$C_3 := \{bE_2+cE_3+dE_4+eE_5+fE_6 : b(\alpha_1 c+\beta_1 d+\beta_2 e+\beta_3 f)+c(\alpha_2 d+\beta_4 e+\beta_5 f)+ d(\alpha_3 e+\beta_6 f)+e\alpha_4 f=0, b,c,d,e,f\in\mathbb{R}\}$, *such that at least two of the numbers b,c,d,e,f are non-zero with exception of the cases:*

1. $b=c=d=0$, 2. $b=c=f=0$, 3. $d=e=f=0$,
4. $b=c=e=0$ *with* $\beta_6\neq 0$, 5. $c=d=f=0$ *with* $\beta_2\neq 0$,
6. $c=d=e=0$, *with* $\beta_3\neq 0$.

Proof. Using the commutators (1.6) and the claim (1.2) we obtain that the non-zero element $Y=aE_1+bE_2+cE_3+dE_4+eE_5+fE_6\in\mathfrak{n}_{6,17}(\alpha_i,\beta_j)$ is geodesic if and only if for the real numbers a,b,c,d,e,f with respect to a basis $\{E_1,E_2,E_3,E_4,E_5,E_6\}$ the system of equations

$$\left\{\begin{array}{l} af=0,\ ae=0,\\ a\alpha_2 d+b\alpha_5 f=0,\\ a(\alpha_1 c+\beta_1 d)-c\alpha_5 f=0,\\ b(\alpha_1 c+\beta_1 d+\beta_2 e+\beta_3 f)+c(\alpha_2 d+\beta_4 e+\beta_5 f)+d(\alpha_3 e+\beta_6 f)+e\alpha_4 f=0\end{array}\right. \tag{4.1}$$

is satisfied. If $a=0$, then the third and the fourth equations of (4.1) give $bf=0=cf$. In the case $b=c=0=a$, the vector $Y=dE_4+eE_5+fE_6$ is geodesic of $(\mathfrak{n}_{6,17}(\alpha_i,\beta_j),\langle .,.\rangle)$ if and only if it lies in the set $\overline{C_1}:=\{dE_4+eE_5+fE_6 : d(\alpha_3 e+\beta_6 f)+e\alpha_4 f=0\}$, $d,e,f\in\mathbb{R}$. For $f=0$ the element $Y=dE_4+eE_5\in\overline{C_1}$, $d,e\in\mathbb{R}$, satisfies the condition $\alpha_3 de=0$. Since $\alpha_3\neq 0$ we obtain that either $d=0$ and the element $Y=eE_5$ is in $\mathfrak{a}_4$, or $e=0$ and the element $Y=dE_4$ is in $\mathfrak{a}_3$. If $d=0$, then for the element $Y=eE_5+fE_6\in\overline{C_1}$, $e,f\in\mathbb{R}$, the condition $\alpha_4 ef=0$ holds. Since $\alpha_4\neq 0$ we receive that either $e=0$ and the element $Y=fE_6$ is in ζ, or $f=0$ and the element $Y=eE_5$ is in $\mathfrak{a}_4$. If $e=0$ and $\beta_6\neq 0$, then for the element $Y=dE_4+fE_6\in\overline{C_1}$, $d,f\in\mathbb{R}$, the condition $df=0$ is satisfied. Hence we get either $d=0$ and the element $Y=fE_6$ is in ζ, or $f=0$ and the element $Y=dE_4$ is in $\mathfrak{a}_3$. Therefore the conditions for the set C_1 in the theorem are proved.

In the case $f=0=a$ the non-zero vector $Y=bE_2+cE_3+dE_4+eE_5$ is geodesic of $(\mathfrak{n}_{6,17}(\alpha_i,\beta_j),\langle .,.\rangle)$ if and only if it lies in the set $\overline{C_2}:=\{bE_2+cE_3+dE_4+eE_5 : b(\alpha_1 c+\beta_1 d+\beta_2 e)+c(\alpha_2 d+\beta_4 e)+d\alpha_3 e=0\}$, $b,c,d,e\in\mathbb{R}$. If $b=c=0$, then the element $Y=dE_4+eE_5\in\overline{C_2}$, $d,e\in\mathbb{R}$, satisfies the condition $\alpha_3 de=0$. Since $\alpha_3\neq 0$ we obtain that either $d=0$ and the element $Y = eE_5$ is in $\mathfrak{a}_4$, or $e = 0$ and the element $Y = dE_4$ is in $\mathfrak{a}_3$. If $b = e = 0$, then for the element $Y = cE_3 + dE_4 \in \overline{C_2}$, $c, d \in \mathbb{R}$, the condition $\alpha_2 cd = 0$ holds. Since $\alpha_2 \neq 0$ we receive that either $c = 0$ and the element $Y = dE_4$ is in $\mathfrak{a}_3$, or $d = 0$ and the element $Y = cE_3$ is in $\mathfrak{a}_2$. If $d = e = 0$, then for the element $Y = bE_2 + cE_3 \in \overline{C_2}$, $b, c \in \mathbb{R}$, the condition $\alpha_1 bc = 0$ is satisfied. As $\alpha_1 \neq 0$ we get either $b = 0$ and the element $Y = cE_3$ is in $\mathfrak{a}_2$, or $c = 0$ and the element $Y = bE_2$ is in $\mathfrak{a}_1$. If $b = d = 0$ and $\beta_4 \neq 0$, then for the element $Y = cE_3 + eE_5 \in \overline{C_2}$ the condition $ce = 0$ holds. Hence we have either $c = 0$ and the element $Y = eE_5$ is in $\mathfrak{a}_4$, or $e = 0$ and the element $Y = cE_3$ is in $\mathfrak{a}_2$. If $c = d = 0$ and $\beta_2 \neq 0$, then for the element $Y = bE_2 + eE_5 \in \overline{C_2}$, $b, e \in \mathbb{R}$ the condition $be = 0$ holds. Therefore we get either $b = 0$ and the element $Y = eE_5$ is in $\mathfrak{a}_4$, or $e = 0$ and the element $Y = bE_2$ is in $\mathfrak{a}_1$. Finally, if $c = e = 0$ and $\beta_1 \neq 0$, then for the element $Y = bE_2 + dE_4 \in \overline{C_2}$, $b, d \in \mathbb{R}$ the condition $bd = 0$ holds. It follows that either $b = 0$ and the element $Y = dE_4$ is in $\mathfrak{a}_3$, or $d = 0$ and the element $Y = bE_2$ is in $\mathfrak{a}_1$. This proves the conditions for the set C_2 in the theorem.

If $f = e = 0$, then from the third and fourth equations of (4.1) we obtain that $ad = 0 = ac$. The case $a = 0 = f = e$ is discussed above. In the case $d = c = 0 = f = e$ any vector $Y = aE_1 + bE_2$, $a, b \in \mathbb{R}$ is geodesic because it lies in $\mathfrak{a}_1$. The intersection $C_1 \cap C_2$ is empty, because for any element of C_1 one has $b = c = 0$ and any element of C_2 one gets $f = 0$. Hence the claim above the set of the geodesic vectors of $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ is shown.

In the case of the metric Lie algebra $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$ the system (4.1) of equations is satisfied with $\alpha_5 = 0$. Therefore we obtain that $af = 0 = ae = ad = ac$. The case $f = e = d = c = 0$ is discussed above. In the case $a = 0$ the vector $Y = bE_2 + cE_3 + dE_4 + eE_5 + fE_6$ is geodesic of $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$ precisely if it lies in the set $\overline{C_3} := \{bE_2 + cE_3 + dE_4 + eE_5 + fE_6 : b(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) + c(\alpha_2 d + \beta_4 e + \beta_5 f) + d(\alpha_3 e + \beta_6 f) + e\alpha_4 f = 0, b, c, d, e, f \in \mathbb{R}\}$. If $b = c = d = 0$, then the element $Y = eE_5 + fE_6 \in \overline{C_3}$, $e, f \in \mathbb{R}$, satisfies the condition $\alpha_4 ef = 0$. Since $\alpha_4 \neq 0$ we obtain that either $e = 0$ and the element $Y = fE_6$ is in ζ, or $f = 0$ and the element $Y = eE_5$ is in $\mathfrak{a}_4$. If $b = c = f = 0$, then for the element $Y = dE_4 + eE_5 \in \overline{C_3}$, $d, e \in \mathbb{R}$, the condition $\alpha_3 de = 0$ holds. Since $\alpha_3 \neq 0$ we receive that either $d = 0$ and the element $Y = eE_5$ is in $\mathfrak{a}_4$, or $e = 0$ and the element $Y = dE_4$ is in $\mathfrak{a}_3$. If $d = e = f = 0$, then for the element $Y = bE_2 + cE_3 \in \overline{C_3}$, $b, c \in \mathbb{R}$, the condition $\alpha_1 bc = 0$ is satisfied. As $\alpha_1 \neq 0$ we get either $b = 0$ and the element $Y = cE_3$ is in $\mathfrak{a}_2$, or $c = 0$ and the element $Y = bE_2$ is in $\mathfrak{a}_1$. In the case $b = c = e = 0$ and $\beta_6 \neq 0$, then for the element $Y = dE_4 + fE_6 \in \overline{C_3}$, $d, f \in \mathbb{R}$, the condition $df = 0$ holds. Hence we have either $d = 0$ and the element $Y = fE_6$ is in ζ, or $f = 0$ and the element $Y = dE_4$ is in $\mathfrak{a}_3$. If $c = d = f = 0$ and $\beta_2 \neq 0$, then for the element $Y = bE_2 + eE_5 \in \overline{C_3}$, $b, e \in \mathbb{R}$, the condition $be = 0$ holds. Therefore we get either $b = 0$ and the element $Y = eE_5$ is in $\mathfrak{a}_4$, or $e = 0$ and the element $Y = bE_2$ is in $\mathfrak{a}_1$. Finally, if $c = d = e = 0$ and $\beta_3 \neq 0$, then for the element $Y = bE_2 + fE_6 \in \overline{C_3}$, $b, f \in \mathbb{R}$ the condition $bf = 0$ holds. It follows that either $b = 0$ and the element $Y = fE_6$ is in ζ, or $f = 0$ and the element $Y = bE_2$ is in $\mathfrak{a}_1$. This shows the conditions for the set C_3 in the theorem. Hence Theorem 4.1 is proved. $\square$

The flat totally geodesic subalgebras of $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ are given in the following theorem.

