A class of solvable binary Lie algebras of dimension 5

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Abstract

We study anti-commutative algebras, which are extensions of a one-dimensional algebra by a four-dimensional nilpotent Lie algebra and at the same time extensions of the two-dimensional non-abelian Lie algebra by an abelian algebra. These algebras, just like the five-dimensional solvable Malcev algebras, have a flag of subalgebras and can therefore be considered their closest relatives. We characterize binary Lie algebras in this class, called Malcev-like algebras, playing interesting role in non-associative Lie theory. We find normal forms of their multiplications, and determine their isomorphism classes.

1 Introduction

The power series terms of local loop multiplications at the unit element define several multilinear operations in the tangent space of the local loop. Particularly interesting is the case when these multilinear operations are determined only by a bilinear operation. The simplest such loops are the Moufang loops and the more general diassociative (or alternative) loops (any two elements generate a subgroup). The corresponding generalizations of Lie algebras are the Malcev algebras and the binary Lie algebras (any two elements generate a Lie subalgebra), which play an important role in non-associative Lie theory. Their outstanding property is that the classical Campbell-Hausdorff power series formed by their bilinear operation determines the analytic local loop multiplication, the tangent algebra of which is the given Malcev algebra or the binary Lie algebra, (cf. [1], Ch. 4. §5 in [6], [15]). Moreover, any local analytic Moufang loop can be uniquely embedded into a connected, simply connected

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global loop (cf. [12], [13], [19]), but this statement does not generally apply to local analytic diassociative loops (cf. [9]).

The correspondence between local analytic Moufang and diassociative loops and their tangent algebras was the main motivation of A. I. Malcev for introducing in 1955 the concepts of Malcev algebras (called them Moufang-Lie algebras) and binary Lie algebras. His work [17] became the starting point for the foundation of non-associative Lie theory and had a major impact on the development of mathematics (cf. [20]). Over the past decades, the theory of differentiable Moufang loops and their tangent Malcev algebras has developed significantly almost to the level of the theory of Lie groups and algebras, but there are many interesting open questions about diassociative loops and their tangent binary Lie algebras. Malcev algebras and the corresponding Moufang loops of dimension ≤ 5 were classified by E. N. Kuzmin in [14], 1970: there is one non-Lie Malcev algebra in dimension 4 and there are one nilpotent algebra, five solvable algebras and one non-solvable algebra in dimension 5. The systematic study of binary Lie algebras began with the works of A. T. Gainov [3], 1957. He found that the minimum of the dimension of non-Lie binary Lie algebras is 4 and classified these algebras in [4], 1963. The study of low-dimensional algebras (cf. for example [4], [14], [16]) provided models for further research and thus contributed to an important advancement in the general theory of Malcev and binary Lie algebras (e.g. [2], [7], [8], [10], [23], [24]). The history of non-associative Lie theory is well described in the paper [22] by L. V. Sabinin.

Binary Lie algebras of dimension 4 are extensions of a one-dimensional algebra by a nilpotent Lie algebra and at the same time are extensions of a two-dimensional Lie algebra by an abelian algebra. We classified in [2] solvable 4-dimensional anti-commutative algebras with analogous decomposition and determined their automorphism groups. The solvable 5-dimensional Malcev algebras have very similar decomposition properties as the 4-dimensional binary Lie algebras, they are extensions of a 1-dimensional algebra by a nilpotent Lie algebra and simultaneously extensions of a two-dimensional Lie algebra by an abelian algebra. We call the 5-dimensional anti-commutative algebras that have the same decomposition properties as solvable Malcev algebras *Malcev-like algebras* or $\mathcal{M}^5$-*algebras*, these algebras can be regarded as close relatives of solvable Malcev algebras. The aim of this work is to study binary Lie algebras in the class of Malcev-like algebras, and find their normal forms and isomorphism classes over a commutative field $\mathbb{K}$ of characteristic 0. The study of this natural class of low-dimensional binary Lie algebras can make a significant contribution to the development of Lie theory of analytic diassociative loops and their tangent algebras.

The derived algebras of non-commutative extensions of a two-dimensional Lie algebra by an abelian algebra are, according to [7], nilpotent algebras. The 4-dimensional nilpotent Lie algebras have 3 isomorphism classes. If $\{c_{ij}^k(X)\}$ is the system of structure constants given by $e_r e_s = \sum_t c_{rs}^t e_t$ with respect to a basis $\{e_0, e_1, \ldots, e_4\}$, then the function

$$\gamma(x^0, \ldots, x^4; y^0, \ldots, y^4) = \sum_{r,s=0}^{4} (c_{rs}^0 x^r y^s, \ldots, c_{rs}^4 x^r y^s)$$

describes the multiplication expressed by the structure constants and the coordinates. The general linear group $\mathrm{GL}(5, \mathbb{K})$ acts $\gamma \mapsto g \circ \gamma$ on the multiplication functions γ as

$$g \circ \gamma(x, y) = g^{-1}\gamma(gx, gy), \quad x, y \in \mathbb{K}^5 \tag{1}$$

via change of basis by $g \in \mathrm{GL}(5, \mathbb{K})$. The isomorphism classes of anti-commutative algebras correspond to the orbits of the action (1), namely, γ_1 and γ_2 determine isomorphic algebras if and only if $g \circ \gamma_1 = \gamma_2$ for some $g \in \mathrm{GL}(5, \mathbb{K})$. To get a linear instead of the quadratic equation for g, we reduce the action (1) to a transitive subgroup of $\mathrm{GL}(5, \mathbb{K})$ on the multiplication functions using the subalgebra flag of the algebra. We obtain normal forms of multiplications as representatives of isomorphism classes by choosing an appropriate element of each trajectory.

We introduce in §2 the basic concepts on binary Lie and Malcev algebras. In §3 we define Malcev-like algebras, their types in terms of the four-dimensional nilpotent Lie algebras isomorphic to the derived ideal. §4 is devoted to the study of the group of partial automorphisms acting simply transitively on distinguished bases, isomorphism classes of Malcev-like algebras corresponding to the orbits of the action of the group of partial isomorphisms, and the linear systems of equations for partial isomorphisms giving isomorphisms. In §5 we classify Malcev-like algebras with codimension one abelian ideal. It turns out that these algebras are Lie algebras, the constructions of the corresponding global matrix Lie groups (e.g. [5], §3) can serve in the future for the study of diassociative global loops corresponding to Malcev-like binary Lie algebras. In §6 we characterize Lie, Malcev and binary Lie algebras such that the derived ideal contains as subideal the 3-dimensional Heisenberg algebra. In §7 we investigate the ideal properties of subalgebras of these binary Lie algebras and determine the normal forms of the multiplications with different such properties. In §8 Malcev-like algebras with derived ideal isomorphic to the filiform nilpotent Lie algebra are investigated. §9 is devoted to the determination of normal forms of the multiplications of binary Lie algebras with filiform derived ideal. In §10 we summarize the results obtained for the determination and classification of 5-dimensional Malcev-like binary Lie algebras. In Appendix we describe the general anti-commutative split extensions (or semidirect sums) of the two-dimensional Lie algebra by abelian algebras and give a necessary and sufficient condition for anti-commutative semidirect sum to be binary Lie algebra, respectively Malcev algebra.

2 Preliminaries

Let $\mathbb{K}$ be a commutative field of characteristic zero and denote $\mathbb{K}^\circ$ the multiplicative group of the field $\mathbb{K}$. We consider a (non-associative) anti-commutative algebra $\mathfrak{g}$ over the field $\mathbb{K}$. The multiplication of the elements $x, y \in \mathfrak{g}$ is denoted by $x \cdot y$, but the multiplication symbol will be omitted whenever it does not cause confusion. The multiplication symbol $\cdot$ is considered less binding than the juxtaposition, i.e. $xy \cdot u$ is a short form of $(x \cdot y) \cdot u$. $\mathrm{Aut}(\mathfrak{g})$ denotes the group of automorphisms, $\mathrm{End}(\mathfrak{g})$ the algebra of linear maps of $\mathfrak{g}$. $L_x, R_x : \mathfrak{g} \to \mathfrak{g}$ are the left and right multiplication maps: $L_x(t) = xt$ and $R_x(t) = tx,\ x, t \in \mathfrak{g}$. The maps L_x, R_x differ only by sign, a subalgebra in $\mathfrak{g}$ is an *ideal* if it is invariant under the left (and hence the right) multiplication maps. The *derived algebra* $\mathfrak{g}' = \mathfrak{g} \cdot \mathfrak{g}$ is an ideal of $\mathfrak{g}$, the second derived algebra $\mathfrak{g}'' = \mathfrak{g}' \cdot \mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$. The Jacobian $\mathcal{J} : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the map $\mathcal{J} : (x, y, z) \mapsto xy \cdot z + zx \cdot y + yz \cdot x$, the Jacobi identity is $\mathcal{J}(x, y, z) = 0$, $x, y, z \in \mathfrak{g}$. For a Lie algebra $\mathfrak{g}$ the ideal $\mathcal{C}(\mathfrak{g}) = \{z \in \mathfrak{g}; z\mathfrak{g} = \mathfrak{g}z = 0\}$ is the center in $\mathfrak{g}$. An anti-commutative algebra $\mathfrak{g}$ is a *binary Lie algebra* if the identity

$$\mathcal{J}(x, y, xy) = (y \cdot xy)x + (xy \cdot x)y = 0, \quad x, y \in \mathfrak{g} \tag{2}$$

holds. $\mathfrak{g}$ is a *Malcev algebra* if the identity

$$xy \cdot xz = (xy \cdot z)x + (yz \cdot x)x + (zx \cdot x)y, \quad x, y, z \in \mathfrak{g} \tag{3}$$

or equivalently the *Sagle identity*

$$(xy \cdot z)t + (yz \cdot t)x + (zt \cdot x)y + (tx \cdot y)z = xz \cdot yt, \quad x, y, z, t \in \mathfrak{g} \tag{4}$$

is satisfied in $\mathfrak{g}$.