Theorem 4.2. *Let $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ be the metric Lie algebra defined on $\mathbb{E}^6$ by non-vanishing commutators given by (1.6). The flat totally geodesic subalgebras of dimension > 1 in the metric Lie algebra $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ are the 2-dimensional subalgebras:*

(1) $\mathfrak{h}_9 = span(E_4 - \frac{\alpha_3}{\alpha_4} E_6, \quad E_5)$ in the case $\beta_6 = 0$,

(2) $\mathfrak{h}_2 = span(E_2 + k_1 E_3 - \frac{\beta_2+\beta_4 k_1}{\alpha_3} E_4, \quad E_5)$*, where* k_1 *is a solution of the equation*

$$\alpha_2\beta_4 k_1^2 + (\beta_1\beta_4 + \alpha_2\beta_2 - \alpha_1\alpha_3)k_1 + \beta_1\beta_2 = 0, \tag{4.2}$$

(3) $\mathfrak{h}_4 = span\,(E_3, E_5)$ *in the case* $\beta_4 = 0$*,*
(4) $\mathfrak{h}_1 = span(E_2 - \frac{\beta_1+\alpha_3 k_2}{\alpha_2} E_3 + k_2 E_5, \quad E_4)$*, where* k_2 *is a solution of the equation*

$$\alpha_3\beta_4 k_2^2 + (\alpha_1\alpha_3 + \beta_1\beta_4 - \alpha_2\beta_2)k_2 + \alpha_1\beta_1 = 0, \tag{4.3}$$

(5) $\mathfrak{h}_3 = span(E_3 - \frac{\alpha_2}{\alpha_3} E_5, \quad E_4)$ *in the case* $\beta_4 = 0$*,*
(6) $\mathfrak{h}_5 = span\,(E_4, E_6)$ *in the case* $\beta_6 = 0$*.*

Proof. As pointed out in [1, Theorem 1.19], the metric Lie algebra $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ does not have a totally geodesic subalgebra greater than 2. Hence we compute only the 2-dimensional abelian subalgebras in the Lie algebra $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$. These subalgebras are the following:

$$\begin{aligned}
\mathfrak{h}_1 &= \mathrm{span}\,(E_2 + k_1E_3 + k_2E_5 + k_3E_6, \quad E_4 + l_1E_5 + l_2E_6)\,,\\
\mathfrak{h}_2 &= \mathrm{span}\,(E_2 + k_1E_3 + k_2E_4 + k_3E_6, \quad E_5 + l_1E_6)\,,\\
\mathfrak{h}_3 &= \mathrm{span}\,(E_3 + k_1E_5 + k_2E_6, \quad E_4 + l_1E_5 + l_2E_6)\,,\\
\mathfrak{h}_4 &= \mathrm{span}\,(E_3 + k_1E_4 + k_2E_6, \quad E_5 + l_1E_6)\,,\\
\mathfrak{h}_5 &= \mathrm{span}\,(E_4 + k_1E_5, \quad E_6)\,,\\
\mathfrak{h}_6 &= \mathrm{span}\,(E_1 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_5, \quad E_6)\,,\\
\mathfrak{h}_7 &= \mathrm{span}\,(E_2 + k_1E_3 + k_2E_4 + k_3E_5, \quad E_6)\,,\\
\mathfrak{h}_8 &= \mathrm{span}\,(E_3 + k_1E_4 + k_2E_5, \quad E_6)\,,\\
\mathfrak{h}_9 &= \mathrm{span}\,(E_4 + k_1E_6, \quad E_5 + l_1E_6)\,,\\
\mathfrak{h}_{10} &= \mathrm{span}\,(E_5, E_6)\,,
\end{aligned}$$

where $k_1, k_2, k_3, l_1, l_2 \in \mathbb{R}$.

The subalgebra $\mathfrak{h}_{10}$ is not flat totally geodesic because for the vector $E_5 + E_6$ the fifth equation of (4.1) gives the contradiction $\alpha_4 = 0$. Hence the subalgebra $\mathfrak{h}_{10}$ is excluded. Since for the vector $Y = E_1 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_5 + E_6 \in \mathfrak{h}_6$ the first equation of the system (4.1) yields the contradiction $\alpha_4 = 0$ the subalgebra $\mathfrak{h}_6$ is not flat totally geodesic (see Lemma 2.1). Therefore the subalgebra $\mathfrak{h}_6$ is excluded. the vector $E_2 + k_1E_3 + k_2E_4 + k_3E_5 + E_6 \in \mathfrak{h}_7$ is not geodesic since the third equation of (4.1) gives the contradiction $\alpha_5 = 0$. Therefore the subalgebra $\mathfrak{h}_7$ is not flat totally geodesic. The non-zero vector $Y = E_3 + k_1E_4 + k_2E_5 + E_6 \in \mathfrak{h}_8$ is not geodesic because the fourth equation of (4.1) leads to the contradiction $\alpha_5 = 0$. Hence the subalgebra $\mathfrak{h}_8$ is not flat totally geodesic.

The non-zero vector $E_5 + l_1E_6 \in \mathfrak{h}_2 \cap \mathfrak{h}_4 \cap \mathfrak{h}_9$ is geodesic if for $a = b = c = d = 0$, $e = 1, f = l_1$ the system (4.1) of equations is satisfied. From the fifth equation of (4.1) one has $\alpha_4 l_1 = 0$. As $\alpha_4 \neq 0$ we get $l_1 = 0$.

Now, we consider the subalgebra $\mathfrak{h}_9$. The non-zero vector $E_4 + k_1E_6 \in \mathfrak{h}_9$ is geodesic precisely if for $a = b = c = e = 0$, $d = 1$, $f = k_1$ the system (4.1) of equations holds. From the fifth equation we obtain

$$\beta_6 k_1 = 0. \tag{4.4}$$

Furthermore, the element $E_4+k_1E_6+E_5\in\mathfrak{h}_9$ is geodesic if for $a=b=c=0$, $d=e=1, f=k_1$ the system (4.1) of equations is valid. It follows from the fifth equation of (4.1) that

$$\alpha_3+\beta_6k_1+\alpha_4k_1=0. \tag{4.5}$$

Taking into account (4.4) equation (4.5) reduces to

$$\alpha_3+\alpha_4k_1=0. \tag{4.6}$$

The equation (4.6) gives $k_1=-\frac{\alpha_3}{\alpha_4}$. Putting this expression into (4.4) on has $\beta_6=0$. Therefore the case (1) is proved.

Next we deal with the subalgebra $\mathfrak{h}_2$. The element $E_2+k_1E_3+k_2E_4+k_3E_6\in\mathfrak{h}_2$ is geodesic precisely if for $a=e=0$, $b=1, c=k_1, d=k_2, f=k_3$ the system (4.1) of equations holds. From the third equation of (4.1) we obtain $\alpha_5k_3=0$. Since $\alpha_5\neq0$ we receive $k_3=0=f$. Using this the fifth equation of (4.1) gives

$$\alpha_1k_1+\beta_1k_2+k_1\alpha_2k_2=0. \tag{4.7}$$

Moreover, the element $E_2+k_1E_3+k_2E_4+E_5\in\mathfrak{h}_2$ is geodesic if and only if for $a=f=0, b=e=1, c=k_1, d=k_2$ the system (4.1) of equations is valid. From the fifth equation of (4.1) we receive

$$\alpha_1k_1+\beta_1k_2+\beta_2+k_1\alpha_2k_2+k_1\beta_4+k_2\alpha_3=0. \tag{4.8}$$

Taking into account (4.7) equation (4.8) reduces to

$$\beta_2+\beta_4k_1+\alpha_3k_2=0. \tag{4.9}$$

From (4.9) one has $k_2=-\frac{\beta_2+\beta_4k_1}{\alpha_3}$. Putting this expression into (4.7) we receive the second order equation (4.2). Therefore the case (2) is proved.

Now we treat the subalgebra $\mathfrak{h}_4$. The element $E_3+k_1E_4+k_2E_6\in\mathfrak{h}_4$ is geodesic if and only if for $a=b=e=0, c=1, d=k_1, f=k_2$ the system (4.1) of equation is satisfied. From the fourth equation of (4.1) we receive $\alpha_5k_2=0$. As $\alpha_5\neq0$ we get $k_2=f=0$. Using this the fifth equation of (4.1) gives $\alpha_2k_1=0$. Since $\alpha_2\neq0$ we obtain $k_1=0$. The vector $E_3+E_5\in\mathfrak{h}_4$ is geodesic precisely if for $a=b=d=f=0$, $c=e=1$, the system (4.1) of equation holds. From the fifth equation of (4.1) we get $\beta_4=0$. This gives the case (3).

The non-zero vector $E_4+l_1E_5+l_2E_6\in\mathfrak{h}_1\cap\mathfrak{h}_3$ is geodesic if for $a=b=c=0, d=1, e=l_1, f=l_2$ the system (4.1) of equation is satisfied. The fifth equation of (4.1) yields

$$\alpha_3l_1+\beta_6l_2+l_1\alpha_4l_2=0. \tag{4.10}$$

Let us consider the subalgebra $\mathfrak{h}_1$. The element $E_2+k_1E_3+k_2E_5+k_3E_6\in\mathfrak{h}_1$ is geodesic if and only if for $a=d=0, b=1, c=k_1, e=k_2, f=k_3$ the system (4.1) of equation is valid. From the third equation of (4.1) one has $\alpha_5k_3=0$. As $\alpha_5\neq0$ we receive $k_3=f=0$. Using this the fifth equation of (4.1) gives

$$\alpha_1k_1+\beta_2k_2+k_1\beta_4k_2=0. \tag{4.11}$$

Additionally, the non-zero vector $E_2+E_4+k_1E_3+(k_2+l_1)E_5+l_2E_6\in\mathfrak{h}_1$ is geodesic if for $a=0, b=d=1, c=k_1, e=k_2+l_1, f=l_2$ the system (4.1) of equation is valid. The third equation of (4.1) gives $\alpha_5l_2=0$. Since $\alpha_5\neq0$ we get $l_2=f=0$. Using this from the equation (4.10) we get $\alpha_3l_1=0$. As $\alpha_3\neq0$ we obtain $l_1=e=0$, and from the fifth equation of (4.1) we obtain

$$\alpha_1k_1+\beta_1+\beta_2k_2+k_1\alpha_2+k_1\beta_4k_2+\alpha_3k_2=0. \tag{4.12}$$

Applying (4.11) equation (4.12) reduces to

$$\beta_1 + \alpha_2 k_1 + \alpha_3 k_2 = 0.$$

The above equation gives $k_1 = -\frac{\beta_1 + \alpha_3 k_2}{\alpha_2}$. Putting this expression into (4.11) we receive the second order equation (4.2). Hence the case (4) is proved.