Remark 2.1. Putting $y \mapsto z$ in the Malcev identity (3) or $y \mapsto z$, $x \mapsto t$ in the Sagle identity (4) we obtain identities equivalent to (2) characterizing the binary Lie algebras.

The anti-commutative algebra $\mathfrak{g}$ is called *decomposable* if it is the direct sum of subalgebras. An *extension* $\mathfrak{c}$ of an anti-commutative algebra $\mathfrak{b}$ by an anti-commutative algebra $\mathfrak{a}$ is a short exact sequence

$$0 \to \mathfrak{a} \xrightarrow{\iota} \mathfrak{c} \xrightarrow{\pi} \mathfrak{b} \to 0, \tag{5}$$

where $\iota(\mathfrak{a})$ is an ideal of $\mathfrak{c}$ and π induces an isomorphism of the factor algebra $\mathfrak{c}/\iota(\mathfrak{a})$ to $\mathfrak{b}$. The extension (5) is *splitting* if there exists a subalgebra $\widetilde{\mathfrak{b}}$ of $\mathfrak{c}$ such that $\pi(\widetilde{\mathfrak{b}})$ is isomorphic to $\mathfrak{b}$, in this case we say that $\mathfrak{c}$ is the semidirect sum of $\mathfrak{a}$ and $\widetilde{\mathfrak{b}}$. In the following we will deal with anti-commutative semidirect sums $\mathfrak{a} \oplus_l \mathfrak{b}$ of Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$ defined by the operation

$$(\xi, X) \cdot (\eta, Y) = \big(\xi\eta, XY + l_\xi(Y) - l_\eta(X)\big), \quad \xi, \eta \in \mathfrak{a}, \ X, Y \in \mathfrak{b} \tag{6}$$

on $\mathfrak{a} \oplus \mathfrak{b}$, where $l : \mathfrak{a} \times \mathfrak{b} \to \mathfrak{b}$ is a given bilinear map.
For $\mathcal{B} = \{e_0, e_1, \ldots, e_{n-1}\}$, $\hat{\mathcal{B}} = \{\hat{e}_0, \hat{e}_1, \ldots, \hat{e}_{n-1}\}$ bases of an anti-commutative algebra $\mathfrak{g}$ let $\mathbb{K}^n(\mathcal{B})$ and $\mathbb{K}^n(\hat{\mathcal{B}})$ be the anti-commutative algebras on the vector space $\mathbb{K}^n$, such that the linear coordinate maps $\phi_{\mathcal{B}} : \mathfrak{g} \to \mathbb{K}^n$ and $\phi_{\hat{\mathcal{B}}} : \mathfrak{g} \to \mathbb{K}^n$ are isomorphisms. The composition $\phi_{\hat{\mathcal{B}}} \circ \phi_{\mathcal{B}}^{-1} : \mathbb{K}^n(\mathcal{B}) \to \mathbb{K}^n(\hat{\mathcal{B}})$ is an isomorphism corresponding to the change of the basis $\mathcal{B}$ to $\hat{\mathcal{B}}$. It follows that the general linear group $\mathrm{GL}(5, \mathbb{K})$ acts by (1) on the multiplication functions via change of basis and the isomorphism classes correspond to the orbits of this action.
In what follows, operations of algebras are often given by listing the non-zero products with respect to an appropriately chosen basis $\{e_0, e_1, \ldots, e_{n-1}\}$. Accordingly, the five-dimensional non-decomposable solvable Malcev algebras over $\mathbb{K}$ are classified by the following multiplications (cf. [14]):

$$\begin{aligned}
\mathfrak{m}_1 :& \quad e_1e_2 = e_4,\ e_0e_1 = e_1,\ e_0e_2 = e_2,\ e_0e_3 = -e_3,\ e_0e_4 = e_3 + 2e_4, \\
\mathfrak{m}_2 :& \quad e_1e_2 = e_4,\ e_0e_1 = e_1,\ e_0e_2 = e_2,\ e_0e_3 = -e_2 - 2e_3,\ e_0e_4 = -e_4, \\
\mathfrak{m}_3 :& \quad e_1e_2 = e_4,\ e_0e_1 = e_1,\ e_0e_2 = e_2,\ e_0e_3 = -e_3 - e_4,\ e_0e_4 = -e_4, \\
\mathfrak{m}_4 :& \quad e_1e_2 = e_4,\ e_0e_1 = e_1,\ e_0e_2 = e_2 + e_3,\ e_0e_3 = e_3,\ e_0e_4 = -e_4, \\
\mathfrak{m}_5(\gamma) :& \quad e_1e_2 = e_4,\ e_0e_1 = e_1,\ e_0e_2 = e_2,\ e_0e_3 = \gamma e_3,\ e_0e_4 = -e_4.
\end{aligned} \tag{7}$$

3 $\mathcal{M}^5$-algebras and distinguished bases

Five-dimensional non-decomposable solvable Malcev algebras (7) belong to the following class of solvable anti-commutative algebras:

Definition 3.1. A 5-dimensional non-decomposable anti-commutative algebra $\mathfrak{c}$ is called *Malcev-like algebra*, or shortly $\mathcal{M}^5$*-algebra* if

(i) the derived algebra $\mathfrak{c}'$ is a 4-dimensional nilpotent Lie algebra,

(ii) $\mathfrak{c}$ is an anti-commutative semidirect sum $\mathfrak{l}_2 \oplus_l \mathfrak{i}$ of $\mathfrak{l}_2$, where $\mathfrak{i}$ is an abelian ideal.

The derived algebra $\mathfrak{c}'$ is a nilpotent ideal in $\mathfrak{c}$, hence $\mathfrak{c}$ is a solvable anti-commutative algebra, which is isomorphic to the semidirect sum $\mathbb{K}e_0 \oplus_{\lambda_0} \mathfrak{c}'$, where $\lambda_0 = L_{e_0}|_{\mathfrak{c}'} : \mathfrak{c}' \to \mathfrak{c}'$. We notice, that according to [7], the derived algebra of a solvable binary Lie algebra is nilpotent, hence for such algebras the condition (i) is equivalent to the assumption that the derived algebra is 4-dimensional. Any 4-dimensional nilpotent Lie algebra is isomorphic to one of the following three algebras:

abelian $\mathfrak{a}$,

direct sum $\mathfrak{n} = \mathfrak{n}_3 \oplus \mathbb{K}$, where $\mathfrak{n}_3$ is the 3-dimensional Heisenberg Lie algebra, with non-vanishing multiplication $e_1 e_2 = e_4$,

filiform nilpotent Lie algebra $\mathfrak{f}$ with non-vanishing multiplications $e_1 e_2 = e_3$, $e_1 e_3 = e_4$.

Clearly, the subspace spanned by the vectors e_2, e_3, e_4 is an abelian ideal in $\mathfrak{c}'$, but the ideal property of $\mathfrak{i}$ in $\mathfrak{c}$ is guaranteed by the condition (ii) of Definition 3.1.

Definition 3.2. An $\mathcal{M}^5$-algebra $\mathfrak{c} = \mathfrak{l}_2 \oplus_l \mathfrak{i}$ is called $\mathcal{M}^5(\mathfrak{t})$*-algebra* or $\mathcal{M}^5$*-algebra of type* $\mathfrak{t}$, if $\mathfrak{c}' \cong \mathfrak{t}$, where $\mathfrak{t} \in \{\mathfrak{a}, \mathfrak{n}, \mathfrak{f}\}$.
A basis $\{e_0, e_1, e_2, e_3, e_4\}$ in an $\mathcal{M}^5$-algebra $\mathfrak{c}$ is called *distinguished*, if $\{e_1, e_2, e_3, e_4\}$ is a basis of $\mathfrak{c}'$ such that the multiplication in $\mathfrak{c}'$ satisfies

$\mathcal{M}^5(\mathfrak{a})$-algebra: abelian

$\mathcal{M}^5(\mathfrak{n})$-algebra: $e_1 e_2 = e_4$,

$\mathcal{M}^5(\mathfrak{f})$-algebra: $e_1 e_2 = e_3$, $e_1 e_3 = e_4$.

A distinguished basis $\{e_0, e_1, e_2, e_3, e_4\}$ in $\mathfrak{c} = \mathfrak{l}_2 \oplus_l \mathfrak{i}$ is said to be *special* if $e_0 e_1 = e_1$ holds and the vectors e_2, e_3, e_4 generate an abelian ideal.

For any distinguished basis $\{e_0, e_1, e_2, e_3, e_4\}$ of an $\mathcal{M}^5$-algebra the vectors e_2, e_3, e_4 generate an abelian subalgebra, which is not necessarily an ideal.

Lemma 3.1. An anti-commutative algebra $\mathfrak{c}$ is an $\mathcal{M}^5$-algebra if and only if

(i) the derived algebra $\mathfrak{c}'$ is a 4-dimensional nilpotent Lie algebra,

(ii) $\mathfrak{c}$ has a special distinguished basis.

Proposition 3.2. Let $\mathfrak{c}$ be an $\mathcal{M}^5(\mathfrak{t})$-algebra, where $\mathfrak{t} \in \{\mathfrak{a}, \mathfrak{n}, \mathfrak{f}\}$, and $\{e_0, e_1, e_2, e_3, e_4\}$ a special distinguished basis of $\mathfrak{c}$. The matrix $\mathbf{X}$ of the map $\lambda_0 = L_{e_0}|_{\mathfrak{c}'} : \mathfrak{c}' \to \mathfrak{c}'$ has the form

$$\mathbf{X} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0}^t & X \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0\,0\,0 \end{bmatrix}, \quad l_0 = X = \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix}, \tag{8}$$

where $l_0 = X$ is the matrix of the map $L_{e_0}|_{\mathfrak{i}} : \mathfrak{i} \to \mathfrak{i}$ induced on the abelian ideal $\mathfrak{i}$ satisfying the condition

$$\det(X) \neq 0; \quad \operatorname{rank} \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \end{bmatrix} = 2; \quad \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \tag{9}$$

for $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{f}$, respectively. The matrix of $l_k = L_{e_k}|_{\mathfrak{i}} : \mathfrak{i} \to \mathfrak{i}$, $k = 2, 3, 4$ is the zero matrix. The matrix Y of the map $l_1 = L_{e_1}|_{\mathfrak{i}} : \mathfrak{i} \to \mathfrak{i}$ is given by

$$Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{10}$$

for $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{f}$, respectively.

Proof. The condition (10) follows from the definition of distinguished bases. Since for a special distinguished basis one has $e_0 e_1 = e_1$ and e_2, e_3, e_4 generate an abelian ideal, the matrix of the map $l_0 = L_{e_0}|_{\mathfrak{i}} : \mathfrak{i} \to \mathfrak{i}$ has the form (8). The conditions (9) given for the matrix X hold since $\dim(\mathfrak{c}') = 4$. □

We parametrize $\mathcal{M}^5(\mathfrak{t})$-algebras, $\mathfrak{t} \in \{\mathfrak{a}, \mathfrak{n}, \mathfrak{f}\}$, defined on the vector space $\mathbb{K}^5$ having the canonical basis $\{e_0, e_1, e_2, e_3, e_4\}$ of $\mathbb{K}^5$ as a special distinguished basis by 3×3 matrices.

Proposition 3.3. Let $X = \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix}$ be an arbitrary matrix satisfying the conditions (9) with respect to $\mathfrak{t} \in \{\mathfrak{a}, \mathfrak{n}, \mathfrak{f}\}$. The multiplication determined by the matrices λ_0, l_κ, $\kappa = 0, 1$ given by formulas (8) and (10) defines a unique $\mathcal{M}^5(\mathfrak{t})$-algebra $\mathfrak{c}(X)$ on the vector space $\mathbb{K}^5$, such that the canonical basis $\{e_0, e_1, e_2, e_3, e_4\}$ of $\mathbb{K}^5$ is a special distinguished basis of $\mathfrak{c}(X)$. If (8) is the matrix of the map $\lambda_0 = L_{e_0}|_{\mathfrak{c}'} : \mathfrak{c}' \to \mathfrak{c}'$ for an $\mathcal{M}^5(\mathfrak{t})$-algebra $\mathfrak{c}$ having the properties given by (9), then the $\mathcal{M}^5(\mathfrak{t})$-algebra $\mathfrak{c}(X)$ constructed on the vector space $\mathbb{K}^5$ is isomorphic to $\mathfrak{c}$.

Proof. We define the multiplication with respect to the canonical basis $\{e_0, e_1, e_2, e_3, e_4\}$ of $\mathbb{K}^5$ by the relations $e_0 e_1 = e_1$, $e_0 e_j = \sum_{k=2}^{4} x_j^k e_k$, $e_1 e_j = l_1 e_j$, $j = 2, 3, 4$, where the map $l_1 : e_j \mapsto L_{e_1} e_j$ is determined by the multiplication formulas of an $\mathcal{M}^5(\mathfrak{t})$-algebra $\mathfrak{c}(X)$ with respect to a distinguished basis described in Definition 3.2. We obtain an $\mathcal{M}^5(\mathfrak{t})$-algebra $\mathfrak{c}(X)$, for which $\{e_0, e_1, e_2, e_3, e_4\}$ is a special distinguished basis and the multiplication with the matrix $\mathbf{X}$ gives the map $\lambda_0 = L_{e_0}|_{\mathfrak{c}'(X)} : \mathfrak{c}'(X) \to \mathfrak{c}'(X)$ and the multiplication with X the map l_0. Clearly, for a given $\mathcal{M}^5(\mathfrak{t})$-algebra $\mathfrak{c}$ the $\mathcal{M}^5(\mathfrak{t})$-algebra $\mathfrak{c}(X)$ defined by the above construction is isomorphic to $\mathfrak{c}$. □

Definition 3.3. A 5-dimensional vector space $\mathbf{V}$ with a fixed basis $\mathcal{B}$ can be identified with the vector space $\mathbb{K}^5$. For a matrix X satisfying (9) we denote by $\mathfrak{c}(X)$ the uniquely defined $\mathcal{M}^5(\mathfrak{t})$-algebra on $\mathbf{V}$ with special distinguished basis $\mathcal{B}$ constructed according to Proposition 3.3.

4 Partial isomorphisms of $\mathcal{M}^5$-algebras

Definition 4.1. A map $\varphi : \mathfrak{c} \to \hat{\mathfrak{c}}$ between $\mathcal{M}^5$-algebras $\mathfrak{c}$ and $\hat{\mathfrak{c}}$ is called *partial isomorphism* if the induced map $\varphi|_{\mathfrak{c}'} : \mathfrak{c}' \to \hat{\mathfrak{c}}'$ is an isomorphism. A partial isomorphism $\varphi : \mathfrak{c} \to \mathfrak{c}$ is called *partial automorphism.*

Partial isomorphisms preserve the type $\mathfrak{t} \in \{\mathfrak{a}, \mathfrak{n}, \mathfrak{f}\}$ of $\mathcal{M}^5$-algebras. If $\{e_0, e_1, e_2, e_3, e_4\}$ and $\{\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$ are distinguished bases of $\mathcal{M}^5$-algebras $\mathfrak{c}$ and $\mathfrak{c}^*$ of same type, then the linear map $\varphi : \mathfrak{c} \to \mathfrak{c}^*$ determined by $\varphi(e_i) = \hat{e}_i$, $i = 0, 1, 2, 3, 4$, is a partial isomorphism. We get immediately

Lemma 4.1. A linear map $\varphi : \mathfrak{c} \to \mathfrak{c}^*$ is a partial isomorphism if and only if its matrix with respect to distinguished bases of $\mathfrak{c}$ and $\mathfrak{c}^*$ has the form

$$\begin{bmatrix} u^0 & \mathbf{0} \\ \mathbf{u}^t & \mathbf{A} \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u^1 \; u^2 \; u^3 \; u^4 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0\;0\;0\;0 \end{bmatrix}, \tag{11}$$

where for an

(a) $\mathcal{M}^5(\mathfrak{a})$-algebra: $\det(\mathbf{A}) \neq 0$,

(b) $\mathcal{M}^5(\mathfrak{n})$-algebra: $\mathbf{A} = \begin{bmatrix} p^1 & q^1 & 0 & 0 \\ p^2 & q^2 & 0 & 0 \\ p^3 & q^3 & r^3 & 0 \\ p^4 & q^4 & r^4 & p^1q^2 - p^2q^1 \end{bmatrix}$, $(p^1q^2 - p^2q^1)r^3 \neq 0$,

(c) $\mathcal{M}^5(\mathfrak{f})$-algebra: $\mathbf{A} = \begin{bmatrix} p^1 & 0 & 0 & 0 \\ p^2 & q^2 & 0 & 0 \\ p^3 & q^3 & p^1q^2 & 0 \\ p^4 & q^4 & p^1q^3 & (p^1)^2q^2 \end{bmatrix}$, $p^1q^2 \neq 0$,

$r^3, r^4, p^i, q^i \in \mathbb{K},\ i = 1, 2, 3, 4.$

Isomorphisms between $\mathcal{M}^5$-algebras induce isomorphisms between their derived algebras.

Corollary 4.2. Isomorphisms between $\mathcal{M}^5$-algebras are partial isomorphisms. The automorphism group of an $\mathcal{M}^5$-algebra is a subgroup of the group of its partial automorphisms.