Next we deal with the subalgebra $\mathfrak{h}_3$. The non-zero vector $E_3 + k_1 E_5 + k_2 E_6 \in \mathfrak{h}_3$ is geodesic if for $a = b = d = 0, c = 1, e = k_1, f = k_2$ the system (4.1) of equation is satisfied. From the fourth equation of (4.1) one obtains $\alpha_5 k_2 = 0$. As $\alpha_5 \neq 0$ we get $k_2 = f = 0$. Using this the fifth equation of (4.1) gives

$$\beta_4 k_1 = 0. \tag{4.13}$$

In addition, the element $E_3 + E_4 + (k_1 + l_1)E_5 + l_2 E_6 \in \mathfrak{h}_3$ is geodesic if for $a = b = 0, c = d = 1, e = k_1 + l_1, f = l_2$ the system (4.1) of equation is valid. The third equation of (4.1) gives $\alpha_5 l_2 = 0$. Since $\alpha_5 \neq 0$ one has $l_2 = f = 0$. Using this from the equation (4.10) we get $\alpha_3 l_1 = 0$. As $\alpha_3 \neq 0$ we receive $l_1 = 0$, and from the fifth equation of (4.1) we obtain

$$\alpha_2 + \beta_4 k_1 + \alpha_3 k_1 = 0. \tag{4.14}$$

Applying (4.13) equation (4.14) reduces to

$$\alpha_2 + \alpha_3 k_1 = 0.$$

From the above equation we obtain $k_1 = -\frac{\alpha_2}{\alpha_3}$. Putting this expression into (4.13) we receive $\beta_4 = 0$. This proves the case (5).

Finally we treat the subalgebra $\mathfrak{h}_5$. The vector $E_4 + k_1 E_5 \in \mathfrak{h}_5$ is geodesic if and only if for $a = b = c = f = 0$, $d = 1$, $e = k_1$ the system (4.1) of equations is valid. It follows from the fifth equation of (4.1) that $\alpha_3 k_1 = 0$. As $\alpha_3 \neq 0$ we gets $k_1 = 0$. The vector $E_4 + E_6 \in \mathfrak{h}_5$ is geodesic if and only if for $a = b = c = e = 0$, $d = f = 1$ the system (4.1) of equations is valid. The fifth equation gives $\beta_6 = 0$. Therefore the case (6) is proved. This proves Theorem 4.2. □

Now we determine the flat totally geodesic subalgebras of dimension > 1 in the standard filiform metric Lie algebra $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$.

Theorem 4.3. *Let $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$ be the metric Lie algebra defined on $\mathbb{E}^6$ by non-vanishing commutators given by (1.6) with $\alpha_5 = 0$. The flat totally geodesic subalgebra of dimension > 1 in the metric Lie algebra $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$ as follow:*

(1) The 4-dimensional subalgebras are the following:
 - *(a) $\mathfrak{h}_1 = span(E_2 - \frac{\alpha_1 \alpha_3}{\alpha_2 \alpha_4} E_6, \quad E_3, \quad E_4 - \frac{\alpha_3}{\alpha_4} E_6, \quad E_5)$ in the case $\beta_1 = \beta_3 = \beta_4 = \beta_6 = 0$, $\beta_5 = \frac{\alpha_2 \alpha_4}{\alpha_3}$, $\beta_2 = \frac{\alpha_1 \alpha_3}{\alpha_2}$,*
 - *(b) $\mathfrak{h}_2 = span(E_2, \quad E_3 - \frac{\alpha_2}{\alpha_3} E_5, \quad E_4, \quad E_6)$ in the case $\beta_1 = \beta_3 = \beta_4 = \beta_6 = 0$, $\beta_5 = \frac{\alpha_2 \alpha_4}{\alpha_3}$, $\beta_2 = \frac{\alpha_1 \alpha_3}{\alpha_2}$,*

(2) The 3-dimensional subalgebras are the following:

(a) $\mathfrak{h}_6 = span\,(E_2 + k_1E_5 + k_2E_6, \quad E_3 + l_1E_5 + l_2E_6, \quad E_4 + s_1E_5 + s_2E_6)$ *if and only if the following equations*

$$\beta_2 k_1 + \beta_3 k_2 + k_1\alpha_4 k_2 = 0, \quad \beta_4 l_1 + \beta_5 l_2 + l_1\alpha_4 l_2 = 0,$$
$$\alpha_3 s_1 + \beta_6 s_2 + s_1\alpha_4 s_2 = 0,$$
$$\alpha_1 + \beta_2 l_1 + \beta_3 l_2 + \beta_4 k_1 + \beta_5 k_2 + k_1 l_2\alpha_4 + l_1 k_2\alpha_4 = 0,$$
$$\beta_1 + \beta_2 s_1 + \beta_3 s_2 + \alpha_3 k_1 + \beta_6 k_2 + k_1 s_2\alpha_4 + s_1 k_2\alpha_4 = 0,$$
$$\alpha_2 + \beta_4 s_1 + \beta_5 s_2 + \alpha_3 l_1 + \beta_6 l_2 + l_1 s_2\alpha_4 + s_1 l_2\alpha_4 = 0,$$
$$\beta_2 k_1 + \beta_3 k_2 + \beta_4 l_1 + \beta_5 l_2 + \alpha_3 s_1 + \beta_6 s_2 + k_1 s_2 + k_1 k_2\alpha_4 + l_1 l_2\alpha_4 + s_1 s_2\alpha_4 = 0$$

are satisfied,

(b) $\mathfrak{h}_7 = span\,(E_2 + k_1E_4 + k_2E_6, \quad E_3 + l_1E_4 + l_2E_6, \quad E_5)$ *if and only if* $\beta_2 = 0$ *and the following equations*

$$\beta_1 k_1 + \beta_2 k_2 + k_1\beta_6 k_2 = 0, \quad \alpha_2 l_1 + \beta_5 l_2 + l_1\beta_6 l_2 = 0,$$
$$\alpha_1 + \beta_2 l_1 + \beta_3 l_2 + \alpha_2 k_1 + \beta_5 k_2 + k_1 l_2\beta_6 + l_1 k_2\beta_6 = 0,$$
$$\beta_2 + \alpha_3 k_1 + \alpha_4 k_2 = 0, \quad \beta_4 + \alpha_3 l_1 + \alpha_4 l_2 = 0$$

are satisfied

(c) $\mathfrak{h}_8 = span\,(E_2 + k_1E_4 + k_2E_5, \quad E_3 + l_1E_4 + l_2E_5, \quad E_6)$ *if and only if the following equations*

$$\beta_1 k_1 + \beta_2 k_2 + k_1\alpha_3 k_2 = 0, \quad \alpha_2 l_1 + \beta_4 l_2 + l_1 l_2\alpha_3 = 0,$$
$$\alpha_1 + \beta_1 l_1 + \beta_2 l_2 + \alpha_2 k_1 + \beta_4 k_2 + k_1 l_2\alpha_3 + l_1 k_2\alpha_3 = 0,$$
$$\beta_3 + \beta_6 k_1 + \alpha_4 k_2 = 0, \quad \beta_5 + l_1\beta_6 + l_2\alpha_4 = 0$$

hold,

(d) $\mathfrak{h}_9 = span\left(E_2 + k_1E_3 - \frac{\beta_2 + k_1\beta_4}{\alpha_4}E_6, \quad E_4 - \frac{\alpha_3}{\alpha_4}E_6, \quad E_5\right)$ *such that* k_1 *is a solution of the equation*

$$\beta_4\beta_5 k_1^2 + k_1(\beta_2\beta_5 + \beta_3\beta_4 - \alpha_1\alpha_4) + \beta_2\beta_3 = 0, \tag{4.15}$$

and $\beta_1 = \frac{\alpha_3}{\alpha_4}\beta_3$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$,

(e) $\mathfrak{h}_9 = span\left(E_2 + \frac{\alpha_3\beta_3 - \alpha_4\beta_1}{\alpha_2\alpha_4 - \alpha_3\beta_5}E_3 - \frac{\beta_2(\alpha_2\alpha_4 - \alpha_3\beta_5) + \beta_4(\beta_3\alpha_3 - \beta_1\alpha_4)}{\alpha_4(\alpha_2\alpha_4 - \alpha_3\beta_5)}E_6, \quad E_4 - \frac{\alpha_3}{\alpha_4}E_6, \quad E_5\right)$ *such that* $\beta_5 \neq \frac{\alpha_2\alpha_4}{\alpha_3}$ *and the equation*

$$\begin{aligned}(\alpha_2\alpha_4 - \alpha_3\beta_5)[(\alpha_1\alpha_4 - \beta_3\beta_4 - \beta_2\beta_5)(\beta_3\alpha_3 - \beta_1\alpha_4) - \beta_2\beta_3(\alpha_2\alpha_4 - \alpha_3\beta_5)\\ - \beta_4\beta_5(\beta_3\alpha_3 - \beta_1\alpha_4)^2 = 0\end{aligned} \tag{4.16}$$

holds,

(f) $\mathfrak{h}_{10} = span\left(E_2 + k_1E_3 - \frac{\beta_3 + k_1\beta_5}{\alpha_4}E_5, \quad E_4, \quad E_6\right)$ *such that* k_1 *is a solution of the equation* (4.15) *and* $\beta_1 = \frac{\alpha_3}{\alpha_4}\beta_3$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$,

(g) $\mathfrak{h}_{10} = span\left(E_2 + \frac{\alpha_3\beta_3 - \alpha_4\beta_1}{\alpha_2\alpha_4 - \alpha_3\beta_5}E_3 - \frac{\beta_3(\alpha_2\alpha_4 - \alpha_3\beta_5) + \beta_5(\beta_3\alpha_3 - \beta_1\alpha_4)}{\alpha_4(\alpha_2\alpha_4 - \alpha_3\beta_5)}E_6, \quad E_4, \quad E_5\right)$ *such that* $\beta_5 \neq \frac{\alpha_2\alpha_4}{\alpha_3}$ *and the equation* (4.16) *holds.*

(h) $\mathfrak{h}_{12} = span\left(E_3, \quad E_4 - \frac{\alpha_3}{\alpha_4}E_6, \quad E_5\right)$ *in the case* $\beta_4 = \beta_6 = 0$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$,

(i) $\mathfrak{h}_{13} = span\left(E_3 - \frac{\alpha_2}{\alpha_3}E_5, \quad E_4, \quad E_6\right)$ *in the case* $\beta_4 = \beta_6 = 0$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$,

(3) *The 2-dimensional subalgebras are the following:*

(a) $\mathfrak{h}_{24} = span(E_4 - \frac{\alpha_3}{\alpha_4}E_6, \quad E_5)$ *in the case* $\beta_6 = 0$,
(b) $\mathfrak{h}_{25} = span\,(E_4, E_6)$ *in the case* $\beta_6 = 0$,
(c) $\mathfrak{h}_{18} = span\,(E_2 + k_1E_3 + k_2E_5 + k_3E_6, \quad E_4 + l_1E_5 + l_2E_6)$ *if and only if the following equations*

$$\alpha_3 l_1 + \beta_6 l_2 + l_1\alpha_4 l_1 = 0, \quad \alpha_1 k_1 + \beta_2 k_2 + \beta_3 k_3 + k_1\beta_4 k_2 + k_1\beta_5 k_3 + k_2\alpha_4 k_3 = 0,$$
$$\beta_1 + \beta_2 l_1 + \beta_3 l_2 + \alpha_2 k_1 + k_1\beta_4 l_1 + k_1\beta_5 k_3 + k_1\beta_5 l_2 + \alpha_3 k_2 +$$
$$\beta_6 k_3 + \alpha_4 k_2 l_2 + \alpha_4 l_1 k_3 = 0$$

hold,
(d) $\mathfrak{h}_{21} = span(E_3 + k_1E_5 + k_2E_6, \quad E_4 + l_1E_5 + l_2E_6)$ *if and only if the following equations*

$$\beta_4 k_1 + \beta_5 k_2 + k_1\alpha_4 k_2 = 0, \quad \alpha_3 l_1 + \beta_6 l_2 + l_1\alpha_4 l_2 = 0,$$
$$\alpha_2 + \beta_4 l_1 + \beta_5 l_2 + \alpha_3 k_1 + \beta_6 k_2 + k_1 l_2\alpha_4 + l_1 k_2\alpha_4 = 0$$

are satisfied,
(e) $\mathfrak{h}_{19} = span(E_2 + k_1E_3 + k_2E_4 - \frac{\beta_2 + k_1\beta_4 + k_2\alpha_3}{\alpha_4}E_6, \quad E_5)$ *such that the equation*

$$(\alpha_3 k_2 + \beta_2 + \beta_4 k_1)(k_1\beta_5 + k_2\beta_6 + \beta_3) - \alpha_2\alpha_4 k_1 k_2 - \alpha_1\alpha_4 k_1 - \beta_1\alpha_4 k_2 = 0 \tag{4.17}$$

is satisfied,
(f) $\mathfrak{h}_{22} = span(E_3 + k_1E_4 - \frac{\beta_4 + \alpha_3 k_1}{\alpha_4}E_6, \quad E_5)$, *where* k_1 *is a solution of the equation*

$$\alpha_3\beta_6 k_1^2 + k_1(\alpha_3\beta_5 + \beta_4\beta_6 - \alpha_2\alpha_4) + \beta_4\beta_5 = 0, \tag{4.18}$$

(g) $\mathfrak{h}_{23} = span(E_3 + k_1E_4 - \frac{\beta_5 + \beta_6 k_1}{\alpha_4}E_5, \quad E_6)$, *where* k_1 *is a solution of the equation* (4.18),
(h) $\mathfrak{h}_{20} = span(E_2 + k_1E_3 + k_2E_4 - \frac{\beta_3 + k_1\beta_5 + k_2\beta_6}{\alpha_4}E_5, \quad E_6)$ *such that the equation*

$$(\alpha_3 k_2 + \beta_4 k_1 + \beta_2)(k_1\beta_5 + k_2\beta_6 + \beta_3) - \alpha_2\alpha_4 k_1 k_2 - \alpha_1\alpha_4 k_1 - \beta_1\alpha_4 k_2 = 0 \tag{4.19}$$

holds,
(i) $\mathfrak{h}_{17} = span(E_2 + k_1E_4 + k_2E_5 + k_3E_6, \quad E_3 + l_1E_4 + l_2E_5 + l_3E_6)$ *if and only if the following equations*

$$\begin{gathered}\beta_1 k_1 + \beta_2 k_2 + \beta_3 k_3 + k_1\alpha_3 k_2 + k_1\beta_6 k_3 + k_2\alpha_4 k_3 = 0,\\ \alpha_2 l_1 + \beta_4 l_2 + \beta_5 l_3 + l_1\alpha_3 l_2 + l_1\beta_6 l_3 + l_2\alpha_4 l_3 = 0,\\ \alpha_1 + \beta_1 l_1 + \beta_2 l_2 + \beta_3 l_3 + \alpha_2 k_1 + \beta_4 k_2 + \beta_5 k_3 + k_1 l_2\alpha_3 + l_1 k_2\alpha_3 +\\ k_1 l_3\beta_6 + l_1 k_3\beta_6 + k_2 l_3\alpha_4 + l_2 l_3\alpha_4 = 0.\end{gathered} \tag{4.20}$$

are satisfied.

Proof. According to Proposition 2.3 b) the dimension of the flat totally geodesic subalgebras of the metric Lie algebra $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle .,. \rangle)$ is at most 4. Firstly we list the 4-dimensional, the 3-dimensional and the 2-dimensional abelian subalgebras of the Lie algebra $\mathfrak{n}_{6,18}(\alpha_i, \beta_j)$ defined by the commutators (1.6) such that $\alpha_5 = 0$. The 4-dimensional subalgebras have one of the following forms:

$$\begin{aligned}
\mathfrak{h}_1 &= \operatorname{span}\left(E_2 + k_1E_6, \quad E_3 + k_2E_6, \quad E_4 + k_3E_6, \quad E_5 + s_1E_6\right), \\
\mathfrak{h}_2 &= \operatorname{span}\left(E_2 + k_1E_5, \quad E_3 + k_2E_5, \quad E_4 + k_3E_5, \quad E_6\right), \\
\mathfrak{h}_3 &= \operatorname{span}\left(E_2 + k_1E_4, \quad E_3 + k_2E_4, \quad E_5, \quad E_6\right), \\
\mathfrak{h}_4 &= \operatorname{span}(E_2, E_4, E_5, E_6), \\
\mathfrak{h}_5 &= \operatorname{span}(E_3, E_4, E_5, E_6),
\end{aligned}$$

where $k_1, k_2, k_3, s_1 \in \mathbb{R}$.

The 3-dimensional abelian subalgebras of the Lie algebra $\mathfrak{n}_{6,18}(\alpha_i, \beta_j)$ are:

$$\begin{aligned}
\mathfrak{h}_6 &= \operatorname{span}\left(E_2 + k_1E_5 + k_2E_6, \quad E_3 + l_1E_5 + l_2E_6, \quad E_4 + s_1E_5 + s_2E_6\right), \\
\mathfrak{h}_7 &= \operatorname{span}\left(E_2 + k_1E_4 + k_2E_6, \quad E_3 + l_1E_4 + l_2E_6, \quad E_5 + s_1E_6\right), \\
\mathfrak{h}_8 &= \operatorname{span}\left(E_2 + k_1E_4 + k_2E_5, \quad E_3 + l_1E_4 + l_2E_5, \quad E_6\right), \\
\mathfrak{h}_9 &= \operatorname{span}\left(E_2 + k_1E_3 + k_2E_6, \quad E_4 + k_3E_6, \quad E_5 + s_1E_6\right), \\
\mathfrak{h}_{10} &= \operatorname{span}\left(E_2 + k_1E_3 + k_2E_5, \quad E_4 + k_3E_5, \quad E_6\right), \\
\mathfrak{h}_{11} &= \operatorname{span}\left(E_2 + k_1E_3 + k_2E_4, \quad E_5, \quad E_6\right), \\
\mathfrak{h}_{12} &= \operatorname{span}\left(E_3 + k_2E_6, \quad E_4 + k_3E_6, \quad E_5 + s_1E_6\right), \\
\mathfrak{h}_{13} &= \operatorname{span}\left(E_3 + k_2E_5, \quad E_4 + k_3E_5, \quad E_6\right), \\
\mathfrak{h}_{14} &= \operatorname{span}\left(E_3 + k_2E_4, \quad E_5, \quad E_6\right), \\
\mathfrak{h}_{15} &= \operatorname{span}\left(E_4, E_5, E_6\right),
\end{aligned}$$

where $k_1, k_2, k_3, l_1, l_2, s_1, s_2 \in \mathbb{R}$.