Definition 4.2. A pair $\{\mathbf{V}, \mathfrak{p}\}$ of a 5-dimensional vector space $\mathbf{V}$ and a 4-dimensional nilpotent Lie algebra $\mathfrak{p}$ which is contained as a subspace in $\mathbf{V}$ and is isomorphic to $\mathfrak{t} \in \{\mathfrak{a}, \mathfrak{n}, \mathfrak{f}\}$ is called *partial Lie algebra.*
A *partial Lie-automorphism* of a partial Lie algebra $\{\mathbf{V}, \mathfrak{p}\}$ is a linear automorphism of $\mathbf{V}$ which induces a Lie algebra automorphism of $\mathfrak{p}$. The group of partial Lie-automorphisms of a partial Lie algebra $\{\mathbf{V}, \mathfrak{p}\}$ will be denoted by $\mathrm{Aut}^{\mathfrak{p}}(\mathbf{V})$.
An $\mathcal{M}^5$-algebra defined on $\mathbf{V}$ is said to be *adapted to* the partial Lie algebra $\{\mathbf{V}, \mathfrak{p}\}$ if its derived algebra is $\mathfrak{p}$.

Now, we fix a partial Lie algebra $\{\mathbf{V}, \mathfrak{p}\}$. It is clear that the partial automorphisms of any $\mathcal{M}^5$-algebra adapted to $\{\mathbf{V}, \mathfrak{p}\}$ are exactly the partial Lie-automorphisms of $\{\mathbf{V}, \mathfrak{p}\}$. Moreover, if $\mathfrak{p} \cong \mathfrak{t}$, then any $\mathcal{M}^5(\mathfrak{t})$-algebra is isomorphic to some $\mathcal{M}^5(\mathfrak{t})$-algebra adapted to $\{\mathbf{V}, \mathfrak{p}\}$, hence for the investigation of isomorphism classes of $\mathcal{M}^5(\mathfrak{t})$-algebras it is sufficient to consider only $\mathcal{M}^5(\mathfrak{t})$-algebras which are adapted to the partial Lie algebra $\{\mathbf{V}, \mathfrak{p}\}$.
Let $\mathcal{B} = \{e_0, e_1, e_2, e_3, e_4\}$ be fixed basis of $\mathbf{V}$ such that $\{e_1, e_2, e_3, e_4\}$ is a basis of $\mathfrak{p}$ with multiplication:

abelian,	if	$\mathfrak{p} \cong \mathfrak{a}$,
$e_1e_2 = e_4$,	if	$\mathfrak{p} \cong \mathfrak{n}$
$e_1e_2 = e_3$, $e_1e_3 = e_4$,	if	$\mathfrak{p} \cong \mathfrak{f}$.

Consider a vector space $\mathbf{V}$ with a fixed basis $\mathcal{B} = \{e_0, e_1, e_2, e_3, e_4\}$ and all $\mathcal{M}^5(\mathfrak{t})$-algebras on $\mathbf{V}$ having $\mathcal{B}$ as a special distinguished basis. These $\mathcal{M}^5(\mathfrak{t})$-algebras $\mathfrak{c}(X)$ have the same partial automorphism group $\mathrm{Aut}^{\mathfrak{p}}(\mathbf{V})$. Let be $\{c_{ij}^k(X)\}$ the system of structure constants of $\mathfrak{c}(X)$, hence $e_r e_s = \sum_t c_{rs}^t e_t$. Identifying the basis $\mathcal{B}$ with the canonical basis of the vector space $\mathbb{K}^5$, we denote by γ_X the multiplication on $\mathbb{K}^5$ determined by $\{c_{ij}^k(X)\}$. The group $\mathrm{Aut}^{\mathfrak{p}}(\mathbf{V})$ acts on the set Γ of multiplications of $\mathcal{M}^5(\mathfrak{t})$-algebras on $\mathbb{K}^5$ by

$$\gamma_X(x, y) \mapsto M^{-1}\gamma_X(Mx, My), \quad M \in \mathrm{Aut}^{\mathfrak{p}}(\mathbf{V}), \quad x, y \in \mathbb{K}^5. \tag{12}$$

Lemma 4.3. The orbits of the action (12) of the group $\mathrm{Aut}^{\mathfrak{p}}(\mathbf{V})$ on Γ correspond to isomorphism classes of 5-dimensional $\mathcal{M}^5(\mathfrak{t})$-algebras.

Proof. The group $\mathrm{GL}(5, \mathbb{K})$ acts via change of basis on the set of anti-commutative bilinear multiplications on $\mathbb{K}^5$ by $(g \circ \gamma)(x, y) = g^{-1}\gamma(gx, gy)$, $g \in \mathrm{GL}(5, \mathbb{K})$. An orbit $O(\gamma)$ under this action consists of all anti-commutative bilinear multiplications giving isomorphic anti-commutative algebras, (cf. e.g. [**?**]). An isomorphism $\varphi : \mathfrak{c}(X) \to \mathfrak{c}(\hat{X})$ induces isomorphism between the ideals $\mathfrak{c}'(X)$ and $\mathfrak{c}'(\hat{X})$, hence the basis $\{\varphi(e_0), \varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4)\}$ is also a distinguished basis of $\mathfrak{c}(\hat{X})$. It follows that the linear map α defined by $\alpha(e_i) = \hat{e}_i$, $i = 0, 1, \ldots, 4$ is an isomorphism between the multiplications with respect to the bases $\{e_0, e_1, e_2, e_3, e_4\}$ and $\{\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$, respectively. □

In the following we will use Einstein's summation convention, i.e. we sum over all values for repeated free indices in a term. The indices will take the following values:

$$h, i, j, k \in \{2, 3, 4\}, \quad p, q, r, s, t \in \{1, 2, 3, 4\}, \quad \vartheta, \kappa, \lambda, \mu, \nu \in \{0, 1, 2, 3, 4\}.$$

Using Proposition 3.3 we define partial isomorphic $\mathcal{M}^5$-algebras $\mathfrak{c} = \mathfrak{c}(X)$ and $\hat{\mathfrak{c}} = \mathfrak{c}(\hat{X})$ of same type $\mathfrak{t} \in \{\mathfrak{a}, \mathfrak{n}, \mathfrak{f}\}$ determined by the matrices

$$X = \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix} \quad \text{and} \quad \hat{X} = \begin{bmatrix} \hat{x}_2^2 & \hat{x}_3^2 & \hat{x}_4^2 \\ \hat{x}_2^3 & \hat{x}_3^3 & \hat{x}_4^3 \\ \hat{x}_2^4 & \hat{x}_3^4 & \hat{x}_4^4 \end{bmatrix}, \tag{13}$$

of the maps l_0, respectively $\hat{l}_0$, with respect to the special distinguished bases $\{e_0, e_1, e_2, e_3, e_4\}$ and $\{\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$. Let $\psi : \mathfrak{c} \to \hat{\mathfrak{c}}$ be the fixed partial isomorphism given by $\psi(e_\lambda) = \hat{e}_\lambda$. For any other partial isomorphism $\varphi : \mathfrak{c} \to \hat{\mathfrak{c}}$ the map $\alpha = \varphi \circ \psi^{-1} : \hat{\mathfrak{c}} \to \hat{\mathfrak{c}}$ is a partial automorphism, the matrix of which has the form described in Lemma 4.1. Denoting the matrix block $\mathbf{A}$ in (11) by $\{a^r_q\}$ we get

$$\varphi(e_\lambda) = \alpha \circ \psi(e_\lambda) = \alpha(\hat{e}_\lambda), \quad \varphi(e_0) = u^\lambda \hat{e}_\lambda = u^0 \hat{e}_0 + u^t \hat{e}_t, \quad \varphi(e_q) = a^t_q \hat{e}_t. \tag{14}$$

Theorem 4.4. A partial isomorphism $\varphi : \mathfrak{c}(X) \to \hat{\mathfrak{c}}(\hat{X})$ given by (14) is an isomorphism if and only if there is a matrix $\mathbf{A} = \{a^r_q\}$ satisfying the following equations:

(a) for $\mathcal{M}^5(\mathfrak{a})$-algebra:

$$a^1_1 = u^0 a^1_1, \quad a^1_j x^j_h = u^0 a^1_h, \quad a^i_1 = u^0 \hat{x}^i_j a^j_1, \quad a^i_j x^j_h = u^0 \hat{x}^i_j a^j_h,$$

(b) for $\mathcal{M}^5(\mathfrak{n})$-algebra:

$$\begin{aligned} a^1_1 &= u^0 a^1_1, & a^h_1 &= u^0 \hat{x}^h_k a^k_1 + (u^1 a^2_1 - u^2 a^1_1)\delta^h_4, \\ a^1_j x^j_k &= u^0 a^1_k, & a^h_j x^j_k &= u^0 \hat{x}^h_j a^j_k + (u^1 a^2_k - u^2 a^1_k)\delta^h_4. \end{aligned}$$