The 2-dimensional abelian subalgebras of the Lie algebra $\mathfrak{n}_{6,18}(\alpha_i, \beta_j)$ have one of the following forms:

$$\begin{aligned}
\mathfrak{h}_{16} &= \operatorname{span}\left(E_1 + k_2E_2 + k_3E_3 + k_4E_4 + k_5E_5, \quad E_6\right), \\
\mathfrak{h}_{17} &= \operatorname{span}\left(E_2 + k_1E_4 + k_2E_5 + k_3E_6, \quad E_3 + l_1E_4 + l_2E_5 + l_3E_6\right), \\
\mathfrak{h}_{18} &= \operatorname{span}\left(E_2 + k_1E_3 + k_2E_5 + k_3E_6, \quad E_4 + l_1E_5 + l_2E_6\right), \\
\mathfrak{h}_{19} &= \operatorname{span}\left(E_2 + k_1E_3 + k_2E_4 + k_3E_6, \quad E_5 + l_1E_6\right), \\
\mathfrak{h}_{20} &= \operatorname{span}\left(E_2 + k_1E_3 + k_2E_4 + k_3E_5, \quad E_6\right), \\
\mathfrak{h}_{21} &= \operatorname{span}\left(E_3 + k_1E_5 + k_2E_6, \quad E_4 + l_1E_5 + l_2E_6\right), \\
\mathfrak{h}_{22} &= \operatorname{span}\left(E_3 + k_1E_4 + k_2E_6, \quad E_5 + l_1E_6\right), \\
\mathfrak{h}_{23} &= \operatorname{span}\left(E_3 + k_1E_4 + k_2E_5, \quad E_6\right), \\
\mathfrak{h}_{24} &= \operatorname{span}\left(E_4 + k_1E_6, \quad E_5 + l_1E_6\right), \\
\mathfrak{h}_{25} &= \operatorname{span}\left(E_4 + k_1E_5, \quad E_6\right), \\
\mathfrak{h}_{26} &= \operatorname{span}\left(E_5, E_6\right),
\end{aligned}$$

where $k_1, k_2, k_3, k_4, l_1, l_2, l_3 \in \mathbb{R}$.

The vector $E_1 + k_2E_2 + k_3E_3 + k_4E_4 + k_5E_5 + E_6 \in \mathfrak{h}_{16}$ is not geodesic since for $a = f = 1$, $b = k_2$, $c = k_3$, $d = k_4$, $e = k_5$, the first equation of (4.1) gives the contradiction $1 = 0$. Hence the subalgebra $\mathfrak{h}_{16}$ is not flat totally geodesic.

For all elements of the remaining subalgebras we have $a = 0$. Since $\alpha_5 = 0$ for the metric Lie algebra $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle .,. \rangle)$ the system (4.1) of equations reduces to the equation

$$b(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) + c(\alpha_2 d + \beta_4 e + \beta_5 f) + d(\alpha_3 e + \beta_6 f) + e\alpha_4 f = 0. \tag{4.21}$$

The vector $E_5 + E_6 \in \mathfrak{h}_3 \cap \mathfrak{h}_4 \cap \mathfrak{h}_5 \cap \mathfrak{h}_{11} \cap \mathfrak{h}_{14} \cap \mathfrak{h}_{15} \cap \mathfrak{h}_{26}$ is not geodesic because for $b = c = d = 0$, $e = f = 1$ the equation of (4.21) gives the contradiction $\alpha_4 = 0$. Hence the subalgebras $\mathfrak{h}_3$, $\mathfrak{h}_4$, $\mathfrak{h}_5$, $\mathfrak{h}_{11}$, $\mathfrak{h}_{14}$, $\mathfrak{h}_{15}$, $\mathfrak{h}_{26}$ are not flat totally geodesic.

The subalgebra $\mathfrak{h}_1$ is totally geodesic precisely if for all $Y, Z \in \mathfrak{h}_1$ and $X \in \mathfrak{h}_1^{\perp} = \mathrm{span}(E_1, E_6 - \sum_{i=1}^{4} k_i E_{i+1})$ equation (1.1) is satisfied. The subalgebra $\mathfrak{h}_2$ is totally geodesic if and only if for all $Y, Z \in \mathfrak{h}_2$ and $X \in \mathfrak{h}_2^{\perp} = \mathrm{span}(E_1, E_5 - \sum_{i=1}^{3} k_i E_{i+1})$ equation (1.1) is satisfied. Since the commutation relations of the elements $(E_6 - \sum_{i=1}^{4} k_i E_{i+1})$ and the elements of $\mathfrak{h}_1$ as well as of the elements $\left(E_5 - \sum_{i=1}^{3} k_i E_{i+1}\right)$ and the elements of $\mathfrak{h}_2$ are zero, we may assume that $X = E_1$. The element $X = E_1$ lies in the orthogonal complement $\mathfrak{h}_i^{\perp}$ for all $i = 3, \cdots, 15$, too. Using the equation (1.1) we receive the following:

(1) For $Y = Z = E_5 + s_1 E_6 \in \mathfrak{h}_1 \cap \mathfrak{h}_7 \cap \mathfrak{h}_9 \cap \mathfrak{h}_{12}$ we have $2\alpha_4 s_1 = 0$ and hence $s_1 = 0$.

(2) Taking the elements $Y = E_4 + k_3 E_6$, $Z = E_5$ in $\mathfrak{h}_1 \cap \mathfrak{h}_9 \cap \mathfrak{h}_{12}$ we get $\alpha_3 + \alpha_4 k_3 = 0$ and hence $k_3 = -\frac{\alpha_3}{\alpha_4} < 0$.

(3) For $Y = Z = E_4 + k_3 E_6 \in \mathfrak{h}_1 \cap \mathfrak{h}_9 \cap \mathfrak{h}_{12}$ one has $2\beta_6 k_3 = 0$ and hence $\beta_6 = 0$.

(4) The elements $Y = E_3 + k_2 E_6$, $Z = E_4 + k_3 E_6$ in $\mathfrak{h}_1 \cap \mathfrak{h}_{12}$ yields that $\alpha_2 + \beta_5 k_3 + \beta_6 k_2 = \alpha_2 - \beta_5 \frac{\alpha_3}{\alpha_4} = 0$ and hence $\beta_5 = \frac{\alpha_2 \alpha_4}{\alpha_3} > 0$.

(5) For $Y = E_3 + k_2 E_6$, $Z = E_5$ in $\mathfrak{h}_1 \cap \mathfrak{h}_{12}$ we obtain $\beta_4 + \alpha_4 k_2 = 0$ and hence $k_2 = -\frac{\beta_4}{\alpha_4}$.

(6) For $Y = Z = E_3 + k_2 E_6 \in \mathfrak{h}_1 \cap \mathfrak{h}_{12}$, then one has $2\beta_5 k_2 = 0$ and hence $k_2 = 0$ and $\beta_4 = 0$.

(7) For the elements $Y = E_2 + k_1 E_6$, $Z = E_5$ in $\mathfrak{h}_1$ we get $\beta_2 + \alpha_4 k_1 = 0$ and therefore $k_1 = -\frac{\beta_2}{\alpha_4}$.

(8) For the elements $Y = E_2 + k_1 E_6$, $Z = E_3$ in $\mathfrak{h}_1$ we receive $\alpha_1 + \beta_5 k_1 = \alpha_1 - \frac{\alpha_2}{\alpha_3}\beta_2 = 0$ and hence $\beta_2 = \frac{\alpha_1 \alpha_3}{\alpha_2} > 0$, moreover $k_1 = -\frac{\alpha_1 \alpha_3}{\alpha_4 \alpha_2} < 0$.

(9) Taking the elements $Y = Z = E_2 + k_1 E_6 \in \mathfrak{h}_1$ one has $2\beta_3 k_1 = 0$ and hence $\beta_3 = 0$.

(10) For $Y = E_2 + k_1 E_6$, $Z = E_4 + k_3 E_6$ of $\mathfrak{h}_1$ we obtain $\beta_1 + \beta_3 k_3 + \beta_6 k_1 = \beta_1 = 0$.

Taking into account (1)–(10) the subalgebra $\mathfrak{h}_1$ is flat totally geodesic if and only if $\beta_1 = \beta_3 = \beta_4 = \beta_6 = 0$, $\beta_5 = \frac{\alpha_2 \alpha_4}{\alpha_3}$, $\beta_2 = \frac{\alpha_1 \alpha_3}{\alpha_2}$. Hence the case (1a) is proved.

(11) For $Y = Z = E_4 + k_3 E_5 \in \mathfrak{h}_2 \cap \mathfrak{h}_{10} \cap \mathfrak{h}_{13}$ we obtain $\alpha_3 k_3 = 0$ and hence $k_3 = 0$.

(12) For the elements $Y = E_4$, $Z = E_6$ of $\mathfrak{h}_2 \cap \mathfrak{h}_{10} \cap \mathfrak{h}_{13}$ we receive $\beta_6 = 0$.

(13) Taking the elements $Y = E_3 + k_2 E_5$, $Z = E_4$ in $\mathfrak{h}_2 \cap \mathfrak{h}_{13}$ one gets $\alpha_2 + \alpha_3 k_2 = 0$ which implies that $k_2 = -\frac{\alpha_2}{\alpha_3} < 0$.

(14) For $Y = Z = E_3 + k_2 E_5 \in \mathfrak{h}_2 \cap \mathfrak{h}_{13}$ we have $\beta_4 k_2 = 0$ and hence $\beta_4 = 0$.

(15) For the elements $Y = E_3 + k_2 E_5$, $Z = E_6$ of $\mathfrak{h}_2 \cap \mathfrak{h}_{13}$ one obtains $\beta_5 + \alpha_4 k_2 = 0$ which yields $\beta_5 = \frac{\alpha_4 \alpha_2}{\alpha_3}$.

(16) For $Y = E_2 + k_1 E_5$, $Z = E_3 + k_2 E_5$ in $\mathfrak{h}_2$ we receive $\alpha_1 + \beta_2 k_2 + \beta_4 k_1 = 0$ and hence $\beta_2 = \frac{\alpha_1 \alpha_3}{\alpha_2} > 0$.

(17) For $Y = Z = E_2 + k_1 E_5 \in \mathfrak{h}_2$ one has $k_1 \beta_2 = 0$ and hence $k_1 = 0$.

(18) Taking the elements $Y = E_2$, $Z = E_4$ of $\mathfrak{h}_2$ we get $\beta_1 = 0$.

(19) For the elements $Y = E_2$, $Z = E_6$ in $\mathfrak{h}_2$ we receive $\beta_3 = 0$. According to (11)–(19) the subalgebra $\mathfrak{h}_2$ is flat totally geodesic precisely if $\beta_1 = \beta_3 = \beta_4 = \beta_6 = 0$, $\beta_5 = \frac{\alpha_2 \alpha_4}{\alpha_3}$, $\beta_2 = \frac{\alpha_1 \alpha_3}{\alpha_2}$. Hence the assertion (1b) follows.

Taking into account (1)–(12) it follows from equation (4.21) that the subalgebra $\mathfrak{h}_{12}$ in (2h) as well as the subalgebra $\mathfrak{h}_{13}$ in (2i) with the conditions $\beta_4 = \beta_6 = 0$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$ are flat totally geodesic.

Now we consider the subalgebra $\mathfrak{h}_9$. Taking into account (1)–(3) we have $s_1 = \beta_6 = 0$, $k_3 = -\frac{\alpha_3}{\alpha_4}$. The element $E_2 + k_1E_3 + k_2E_6 \in \mathfrak{h}_9$ is geodesic if and only if for $b = 1$, $c = k_1$, $f = k_2$, $d = e = 0$ the equation (4.21) is satisfied. This gives the equation

$$\alpha_1k_1 + \beta_3k_2 + k_1\beta_5k_2 = 0. \tag{4.22}$$

The element $E_2 + k_1E_3 + k_2E_6 + E_5 \in \mathfrak{h}_9$ is geodesic if and only if for $b = e = 1$, $c = k_1$, $f = k_2$, $d = 0$ the equation (4.21) is valid. Using equation (4.22) we obtain the equation

$$\beta_2 + k_1\beta_4 + \alpha_4k_2 = 0. \tag{4.23}$$

Since $\alpha_4 \neq 0$ from equation (4.23) we receive

$$k_2 = -\frac{\beta_2 + k_1\beta_4}{\alpha_4}. \tag{4.24}$$

The element $E_2 + E_4 + k_1E_3 + (k_2 - \frac{\alpha_3}{\alpha_4})E_6 \in \mathfrak{h}_9$ is geodesic precisely if for $b = d = 1$, $c = k_1$, $f = k_2 - \frac{\alpha_3}{\alpha_4}$, $e = 0$ the equation (4.21) holds. Taking into account (4.22) from equation (4.21) we get

$$\beta_1 - \frac{\alpha_3}{\alpha_4}\beta_3 + k_1(\alpha_2 - \beta_5\frac{\alpha_3}{\alpha_4}). \tag{4.25}$$

If $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$, then from (4.25) we obtain $\beta_1 = \frac{\alpha_3}{\alpha_4}\beta_3$ and from (4.22) we get the second order equation (4.15). This proves case (2d).

If $\beta_5 \neq \frac{\alpha_2\alpha_4}{\alpha_3}$, then from (4.25) we have $k_1 = \frac{\beta_3\alpha_3 - \beta_1\alpha_4}{\alpha_2\alpha_4 - \alpha_3\beta_5}$. Putting this into (4.24) we obtain $k_2 = -\frac{\beta_2(\alpha_2\alpha_4 - \alpha_3\beta_5) + \beta_4(\beta_3\alpha_3 - \beta_1\alpha_4)}{\alpha_4(\alpha_2\alpha_4 - \alpha_3\beta_5)}$. Substituting the expression of k_1 and k_2 into (4.22) we receive equation (4.16). Hence case (2e) is shown.

Now we treat the subalgebra $\mathfrak{h}_{10}$. According to (11)–(12) one has $k_3 = \beta_6 = 0$. The element $E_2 + k_1E_3 + k_2E_5 \in \mathfrak{h}_{10}$ is geodesic if and only if for $b = 1$, $c = k_1$, $e = k_2$, $d = f = 0$ the equation (4.21) is satisfied. This gives the equation

$$\alpha_1k_1 + \beta_3k_2 + k_1\beta_4k_2 = 0. \tag{4.26}$$

The element $E_2 + k_1E_3 + k_2E_5 + E_4 \in \mathfrak{h}_{10}$ is geodesic if and only if for $b = d = 1$, $c = k_1$, $e = k_2$, $f = 0$ the equation (4.21) is valid. Using equation (4.26) we obtain the equation

$$\beta_1 + k_1\alpha_2 + \alpha_3k_2 = 0. \tag{4.27}$$

The element $E_2 + k_1E_3 + k_2E_5 + E_6 \in \mathfrak{h}_{10}$ is geodesic precisely if for $b = f = 1$, $c = k_1$, $e = k_2$, $d = 0$ the equation (4.21) holds. Taking into account (4.26) from equation (4.21) we get

$$\beta_3 + k_1\beta_5 + k_2\alpha_4 = 0. \tag{4.28}$$

Hence we obtain $k_2 = -\frac{\beta_3 + k_1\beta_5}{\alpha_4}$. Putting this expression of k_2 into (4.27) we receive

$$k_1(\alpha_2\alpha_4 - \alpha_3\beta_5) = \alpha_3\beta_3 - \alpha_4\beta_1. \tag{4.29}$$

If $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$, then from (4.29) we obtain $\beta_1 = \frac{\alpha_3}{\alpha_4}\beta_3$ and from (4.26) we get the second order equation (4.15). This proves case (2f).

If $\beta_5 \neq \frac{\alpha_2\alpha_4}{\alpha_3}$, then from (4.29) we have $k_1 = \frac{\beta_3\alpha_3-\beta_1\alpha_4}{\alpha_2\alpha_4-\alpha_3\beta_5}$. Putting this into (4.28) we obtain $k_2 = -\frac{\beta_3(\alpha_2\alpha_4-\alpha_3\beta_5)+\beta_5(\beta_3\alpha_3-\beta_1\alpha_4)}{\alpha_4(\alpha_2\alpha_4-\alpha_3\beta_5)}$. Substituting the expression of k_1 and k_2 into (4.26) we receive equation (4.16). Hence case (2g) is proved.

Here we deal with subalgebra $\mathfrak{h}_6$. The element $E_2 + k_1E_5 + k_2E_6 \in \mathfrak{h}_6$ is geodesic if and only if for $b = 1$, $e = k_1$, $f = k_2$, $c = d = 0$ the equation (4.21) holds. From this we obtain

$$\beta_2k_1 + \beta_3k_2 + k_1\alpha_4k_2 = 0. \tag{4.30}$$

The non-zero vector $E_3 + l_1E_5 + l_2E_6 \in \mathfrak{h}_6$ is geodesic if and only if for $c = 1$, $e = l_1$, $f = l_2$, $b = d = 0$ the equation (4.21) is satisfied. This gives

$$\beta_4l_1 + \beta_5l_2 + l_1\alpha_4l_2 = 0. \tag{4.31}$$

The element $E_4 + s_1E_5 + s_2E_6 \in \mathfrak{h}_6$ is geodesic if and only if for $d = 1$, $e = s_1$, $f = s_2$, $b = c = 0$ the equation (4.21) holds. This gives

$$\alpha_3s_1 + \beta_6s_2 + s_1\alpha_4s_2 = 0. \tag{4.32}$$

The element $E_2 + E_3 + (k_1 + l_1)E_5 + (k_2 + l_2)E_6 \in \mathfrak{h}_6$ is geodesic if and only if for $d = c = 1$, $e = k_1 + l_1$, $f = k_2 + l_2$, $d = 0$ the equation (4.21) is satisfied. Using (4.30), (4.31) we receive

$$\alpha_1 + \beta_2l_1 + \beta_3l_2 + \beta_4k_1 + \beta_5k_2 + k_1l_2\alpha_4 + l_1k_2\alpha_4 = 0. \tag{4.33}$$

The element $E_2 + E_4 + (k_1 + s_1)E_5 + (k_2 + s_2)E_6 \in \mathfrak{h}_6$ is geodesic if and only if for $d = b = 1$, $e = k_1 + s_1$, $f = k_2 + s_2$, $c = 0$ the equation (4.21) is satisfied. Using (4.30), (4.32) we get

$$\beta_1 + \beta_2s_1 + \beta_3s_2 + \alpha_3k_1 + \beta_6k_2 + k_1s_2\alpha_4 + s_1k_2\alpha_4 = 0. \tag{4.34}$$

The element $E_3 + E_4 + (l_1 + s_1)E_5 + (l_2 + s_2)E_6 \in \mathfrak{h}_6$ is geodesic if and only if for $d = c = 1$, $e = l_1 + s_1$, $f = l_2 + s_2$, $b = 0$ the equation (4.21) is satisfied. Using (4.31), (4.32) we obtain

$$\alpha_2 + \beta_4s_1 + \beta_5s_2 + \alpha_3l_1 + \beta_6l_2 + l_1s_2\alpha_4 + s_1l_2\alpha_4 = 0. \tag{4.35}$$

The element $E_2 + E_3 + E_4 + (k_1 + l_1 + s_1)E_5 + (k_2 + l_2 + s_2)E_6 \in \mathfrak{h}_6$ is geodesic if and only if for $b = d = c = 1$, $e = k_1 + l_1 + s_1$, $f = k_2 + l_2 + s_2$, $b = 0$ the equation (4.21) is satisfied. Using (4.33), (4.34), and (4.35) we receive

$$\beta_2k_1 + \beta_3k_2 + \beta_4l_1 + \beta_5l_2 + \alpha_3s_1 + \beta_6s_2 + k_1s_2 + k_1k_2\alpha_4 + l_1l_2\alpha_4 + s_1s_2\alpha_4 = 0. \tag{4.36}$$

This gives the case (2a).