(c) for $\mathcal{M}^5(\mathfrak{f})$-algebra: $u^0 = 1$, $a^1_1 \neq 0$, and

$$a^i_1 = \hat{x}^i_j a^j_1 + (u^1 a^2_1 - u^2 a^1_1)\delta^i_3 + (u^1 a^3_1 - u^3 a^1_1)\delta^i_4, \quad a^i_j x^j_k = \hat{x}^i_j a^j_k + u^1 a^2_k \delta^i_3 + u^1 a^3_k \delta^i_4.$$

Proof. We find necessary and sufficient conditions for the partial isomorphism $\varphi = \alpha \circ \psi : \mathfrak{c} \to \hat{\mathfrak{c}}$ to be an isomorphism. The induced map $\varphi|_{\mathfrak{c}'} : \mathfrak{c}' \to \hat{\mathfrak{c}}'$ is an isomorphism, hence we investigate only the identities $\varphi(e_0 e_1) = \varphi(e_0) \star \varphi(e_1)$ and $\varphi(e_0 e_k) = \varphi(e_0) \star \varphi(e_k)$, where $(x, y) \mapsto x \star y$, $x, y \in \hat{\mathfrak{c}}$ denotes the multiplication in $\hat{\mathfrak{c}}$. We have $e_0 e_k = x^j_k e_j$, $\hat{e}_0 \star \hat{e}_k = \hat{x}^j_k \hat{e}_j$ and

$$\varphi(e_0 e_1) = \varphi(e_1) = \alpha(\hat{e}_1) = a^t_1 \hat{e}_t,$$

$$\varphi(e_0) \star \varphi(e_1) = (u^0 \hat{e}_0 + u^t \hat{e}_t) \star (a^1_1 \hat{e}_1 + a^k_1 \hat{e}_k) = u^0 a^1_1 \hat{e}_1 + u^0 a^k_1 \hat{x}^j_k \hat{e}_j - u^j a^1_1 \hat{e}_1 \star \hat{e}_j + u^t a^j_1 \hat{e}_t \star \hat{e}_j.$$

Since $\hat{e}_h \star \hat{e}_j = 0$, we get

$$a^1_1 = u^0 a^1_1, \quad a^j_1 \hat{e}_j = u^0 a^k_1 \hat{x}^j_k \hat{e}_j + (u^1 a^j_1 - u^j a^1_1)\hat{e}_1 \star \hat{e}_j.$$

Similarly,

$$\varphi(e_0 e_k) = \varphi(x^j_k e_j) = x^j_k \varphi(e_j) = x^j_k a^t_j \hat{e}_t,$$

$$\varphi(e_0) \star \varphi(e_k) = (u^0 \hat{e}_0 + u^s \hat{e}_s) \star (a^1_k \hat{e}_1 + a^j_k \hat{e}_j) = u^0 (a^1_k \hat{e}_1 + a^j_k \hat{x}^h_j \hat{e}_h) + (u^1 a^j_k - u^j a^1_k)\hat{e}_1 \star \hat{e}_j,$$

consequently,

$$x^j_k a^1_j = u^0 a^1_k, \quad x^j_k a^h_j \hat{e}_h = u^0 a^j_k \hat{x}^h_j \hat{e}_h + (u^1 a^j_k - u^j a^1_k)\hat{e}_1 \star \hat{e}_j.$$

Putting into $\hat{e}_1 \star \hat{e}_j$ the multiplication formulas of the nilpotent derived $\mathcal{M}^5(\mathfrak{t})$-algebras, $\mathfrak{t} \in \{\mathfrak{a}, \mathfrak{n}, \mathfrak{f}\}$ we get the assertions. $\square$

5 $\mathcal{M}^5(\mathfrak{a})$-algebras

If we rewrite the equations of Theorem 4.4 (a) in matrix form, we get

Lemma 5.1. The $\mathcal{M}^5(\mathfrak{a})$-algebras $\mathfrak{c} = \mathfrak{c}(X)$ and $\hat{\mathfrak{c}} = \mathfrak{c}(\hat{X})$ given by the matrices (13) are isomorphic if and only if there exist $0 \neq u^0 \in \mathbb{K}$ and a non-singular matrix $\mathbf{A} = \begin{bmatrix} a_1^1 & \mathbf{c} \\ \mathbf{b}^t & A \end{bmatrix}$ with

$$\mathbf{b} = \begin{bmatrix} a_1^2 & a_1^3 & a_1^4 \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} a_2^1 & a_3^1 & a_4^1 \end{bmatrix}, \ A = \begin{bmatrix} a_2^2 & a_3^2 & a_4^2 \\ a_2^3 & a_3^3 & a_4^3 \\ a_2^4 & a_3^4 & a_4^4 \end{bmatrix}, \ a_q^p \in \mathbb{K},$$

such that

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0}^t & \widehat{X} \end{bmatrix} = \frac{1}{u^0} \begin{bmatrix} a_1^1 & \mathbf{c} \\ \mathbf{b}^t & A \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0}^t & X \end{bmatrix} \begin{bmatrix} a_1^1 & \mathbf{c} \\ \mathbf{b}^t & A \end{bmatrix}^{-1} = \mathbf{A} \frac{1}{u^0} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0}^t & X \end{bmatrix} \mathbf{A}^{-1}. \tag{15}$$

Equivalently,

$$a_1^1 = u^0 a_1^1, \quad \mathbf{c}X = u^0 \mathbf{c}, \quad \widehat{X}\mathbf{b}^t = \frac{1}{u^0}\mathbf{b}^t, \quad \widehat{X} = \frac{1}{u^0} AXA^{-1}. \tag{16}$$

Putting $u^0 = 1$, $\mathbf{b} = \mathbf{c} = \mathbf{0}$, we obtain

Corollary 5.2. For any $A \in \mathrm{GL}(3, \mathbb{K})$ the $\mathcal{M}^5(\mathfrak{a})$-algebras $\mathfrak{c} = \mathfrak{c}(X)$ and $\hat{\mathfrak{c}} = \mathfrak{c}(AXA^{-1})$ are isomorphic.

Using the Jordan, respectively, rational normal forms of matrices (cf. for example chapters 6 and 7 in [11]) we obtain

Proposition 5.3. Any $\mathcal{M}^5(\mathfrak{a})$-algebra is a Lie algebra isomorphic to an $\mathcal{M}^5(\mathfrak{a})$-algebra $\mathfrak{c}(X)$ determined by one of the following matrices X:

$$\begin{bmatrix} \kappa & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}, \ \kappa\lambda\mu \neq 0, \qquad \begin{bmatrix} \kappa & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix}, \ \kappa\lambda \neq 0, \qquad \begin{bmatrix} \kappa & 0 & 0 \\ 1 & \kappa & 0 \\ 0 & 1 & \kappa \end{bmatrix}, \ \kappa \neq 0, \tag{17}$$

$$\begin{aligned} &\begin{bmatrix} \kappa & 0 & 0 \\ 0 & 0 & -\lambda \\ 0 & 1 & -\mu \end{bmatrix}, \ \kappa\lambda \neq 0, \ t^2 + \mu t + \lambda \text{ is irreducible}, \\ &\begin{bmatrix} 0 & 0 & -\kappa \\ 1 & 0 & -\lambda \\ 0 & 1 & -\mu \end{bmatrix}, \ \kappa \neq 0, \ t^3 + \mu t^2 + \lambda t + \kappa \text{ is irreducible}. \end{aligned} \tag{18}$$

Proof. According to Proposition 3.2 in the case of $\mathcal{M}^5(\mathfrak{a})$-algebra and to Theorem 10.5 (c) any $\mathcal{M}^5(\mathfrak{a})$-algebra is a Lie algebra. Corollary 5.2 yields that the $\mathcal{M}^5(\mathfrak{a})$-algebras $\mathfrak{c} = \mathfrak{c}(X)$ and $\hat{\mathfrak{c}} = \mathfrak{c}(AXA^{-1})$ are isomorphic for any $A \in \mathrm{GL}(3, \mathbb{K})$. If the roots of the characteristic polynomial $\chi(t)$ of X are contained in the field $\mathbb{K}$, then the matrix X has Jordan normal form given in (17). If the characteristic polynomial $\chi(t) = \det(tE - X)$ has an irreducible non-linear factor in $\mathbb{K}$, then the matrices (18) are given by the corresponding companion matrices. □

Definition 5.1. The Lie $\mathcal{M}^5(\mathfrak{a})$-algebras of *normal form* are the algebras determined by the matrices (17) and (18), which are given by the multiplications

$$\mathfrak{bl}_1^{\mathfrak{a}}(\kappa,\lambda,\mu): \quad e_0e_1 = e_1, \quad e_0e_2 = \kappa e_2, \quad e_0e_3 = \lambda e_3, \quad e_0e_4 = \mu e_4,$$

$$\mathfrak{bl}_2^{\mathfrak{a}}(\kappa,\lambda): \quad e_0e_1 = e_1, \quad e_0e_2 = \kappa e_2, \quad e_0e_3 = \lambda e_3 + e_4, \quad e_0e_4 = \lambda e_4,$$

$$\mathfrak{bl}_3^{\mathfrak{a}}(\kappa): \quad e_0e_1 = e_1, \quad e_0e_2 = \kappa e_2 + e_3, \quad e_0e_3 = \kappa e_3 + e_4, \quad e_0e_4 = \kappa e_4,$$

$$\mathfrak{bl}_4^{\mathfrak{a}}(\kappa,\lambda,\mu): \quad e_0e_1 = e_1, \quad e_0e_2 = \kappa e_2, \quad e_0e_3 = e_4, \quad e_0e_4 = -\lambda e_3 - \mu e_4,$$

$$\mathfrak{bl}_5^{\mathfrak{a}}(\kappa,\lambda,\mu): \quad e_0e_1 = e_1, \quad e_0e_2 = e_3, \quad e_0e_3 = e_4, \quad e_0e_4 = -\kappa e_2 - \lambda e_3 - \mu e_4$$

with respect to a special distinguished basis.

Remark 5.4. For an algebraically closed field $\mathbb{K}$ any $\mathcal{M}^5(\mathfrak{a})$-algebra is isomorphic to one of the algebras of normal form given by $\mathfrak{bl}_1^{\mathfrak{a}}(\kappa,\lambda,\mu)$, $\mathfrak{bl}_2^{\mathfrak{a}}(\kappa,\lambda)$, $\mathfrak{bl}_3^{\mathfrak{a}}(\kappa)$. For ordered fields $\mathbb{K}$ there exist algebras $\mathfrak{bl}_4^{\mathfrak{a}}(\kappa,\lambda,\mu)$ of normal form, since the inequality $\mu^2 < 4\lambda$ implies that the polynomial $t^2 + \mu t + \lambda$ is irreducible. In the following we mark the algebras $\mathfrak{bl}_4^{\mathfrak{a}}(\kappa,\lambda,\mu)^*$ and $\mathfrak{bl}_5^{\mathfrak{a}}(\kappa,\lambda,\mu)^*$ with an asterisk, if their existence depends on the properties of the underlying field.

Since isomorphisms of $\mathcal{M}^5(\mathfrak{a})$-algebras preserve the characteristic polynomials, isomorphic $\mathcal{M}^5(\mathfrak{a})$-algebras of normal form have the same type of normal forms, $\mathfrak{bl}_x^{\mathfrak{a}}$, $x \in \{1, 2, \ldots, 5\}$.

Proposition 5.5. The $\mathcal{M}^5(\mathfrak{a})$-algebras of normal form $\mathfrak{bl}_3^{\mathfrak{a}}(\kappa)$ and $\mathfrak{bl}_5^{\mathfrak{a}}(\kappa,\lambda,\mu)^*$ are uniquely determined in the isomorphism class. Among $\mathcal{M}^5(\mathfrak{a})$-algebras of normal form there are the following isomorphisms:

$$\mathfrak{bl}_1^{\mathfrak{a}}(\pi(\kappa),\pi(\lambda),\pi(\mu)) \cong \mathfrak{bl}_1^{\mathfrak{a}}(\tfrac{1}{\rho(\kappa)}, \tfrac{\rho(\lambda)}{\rho(\kappa)}, \tfrac{\rho(\mu)}{\rho(\kappa)}) \cong \mathfrak{bl}_1^{\mathfrak{a}}(\tfrac{\sigma(\mu)}{\sigma(\kappa)}, \tfrac{1}{\sigma(\kappa)}, \tfrac{\sigma(\lambda)}{\sigma(\kappa)}) \cong \mathfrak{bl}_1^{\mathfrak{a}}(\tfrac{\tau(\lambda)}{\tau(\kappa)}, \tfrac{\tau(\mu)}{\tau(\kappa)}, \tfrac{1}{\tau(\kappa)}),$$
$$\text{for any } \pi,\rho,\sigma,\tau \in \mathrm{Sym}(\{\kappa,\lambda,\mu\}), \ \kappa\lambda\mu \neq 0,$$

$$\mathfrak{bl}_2^{\mathfrak{a}}(\kappa,\lambda) \cong \mathfrak{bl}_2^{\mathfrak{a}}(\tfrac{1}{\kappa}, \tfrac{\lambda}{\kappa}),$$

$$\mathfrak{bl}_4^{\mathfrak{a}}(\kappa,\lambda,\mu)^* \cong \mathfrak{bl}_4^{\mathfrak{a}}(\tfrac{1}{\kappa}, \tfrac{\lambda}{\kappa}, \tfrac{\mu}{\kappa})^*.$$

Proof. It follows from (16) that for isomorphic $\mathcal{M}^5(\mathfrak{a})$-algebras $\mathfrak{c} = \mathfrak{c}(X)$ and $\mathfrak{c}(\hat{X})$ one has $\widehat{X} = \frac{1}{u^0} AXA^{-1}$. Moreover, according to Corollary 5.2, the $\mathcal{M}^5(\mathfrak{a})$-algebras $\mathfrak{c} = \mathfrak{c}(X)$ and $\hat{\mathfrak{c}} = \mathfrak{c}(AXA^{-1})$ are isomorphic for any $A \in \mathrm{GL}(3,\mathbb{K})$. The algebras (**??**) are given by Jordan or companion matrices, they are determined by conjugation $X \mapsto AXA^{-1}$, $A \in \mathrm{GL}(3,\mathbb{K})$. Hence we can assume that $\mathfrak{c} = \mathfrak{c}(X)$ and $\mathfrak{c}(\hat{X})$ belong to algebras (**??**). Now, we investigate isomorphisms determined by (15) with $u^0 \neq 1$. Applying the primary decomposition theorem (see Section 6.8 in [11]) we get direct sum decomposition of the extended matrix $\frac{1}{u^0}\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0}^t & X \end{bmatrix}$ into invariant subspaces defined uniquely up to order. For the algebras (**??**) there is one two- or three-dimensional invariant subspace listed last in their order. The conjugations exchanging one-dimensional invariant subspaces when there are more of them, and leaving fixed the two-dimensional or three-dimensional invariant subspaces, we can obtain isomorphisms preserving the direct sum decomposition. All extended matrices of (17) and (18) have $\mathbb{K}e_1$ as a one-dimensional invariant subspace such that the complementary subspace is also invariant. For the algebras $\mathfrak{bl}_3^{\mathfrak{a}}(\kappa)$ and $\mathfrak{bl}_5^{\mathfrak{a}}(\kappa,\lambda,\mu)^*$ the invariant subspace $\mathbb{K}e_1$ is the only one-dimensional direct sum component subspace, hence these algebras are determined uniquely in the isomorphism class. For $\mathfrak{bl}_2^{\mathfrak{a}}(\kappa,\lambda)$ and $\mathfrak{bl}_4^{\mathfrak{a}}(\kappa,\lambda,\mu)^*$ the extended matrix has $\mathbb{K}e_1$, $\mathbb{K}e_2$ as one-dimensional direct sum component subspaces corresponding to the characteristic value 1 and κ. If we swap them according to the formula (15), we get e_1 and e_2 as characteristic vectors with characteristic value $\frac{1}{u^0}$ respectively $\frac{\kappa}{u^0}$, respectively. Since the characteristic value of e_2 should be 1, $u^0=\kappa$ holds and the corresponding characteristic polynomials are $\kappa_2(t)=(t-\frac{1}{\kappa})(t-1)(t-\frac{\lambda}{\kappa})^2$ for $\mathfrak{bl}_2^{\mathfrak{a}}(\kappa,\lambda)$ and $\kappa_4(t)=(t-\frac{1}{\kappa})(t-1)(t^2+\frac{\mu}{\kappa}t+\frac{\lambda}{\kappa^2})$ for $\mathfrak{bl}_4^{\mathfrak{a}}(\kappa,\lambda,\mu)^*$. The polynomial $\kappa_2(t)$ is a product of linear factors and $\kappa_4(t)$ contains the irreducible factor $t^2+\frac{\mu}{\kappa}t+\frac{\lambda}{\kappa^2}$, since $\xi\in\mathbb{K}$ is a root of $t^2+\mu t+\lambda$ if and only if $\frac{\xi}{\kappa}\in\mathbb{K}$ is a root of $t^2+\frac{\mu}{\kappa}t+\frac{\lambda}{\kappa^2}$. The corresponding Jordan, respectively companion matrices of $\widehat{X}$ are

$$\begin{bmatrix}\frac{1}{\kappa} & 0 & 0\\ 0 & \frac{\lambda}{\kappa} & 0\\ 0 & 1 & \frac{\lambda}{\kappa}\end{bmatrix}, \qquad \begin{bmatrix}\frac{1}{\kappa} & 0 & 0\\ 0 & 0 & -\frac{\lambda}{\kappa^2}\\ 0 & 1 & -\frac{\mu}{\kappa}\end{bmatrix},$$

consequently $\mathfrak{bl}_2^{\mathfrak{a}}(\kappa,\lambda)\cong\mathfrak{bl}_2^{\mathfrak{a}}(\frac{1}{\kappa},\frac{\lambda}{\kappa})$ and $\mathfrak{bl}_4^{\mathfrak{a}}(\kappa,\lambda,\mu)^*\cong\mathfrak{bl}_4^{\mathfrak{a}}(\frac{1}{\kappa},\frac{\lambda}{\kappa^2},\frac{\mu}{\kappa})^*$. For $\mathfrak{bl}_1^{\mathfrak{a}}(\kappa,\lambda,\mu)$ each vectors e_1, e_2, e_3, e_4 are characteristic vectors of the extended matrix of X with characteristic values 1, κ, λ, μ, respectively. Conjugations can result in any permutation of e_2, e_3, e_4, leading to permutations of the corresponding characteristic values κ, λ, μ. Hence the $\mathcal{M}^5(\mathfrak{a})$-algebras determined by matrices