Nw we deal with the subalgebra $\mathfrak{h}_7$. Taking into account (1) we get $s_1 = 0$. The element $E_2 + k_1E_4 + k_2E_6 \in \mathfrak{h}_7$ is geodesic if and only if for $b = 1$, $d = k_1$, $f = k_2$, $c = e = 0$ the equation (4.21) holds. This gives the equation

$$\beta_1k_1 + \beta_2k_2 + k_1\beta_6k_2 = 0. \tag{4.37}$$

The element $E_3 + l_1E_4 + l_2E_6 \in \mathfrak{h}_7$ is geodesic precisely if for $c = 1$, $d = l_1$, $f = l_2$, $b = e = 0$ the equation (4.21) holds. Hence we obtain the equation

$$\alpha_2l_1 + \beta_5l_2 + l_1\beta_6l_2 = 0. \tag{4.38}$$

The element $E_2 + E_3 + (k_1 + l_1)E_4 + (k_2 + l_2)E_6 \in \mathfrak{h}_7$ is geodesic if and only if for $b = c = 1$, $d = k_1 + l_1$, $f = k_2 + l_2$, $e = 0$ the equation (4.21) is valid. Using (4.37) and (4.38) we receive the equation

$$\alpha_1 + \beta_2l_1 + \beta_3l_2 + \alpha_2k_1 + \beta_5k_2 + k_1l_2\beta_6 + l_1k_2\beta_6 = 0. \tag{4.39}$$

The element $E_2 + k_1E_4 + k_2E_6 + E_5 \in \mathfrak{h}_7$ is geodesic precisely if for $b = e = 1$, $d = k_1$, $f = k_2$, $c = 0$ the equation (4.21) holds. Using (4.37) one gets the equation

$$\beta_2 + \alpha_3 k_1 + \alpha_4 k_2 = 0. \tag{4.40}$$

The element $E_3 + l_1E_4 + l_2E_6 + E_5 \in \mathfrak{h}_7$ is geodesic precisely if for $c = e = 1$, $d = l_1$, $f = l_2$, $b = 0$ the equation (4.21) is satisfied. Applying (4.38) one obtains the equation

$$\beta_4 + \alpha_3 l_1 + \alpha_4 l_2 = 0. \tag{4.41}$$

The element $E_2+E_3+E_5+(k_1+l_1)E_4+(k_2+l_2)E_6 \in \mathfrak{h}_7$ is geodesic if and only if for $b = c = e = 1$, $d = k_1 + l_1$, $f = k_2 + l_2$ the equation (4.21) is valid. Taking into account (4.37), (4.38), (4.39) we receive the equation

$$\alpha_3 k_1 + \alpha_4 k_2 = 0. \tag{4.42}$$

Comparing the equations (4.40) and (4.42) we get $\beta_2 = 0$ and equations (4.37), (4.38), (4.39), (4.40), (4.41) yield the case (2b).

Now we consider the subalgebra $\mathfrak{h}_8$. The non-zer vector $E_2 + k_1E_4 + k_2E_5 \in \mathfrak{h}_8$ is geodesic if and only if for $b = 1$, $d = k_1$, $e = k_2$, $c = f = 0$ the equation (4.21) is satisfied. This gives the equation

$$\beta_1 k_1 + \beta_2 k_2 + k_1 \alpha_3 k_2 = 0. \tag{4.43}$$

The element $E_3 + l_1E_4 + l_2E_5 \in \mathfrak{h}_8$ is geodesic if and only if for $c = 1$, $d = l_1$, $e = l_2$, $b = f = 0$ the equation (4.21) holds. Hence we receive

$$\alpha_2 l_1 + \beta_4 l_2 + l_1 l_2 \alpha_3 = 0. \tag{4.44}$$

The non-zero vector $E_2+E_3+(k_1+l_1)E_4+(k_2+l_2)E_5 \in \mathfrak{h}_8$ is geodesic if and only if for $b = c = 1$, $d = k_1 + l_1$, $e = k_2 + l_2$, $f = 0$ the equation (4.21) is satisfied. Using (4.43), (4.44) one has the following equation

$$\alpha_1 + \beta_1 l_1 + \beta_2 l_2 + \alpha_2 k_1 + \beta_4 k_2 + k_1 l_2 \alpha_3 + l_1 k_2 \alpha_3 = 0. \tag{4.45}$$

The element $E_2 + E_6 + k_1E_4 + k_2E_5 \in \mathfrak{h}_8$ is geodesic if and only if for $b = f = 1$, $d = k_1$, $e = k_2$, $c = 0$ the equation (4.21) is satisfied. Applying (4.43) one gets

$$\beta_3 + \beta_6 k_1 + \alpha_4 k_2 = 0. \tag{4.46}$$

The element $E_3 + l_1E_4 + l_2E_5 + E_6 \in \mathfrak{h}_8$ is geodesic if and only if for $c = f = 1$, $d = l_1$, $e = l_2$, $b = 0$ the equation (4.21) holds. Using (4.44) we receive

$$\beta_5 + l_1 \beta_6 + l_2 \alpha_4 = 0. \tag{4.47}$$

The non-zero vector $E_2 + E_3 + (k_1 + l_1)E_4 + (k_2 + l_2)E_5 + E_6 \in \mathfrak{h}_8$ is geodesic if and only if for $c = b = f = 1$, $d = k_1 + l_1$, $e = k_2 + l_2$ the equation (4.21) is satisfied. This gives

$$\begin{gathered} \alpha_1 + \beta_1 k_1 + \beta_1 l_1 + \beta_2 k_2 + \beta_2 l_2 + \beta_3 + \alpha_2 k_1 + \alpha_2 l_1 + \beta_4 k_2 + \beta_4 l_2 + \beta_5 + k_1 k_2 \alpha_3 + \\ k_1 l_2 \alpha_3 + l_1 k_2 \alpha_3 + l_1 l_2 \alpha_3 + k_1 \beta_6 + l_1 \beta_6 + k_2 \alpha_4 + l_2 \alpha_4 = 0. \end{gathered} \tag{4.48}$$

Using the equations (4.43), (4.44), (4.45), (4.46), and (4.47), the equation (4.48) holds. Therefore the subalgebra $\mathfrak{h}_8$ is proved. This gives the case (2c)

The subalgebra $\mathfrak{h}_{24}$ coincides with the subalgebra $\mathfrak{h}_9$ in $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$. Hence the case (3a) is valid. Furthermore, the subalgebra $\mathfrak{h}_{25}$ coincides with the subalgebra $\mathfrak{h}_5$ in $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$. Therefore the case (3b) is shown.

The non-zero vector $E_4+l_1E_5+l_2E_6 \in \mathfrak{h}_{18}\cap\mathfrak{h}_{21}$ coincides with the element $E_4+l_1E_5+l_2E_6 \in \mathfrak{h}_1$ in the filiform metric Lie algebra $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle .,. \rangle)$. It is geodesic if the equation

$$\alpha_3 l_1 + \beta_6 l_2 + l_1 \alpha_4 l_2 = 0. \tag{4.49}$$

holds. Now we treat the subalgebra $\mathfrak{h}_{18}$. The element $E_2 + k_1E_3 + k_2E_5 + k_3E_6 \in \mathfrak{h}_{18}$ is geodesic if for $d = 0, b = 1, c = k_1, e = k_2, f = k_3$ the equation (4.21) is valid. Hence we get

$$\alpha_1 k_1 + \beta_2 k_2 + \beta_3 k_3 + k_1\beta_4 k_2 + k_1\beta_5 k_3 + k_2\alpha_4 k_3 = 0. \tag{4.50}$$

In addition, the non-zero vector $E_2 + E_4 + k_1E_3 + (k_2 + l_1)E_5 + (k_3 + l_2)E_6 \in \mathfrak{h}_{18}$ is geodesic if for $b = d = 1, c = k_1, e = k_2 + l_1, f = k_3 + l_2$ the system (4.21) is satisfied. Therefore we obtain

$$\begin{aligned} \alpha_1 k_1 + \beta_1 + \beta_2 k_2 + \beta_2 l_1 + \beta_3 k_3 + \beta_3 l_2 + k_1\alpha_2 + k_1\beta_4 k_2 + k_1\beta_4 l_1 + k_1\beta_5 k_3 + k_1\beta_5 l_2 + \\ \alpha_3 k_2 + \alpha_3 l_1 + \beta_6 k_3 + \beta_6 l_2 + k_2k_3\alpha_4 + k_2l_2\alpha_4 + l_1k_3\alpha_4 + l_1l_2\alpha_4 = 0. \end{aligned} \tag{4.51}$$

Due to the equations (4.49) and (4.50), equation (4.51) becomes

$$\beta_1 + \beta_2 l_1 + \beta_3 l_2 + \alpha_2 k_1 + k_1\beta_4 l_1 + k_1\beta_5 k_3 + k_1\beta_5 l_2 + \alpha_3 k_2 + \beta_6 k_3 + \alpha_4 k_2 l_2 + \alpha_4 l_1 k_3 = 0.$$

This proves case (3c).

Here we consider the case $\mathfrak{h}_{21}$. The element $E_3 + k_1E_5 + k_2E_6 \in \mathfrak{h}_{21}$ is geodesic if for $b = d = 0, c = 1, e = k_1, f = k_2$ the equation (4.21) is satisfied. From this we obtain

$$\beta_4 k_1 + \beta_5 k_2 + k_1\alpha_4 k_2 = 0. \tag{4.52}$$

Moreover, the non-zero vector $E_3 + E_4 + (k_1 + l_1)E_5 + (k_2 + l_2)E_6 \in \mathfrak{h}_{21}$ is geodesic if for $b = 0, c = 1, d = 1, e = k_1 + l_1, f = k_2 + l_2$ the equation (4.21) is valid. This gives

$$\begin{aligned} \alpha_2 + \beta_4 k_1 + \beta_4 l_1 + \beta_5 k_2 + \beta_5 l_2 + \alpha_3 k_1 + \alpha_3 l_1 + \beta_6 k_2 + \beta_6 l_2 + \\ k_1k_2\alpha_4 + k_1l_2\alpha_4 + l_1k_2\alpha_4 + l_1l_2\alpha_4 = 0. \end{aligned} \tag{4.53}$$

Exploiting equations (4.49) and (4.52), equation (4.53) reduces to

$$\alpha_2 + \beta_4 l_1 + \beta_5 l_2 + \alpha_3 k_1 + \beta_6 k_2 + k_1 l_2\alpha_4 + l_1 k_2\alpha_4 = 0.$$

This proves the case (3d).