$$\begin{bmatrix}\pi(\kappa) & 0 & 0\\ 0 & \pi(\lambda) & 0\\ 0 & 0 & \pi(\mu)\end{bmatrix}, \quad \pi\in\mathrm{Sym}(\kappa,\lambda,\mu)$$

are isomorphic, where $\mathrm{Sym}(\kappa,\lambda,\mu)$ denotes the permutation group of the set $\{\kappa,\lambda,\mu\}$. Similarly to the case of $\mathfrak{bl}_2^{\mathfrak{a}}(\kappa,\lambda)$ and $\mathfrak{bl}_4^{\mathfrak{a}}(\kappa,\lambda,\mu)^*$, a conjugation (15) can exchange any of e_2, e_3, e_4 with e_1, and we get

$$\frac{1}{\kappa},\frac{\lambda}{\kappa},\frac{\mu}{\kappa},\quad \frac{\lambda}{\kappa},\frac{1}{\kappa},\frac{\mu}{\kappa},\quad \frac{\lambda}{\kappa},\frac{\mu}{\kappa},\frac{1}{\kappa},$$

respectively, as characteristic values. If we combine the exchanges e_2, e_3, e_4 with e_1 and the permutations of the vectors e_2, e_3, e_4, we get the isomorphisms for $\mathfrak{bl}_1^{\mathfrak{a}}(\kappa,\lambda,\mu)$). Hence the assertion is completely proved. □

Corollary 5.6. Any $\mathcal{M}^5(\mathfrak{a})$-algebra over an ordered field $\mathbb{K}$ is a Lie algebra isomorphic to a unique $\mathcal{M}^5(\mathfrak{a})$-algebra of normal form satisfying the inequalities

(1) $\mathfrak{bl}_1^{\mathfrak{a}}(\kappa,\lambda,\mu)$, $\quad -1\le\mu\le\lambda\le\kappa\le 1,\ \kappa\lambda\mu\neq 0$,

(2) $\mathfrak{bl}_2^{\mathfrak{a}}(\kappa,\lambda)$, $\quad 0<|\kappa|\le 1,\ \lambda\neq 0$,

(3) $\mathfrak{bl}_3^{\mathfrak{a}}(\kappa)$, $\quad \kappa\neq 0$,

(4) $\mathfrak{bl}_4^{\mathfrak{a}}(\kappa,\lambda,\mu)$, $\quad 0<|\kappa|\le 1,\ \lambda\neq 0$,

(5) $\mathfrak{bl}_5^{\mathfrak{a}}(\kappa,\lambda,\mu)^*$, $\quad \kappa\neq 0$.

We notice that the real $\mathcal{M}^5(\mathfrak{a})$-algebras belonging to the class of non-decomposable 5-dimensional solvable Lie algebras with codimension one nilradicals are classified, (cf. p. 105 in [18]; p. 989 in [**?**]; pp. 120–121 in [21]).

6 $\mathcal{M}^5(\mathfrak{n})$-algebras

An $\mathcal{M}^5(\mathfrak{n})$-algebra is an anti-commutative semidirect sum $\mathbb{K}e_0 \oplus_{\lambda_0} \mathfrak{c}'$, such that the ideal $\mathfrak{c}' \cong \tilde{\mathfrak{n}}$ and the subspaces $\mathfrak{c}''$, $\mathcal{C}(\mathfrak{c}')$ are preserved by all elements of $\mathrm{Aut}(\mathfrak{c})$. Considering $\mathcal{M}^5(\mathfrak{n})$-algebras with partial isomorphisms described in Lemma 4.1 (b) we obtain the following assertion from Theorem 4.4 (b), whose 4-dimensional version is proved in [2].

Theorem 6.1. Let $\{e_0, e_1, e_2, e_3, e_4\}$ and $\{\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$ be special distinguished bases of $\mathcal{M}^5(\mathfrak{n})$-algebras $\mathfrak{c} = \mathfrak{c}(X)$ and $\hat{\mathfrak{c}} = \mathfrak{c}(\hat{X})$, respectively, determined by the matrices (13). The partial isomorphism $\varphi : \mathfrak{c} \to \hat{\mathfrak{c}}$ given by (14) is an isomorphism if and only if there is a matrix $\mathbf{A}$ of the form given in Lemma 4.1 (b) satisfying the matrix equation

$$\begin{bmatrix} p^1 & q^1x_2^2 & q^1x_3^2 & q^1x_4^2 \\ p^2 & q^2x_2^2 & q^2x_3^2 & q^2x_4^2 \\ p^3 & q^3x_2^2 + r^3x_2^3 & q^3x_3^2 + r^3x_3^3 & q^3x_4^2 + r^3x_4^3 \\ p^4 & \begin{matrix} q^4x_2^2 + r^4x_2^3 + \\ +(p^1q^2 - p^2q^1)x_2^4 \end{matrix} & \begin{matrix} q^4x_3^2 + r^4x_3^3 + \\ +(p^1q^2 - p^2q^1)x_3^4 \end{matrix} & \begin{matrix} q^4x_4^2 + r^4x_4^3 + \\ +(p^1q^2 - p^2q^1)x_4^4 \end{matrix} \end{bmatrix} = \tag{19}$$

$$= \begin{bmatrix} u^0p^1 & u^0q^1 & 0 & 0 \\ u^0\hat{x}_j^2p^j & u^0\hat{x}_j^2q^j & u^0\hat{x}_j^2r^j & u^0\hat{x}_4^2(p^1q^2 - p^2q^1) \\ u^0\hat{x}_j^3p^j & u^0\hat{x}_j^3q^j & u^0\hat{x}_j^3r^j & u^0\hat{x}_4^3(p^1q^2 - p^2q^1) \\ u^0\hat{x}_j^4p^j + u^1p^2 - u^2p^1 & u^0\hat{x}_j^4q^j + u^1q^2 - u^2q^1 & u^0\hat{x}_j^4r^j & u^0\hat{x}_4^4(p^1q^2 - p^2q^1) \end{bmatrix}.$$

Lemma 6.2. A basis $\{e_0, e_1, e_2, e_3, e_4\}$ of the $\mathcal{M}^5(\mathfrak{n})$-algebra $\mathfrak{c} = \mathfrak{c}(X)$ is a special distinguished basis if and only if

(a) $\{e_1, e_2, e_3, e_4\}$ is a basis of the nilpotent ideal $\mathfrak{c}'$ such that $e_3 \in \mathcal{C}(\mathfrak{c}')$, $e_4 \in \mathfrak{c}''$ and $e_1e_2 = e_4$,

(b) the matrix $\mathbf{X}$ of the map $\lambda_0 = L_{e_0}|_{\mathfrak{c}'} : \mathfrak{c}' \to \mathfrak{c}'$ has the block shape

$$\mathbf{X} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0}^t & X \end{bmatrix}, \quad X = \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix}, \quad \text{where} \quad \operatorname{rank} \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \end{bmatrix} = 2, \tag{20}$$

with respect to the basis $\{e_2, e_3, e_4\}$ of $\mathfrak{i}$.

Proof. The form (20) of the matrix $\mathbf{X}$ means

$$e_0e_1 = e_1, \quad \text{and} \quad \mathbf{X}(\mathbb{K}e_2 + \mathbb{K}e_3 + \mathbb{K}e_4) \subseteq (\mathbb{K}e_2 + \mathbb{K}e_3 + \mathbb{K}e_4),$$

hence it follows from $e_1e_2 = e_4$ that the subspace $\mathfrak{i} = \mathbb{K}e_2 + \mathbb{K}e_3 + \mathbb{K}e_4$ is an abelian ideal. The vectors e_4, e_0e_i span $\mathfrak{i}$ if and only if the cosets $e_0e_i + \mathbb{K}e_4$ span the factor space $\mathfrak{i}/\mathbb{K}e_4$, or equivalently, the first and second rows of X are linearly independent, hence $\mathfrak{c}'$ is isomorphic to $\tilde{\mathfrak{n}}$. Consequently, the conditions (a) and (b) imply that $\{e_0, e_1, e_2, e_3, e_4\}$ is a special distinguished basis. The converse statement is clear. □

According to the definitions formulated in Section 4, an $\mathcal{M}^5(\mathfrak{n})$-algebra $\mathfrak{c}$ is a semidirect sum $\mathfrak{c}(X) = \mathfrak{l}_2 \oplus_l \mathfrak{i}$ of the 2-dimensional non-abelian Lie algebra $\mathfrak{l}_2$ and a 3-dimensional abelian ideal $\mathfrak{i} = \mathbb{K}e_2 \oplus \mathbb{K}e_3 \oplus \mathbb{K}e_4$ with endomorphisms l_0, l_1 whose matrices X and Y, respectively, have the form

$$X = \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix} \text{ with rank} \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \end{bmatrix} = 2, \quad \text{and} \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{21}$$