The non-zero vector $E_5 + l_1E_6 \in \mathfrak{h}_{19} \cap \mathfrak{h}_{22}$ coincides with the element $E_5 + l_1E_6 \in \mathfrak{h}_2$ in the filiform metric Lie algebra $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle .,. \rangle)$. It is geodesic if one has $l_1 = 0$. Now we deal with the subalgebra $\mathfrak{h}_{19}$. The element $E_2 + k_1E_3 + k_2E_4 + k_3E_6 \in \mathfrak{h}_{19}$ is geodesic precisely if for $a = e = 0, b = 1, c = k_1, d = k_2, f = k_3$ the equation (4.21) is satisfied. This yields

$$\alpha_1 k_1 + \beta_1 k_2 + \beta_3 k_3 + k_1\alpha_2 k_2 + k_1\beta_5 k_3 + k_2\beta_6 k_3 = 0. \tag{4.54}$$

Furthermore, the non-zero vector $E_2 + k_1E_3 + k_2E_4 + k_3E_6 + E_5 \in \mathfrak{h}_{19}$ is geodesic if and only if for $b = e = 1, c = k_1, d = k_2, f = k_3$ the equation (4.21) holds. It follows from the fifth equation of (4.1) that

$$\alpha_1 k_1 + \beta_1 k_2 + \beta_2 + \beta_3 k_3 + k_1\alpha_2 k_2 + k_1\beta_4 + k_1\beta_5 k_3 + k_2\alpha_3 + k_2\beta_6 k_3 + \alpha_4 k_3 = 0. \tag{4.55}$$

Taking into account (4.54), equation (4.55) can be written as follows

$$\beta_2 + k_1\beta_4 + k_2\alpha_3 + \alpha_4 k_3 = 0. \tag{4.56}$$

From equation (4.56) we obtain that $k_3 = -\frac{\beta_2 + k_1\beta_4 + k_2\alpha_3}{\alpha_4}$. Putting this expression into equation (4.54) we receive equation (4.17). This proves the case (3e).

Next we consider the subalgebra $\mathfrak{h}_{22}$. The non-zero vector $E_3 + k_1E_4 + k_2E_6 \in \mathfrak{h}_{22}$ is geodesic precisely if for $b = e = 0, c = 1, d = k_1, f = k_2$ the equation 4.21 holds. This gives

$$\alpha_2k_1 + \beta_5k_2 + k_1\beta_6k_2 = 0. \tag{4.57}$$

Additionally, the element $E_3 + k_1E_4 + k_2E_6 + E_5 \in \mathfrak{h}_{22}$ is geodesic if and only if for $b = 0$, $c = 1, d = k_1, f = k_2$ the equation 4.21 is satisfied. From this it follows that

$$\alpha_2k_1 + \beta_4 + \beta_5k_2 + k_1\alpha_3 + k_1\beta_6k_2 + \alpha_4k_2 = 0. \tag{4.58}$$

Comparing with equation (4.57), equation (4.58) reduces to

$$\beta_4 + \alpha_3k_1 + \alpha_4k_2 = 0. \tag{4.59}$$

From (4.59) one has $k_2 = -\frac{\beta_4+\alpha_3k_1}{\alpha_4}$. Substituting this expression into (4.57) we have the second order equation (4.18) for k_1. This gives the case (3f).

Now we consider the subalgebra $\mathfrak{h}_{23}$. The element $E_3 + k_1E_4 + k_2E_5 \in \mathfrak{h}_{23}$ is geodesic precisely if for $b = f = 0, c = 1, d = k_1, e = k_2$ the system (4.1) of equation is satisfied. Hence we receive

$$\alpha_2k_1 + \beta_4k_2 + k_1\alpha_3k_2 = 0. \tag{4.60}$$

Moreover, the non-zero vector $E_3 + k_1E_4 + k_2E_5 + E_6 \in \mathfrak{h}_7$ is geodesic if and only if for $b = 0$, $c = f = 1, d = k_1, e = k_2$ the equation (4.21) holds. This gives

$$\alpha_2k_1 + \beta_4k_2 + \beta_5 + k_1\alpha_3k_2 + k_1\beta_6 + \alpha_4k_2 = 0. \tag{4.61}$$

Using equation (4.60) equation (4.61) reduces to

$$\beta_5 + \beta_6k_1 + \alpha_4k_2 = 0. \tag{4.62}$$

From (4.62) we obtain $k_2 = -\frac{\beta_5+\beta_6k_1}{\alpha_4}$. Putting this expression into (4.60) we have the second order equation (4.18) for k_1. Thus, the case (3g) is proved.

Now we deal with the subalgebra $\mathfrak{h}_{20}$. The element $E_2 + k_1E_3 + k_2E_4 + k_3E_5 \in \mathfrak{h}_{20}$ is geodesic precisely if for $f = 0, b = 1, c = k_1, d = k_2, e = k_3$ the equation (4.21) is satisfied. From this we get

$$\alpha_1k_1 + \beta_1k_2 + \beta_2k_3 + k_1\alpha_2k_2 + k_1\beta_4k_3 + k_2\alpha_3k_3 = 0. \tag{4.63}$$

Additionally, the non-zero vector $E_2 + k_1E_3 + k_2E_4 + k_3E_5 + E_6 \in \mathfrak{h}_4$ is geodesic if and only if for $b = f = 1$, $c = k_1, d = k_2, e = k_3$, the equation (4.21) holds. Hence one obtains

$$\alpha_1k_1 + \beta_1k_2 + \beta_2k_3 + \beta_3 + k_1\alpha_2k_2 + k_1\beta_4k_3 + k_1\beta_5 + k_2\alpha_3k_3 + k_2\beta_6 + k_3\alpha_4 = 0. \tag{4.64}$$

Applying (4.63) equation (4.64) reduces to

$$\beta_3 + k_1\beta_5 + k_2\beta_6 + \alpha_4k_3 = 0. \tag{4.65}$$

From (4.65) we obtain $k_3 = -\frac{\beta_3+k_1\beta_5+k_2\beta_6}{\alpha_4}$. Putting this expression into (4.63) we obtain equation ((4.19)). Hence the case (3h) is shown.

Finally, we consider the subalgebra $\mathfrak{h}_{17}$. The non-zero vector $E_2 + k_1E_4 + k_2E_5 + k_3E_6 \in \mathfrak{h}_{17}$ is geodesic precisely if for $c = 0, b = 1, d = k_1, e = k_2, f = k_3$ the equation (4.21) is valid. Hence we get

$$\beta_1k_1 + \beta_2k_2 + \beta_3k_3 + k_1\alpha_3k_2 + k_1\beta_6k_3 + k_2\alpha_4k_3 = 0. \tag{4.66}$$

The element $E_3 + l_1E_4 + l_2E_5 + l_3E_6 \in \mathfrak{h}_{17}$ is geodesic if and only if for $b = 0$, $c = 1, d = l_1, e = l_2, f = l_3$ the equation (4.21) holds. From this we receive

$$\alpha_2l_1 + \beta_4l_2 + \beta_5l_3 + l_1\alpha_3l_2 + l_1\beta_6l_3 + l_2\alpha_4l_3 = 0, \tag{4.67}$$

Furthermore, the non-zero vector $E_2 + E_3 + (k_1 + l_1)E_4 + (k_2 + l_2)E_5 + (k_3 + l_3)E_6 \in \mathfrak{h}_{17}$ is geodesic precisely if for $b = c = 1$, $d = k_1 + l_1, e = k_2 + l_2, f = k_3 + l_3$ the equation (4.21) is satisfied. Therefore one obtains

$$\begin{aligned}\alpha_1 + \beta_1 k_1 + \beta_1 l_1 + \beta_2 k_2 + \beta_2 l_2 + \beta_3 k_3 + \beta_3 l_3 + \alpha_2 k_1 + \alpha_2 l_1 + \beta_4 k_2 + \beta_4 l_2 + \beta_5 k_3 +\\ \beta_5 l_3 + k_1 k_2 \alpha_3 + k_1 l_2 \alpha_3 + l_1 k_2 \alpha_3 + l_1 l_2 \alpha_3 + k_1 k_3 \beta_6 + k_1 l_3 \beta_6 + l_1 k_3 \beta_6 + l_1 l_3 \beta_6 +\\ k_2 k_3 \alpha_4 + k_2 l_3 \alpha_4 + l_2 k_3 \alpha_4 + l_2 l_3 \alpha_4 = 0.\end{aligned} \tag{4.68}$$

Taking into account (4.66) and (4.67), equation (4.68) reduces to

$$\begin{aligned}\alpha_1 + \beta_1 l_1 + \beta_2 l_2 + \beta_3 l_3 + \alpha_2 k_1 + \beta_4 k_2 + \beta_5 k_3 + k_1 l_2 \alpha_3 + l_1 k_2 \alpha_3 +\\ k_1 l_3 \beta_6 + l_1 k_3 \beta_6 + k_2 l_3 \alpha_4 + l_2 l_3 \alpha_4 = 0.\end{aligned} \tag{4.69}$$

This gives (3i). Hence Theorem 4.3 is proved. □

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Declarations

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References

[1] G. Cairns, A. H. Galić, and Y. Nikolayevsky, Totally geodesic subalgebras of nilpotent Lie algebras, *J. Lie Theory*, 23 (2013), no. 4, 1023-1049

[2] G. Cairns, A. H. Galić, and Y. Nikolayevsky, Totally geodesic subalgebras of filiform nilpotent Lie algebras, *J. Lie Theory*, 23 (2013), no. 4, 1051-1074

[3] D. V. Millionschikov, Graded filiform Lie algebras and symplectic nilmanifolds, in: Geometry, Topology, and Mathematical Physics *Amer. Math. Soc. Transl. Ser. 2*, 212 (2004)

[4] P. Eberlein, Geometry of 2-step nilpotent groups with a left invariant metric II, *Trans. Amer. Math. Soc.* 343 (1994), no. 2, 805–828

[5] P. Eberlein, Left invariant geometry of Lie groups, *Cubo, CUBO, A Mathematical Journal*, 6 (2004), no. 1, 427–510

[6] Á. Figula and P.T. Nagy, Isometry classes of simply connected nilmanifolds, *J. Geom. Phys.* 132 (2018), 370–381

[7] Á. Figula and S. A. Abbas, Isometry groups of six-dimensional filiform nilmanifold, *Int. J. Group Theory*, 12 (2023), no. 2, 67-80

[8] P. T. Nagy and S. Homolya, Geodesic vectors and subalgebras in two-step nilpotent metric Lie algebras, *Adv. Geom.*, 15 (2015), no. 1, 121–126

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