Proposition 6.3. An $\mathcal{M}^5(\mathfrak{n})$-algebra $\mathfrak{c}(X) = \mathfrak{l}_2 \oplus_l \mathfrak{i}$ is a

1) Lie algebra if and only if $X = \begin{bmatrix} x_2^2 & 0 & 0 \\ x_2^3 & x_3^3 & 0 \\ x_2^4 & x_3^4 & x_2^2+1 \end{bmatrix}, \quad x_2^2 x_3^3 \neq 0,$

2) non-Lie binary Lie algebra if and only if X has one of the following forms:

$$\text{(a)} \quad X = \begin{bmatrix} 1 & 0 & 0 \\ x_2^3 & x_3^3 & x_4^3 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix}, \quad x_3^3(x_4^4 - 2) \neq 0 \text{ for } x_4^3 = 0,$$

$$\text{(b)} \quad X = \begin{bmatrix} x_2^2 & x_3^2 & 0 \\ \frac{1}{x_3^2}(x_2^2 - 1)(x_3^3 - 1) & x_3^3 & 0 \\ x_2^4 & x_3^4 & x_2^2 + x_3^3 \end{bmatrix}, \quad (x_2^2 + x_3^3 - 1)x_3^2 \neq 0,$$

3) non-Lie Malcev algebra if and only if X has one of the following forms:

$$\text{(i)} \quad X = \begin{bmatrix} 1 & 0 & 0 \\ x_2^3 & x_3^3 & 0 \\ x_2^4 & x_3^4 & -1 \end{bmatrix}, \quad x_3^3 \neq 0,$$

$$\text{(ii)} \quad X = \begin{bmatrix} 1 & 0 & 0 \\ x_2^3 & 1 - x_4^4 & x_4^3 \\ x_2^4 & \frac{1}{x_4^3}(2 + x_4^4 - x_4^4 x_4^4) & x_4^4 \end{bmatrix}, \quad x_4^3 \neq 0,$$

$$\text{(iii)} \quad X = \begin{bmatrix} x_2^2 & x_3^2 & 0 \\ -\frac{1}{x_3^2}(x_2^2 x_2^2 + x_2 - 2) & -(x_2^2 + 1) & 0 \\ x_2^4 & x_3^4 & -1 \end{bmatrix}, \quad x_3^2 \neq 0.$$

Proof. We identify the maps l_0 and l_1 with their matrices, then we have

$$l_1^2 = 0, \; l_1 l_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_2^2 & x_3^2 & x_4^2 \end{bmatrix}, \; l_0 l_1 = \begin{bmatrix} x_4^2 & 0 & 0 \\ x_4^3 & 0 & 0 \\ x_4^4 & 0 & 0 \end{bmatrix}, \; l_0 l_1 l_0 = \begin{bmatrix} x_4^2 x_2^2 & x_4^2 x_3^2 & x_4^2 x_4^2 \\ x_4^3 x_2^2 & x_4^3 x_3^2 & x_4^3 x_4^2 \\ x_4^4 x_2^2 & x_4^4 x_3^2 & x_4^4 x_4^2 \end{bmatrix}, \tag{22}$$

$$l_1 l_0 l_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_4^2 & 0 & 0 \end{bmatrix}, \; l_1 l_0^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_j^2 x_2^j & x_j^2 x_3^j & x_j^2 x_4^j \end{bmatrix}, \; l_0^2 l_1 = \begin{bmatrix} x_j^2 x_4^j & 0 & 0 \\ x_j^3 x_4^j & 0 & 0 \\ x_j^4 x_4^j & 0 & 0 \end{bmatrix}. \tag{23}$$

According to Theorem 10.5 (c) $\mathfrak{c}$ is a Lie algebra if and only if $l_1 = l_0 l_1 - l_1 l_0$. This gives the matrix equation

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x_4^2 & 0 & 0 \\ x_4^3 & 0 & 0 \\ x_4^4 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_2^2 & x_3^2 & x_4^2 \end{bmatrix} = \begin{bmatrix} x_4^2 & 0 & 0 \\ x_4^3 & 0 & 0 \\ x_4^4 - x_2^2 & -x_3^2 & -x_4^2 \end{bmatrix},$$

hence $l_0 = \begin{bmatrix} x_2^2 & 0 & 0 \\ x_2^3 & x_3^3 & 0 \\ x_2^4 & x_3^4 & 1 + x_2^2 \end{bmatrix}$, proving the first assertion.
Theorem 10.5 (a) yields that $\mathfrak{c}$ is a binary Lie algebra if and only if $l_1 l_0 l_1 = 0$, equivalently $x_4^2 = 0$, and $l_0 l_1 l_0 + l_1 - l_1 l_0^2 - l_0 l_1 = 0$. We get the matrix equation

$$\begin{bmatrix} 0 & 0 & 0 \\ x_4^3 x_2^2 & x_4^3 x_3^2 & 0 \\ x_4^4 x_2^2 & x_4^4 x_3^2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_j^2 x_2^j & x_j^2 x_3^j & x_j^2 x_4^j \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ x_4^3 & 0 & 0 \\ x_4^4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (24)$$

We obtain from the first and second rows

$$x_4^3(x_2^2 - 1) = x_4^3 x_3^2 = 0. \quad (25)$$

If $x_4^3 \neq 0$, then $x_2^2 = 1$, $x_3^2 = 0$, and using $x_4^2 = 0$ we get that the equations in the third row of (24) are satisfied. Hence the matrix of l_0 has the shape

$$X = \begin{bmatrix} 1 & 0 & 0 \\ x_2^3 & x_3^3 & x_4^3 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix}, \quad x_4^3 \neq 0.$$

For $x_4^3 = 0$ the equations (25) are satisfied and the third row of (24) gives

$$(x_2^2 - 1)(x_4^4 - x_2^2 - 1) - x_3^2 x_2^3 = 0, \quad x_3^2(x_4^4 - x_2^2 - x_3^3) = 0.$$

In this case $x_3^2 = 0$ and $x_2^2 \neq 1$ we get the Lie algebra condition, hence we assume $x_3^2 = 0$, $x_2^2 = 1$ and $x_4^4 \neq x_2^2 + 1 = 2$, hence

$$X = \begin{bmatrix} 1 & 0 & 0 \\ x_2^3 & x_3^3 & 0 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix}, \quad x_3^3 \neq 0,\ x_4^4 \neq 2.$$

The assertion 2) (a) is proved.
For $x_4^3 = 0$ and $x_3^2 \neq 0$ we get the conditions

$$x_4^4 = x_2^2 + x_3^3, \quad x_2^3 = \frac{1}{x_3^2}(x_2^2 - 1)(x_3^3 - 1).$$

Consequently

$$X = \begin{bmatrix} x_2^2 & x_3^2 & 0 \\ \frac{1}{x_3^2}(x_2^2 - 1)(x_3^3 - 1) & x_3^3 & 0 \\ x_2^4 & x_3^4 & x_2^2 + x_3^3 \end{bmatrix}, \quad x_3^2 \neq 0,$$

satisfying $\det\begin{bmatrix} x_2^2 & x_3^2 \\ \frac{1}{x_3^2}(x_2^2-1)(x_3^3-1) & x_3^3 \end{bmatrix} = x_2^2 + x_3^3 - 1 \neq 0$, proving assertion 2) (b).

Now, we determine the Malcev algebras among non-Lie binary Lie algebras given in the assertions 2) (a) and 2) (b). According to Theorem 10.5 (b) a binary Lie algebra $\mathfrak{c}$ with $x_4^2 = 0$ is a Malcev algebra if and only if $l_1 l_0^2 - l_0^2 l_1 + l_0 l_1 + l_1 l_0 = 0$. Using expressions (22) and (23) we obtain

$$\begin{bmatrix} -x_3^2 x_4^3 & 0 & 0 \\ x_4^3(1 - x_3^3 - x_4^4) & 0 & 0 \\ x_2^2 x_2^2 + x_3^2 x_2^3 - x_3^4 x_4^3 - x_4^4 x_4^4 + x_2^2 + x_4^4 & (x_2^2 + x_3^3 + 1)x_3^2 & x_3^2 x_4^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

giving the equations

$$\begin{gathered} x_3^2 x_4^3 = 0, \quad x_4^3(1 - x_3^3 - x_4^4) = 0, \quad (x_2^2 + x_3^3 + 1)x_3^2 = 0, \\ x_2^2(x_2^2 + 1) + x_3^2 x_2^3 - x_3^4 x_4^3 + (1 - x_4^4)x_4^4 = 0. \end{gathered} \tag{26}$$

In the case (a) we have $x_2^2 = 1$ and $x_3^2 = 0$, hence from (26) we obtain

$$x_4^3(1 - x_3^3 - x_4^4) = 0,\ 2 - x_3^4 x_4^3 + (1 - x_4^4)x_4^4 = 0.$$

For $x_4^3 = 0$ the roots of the equation $x_4^4 x_4^4 - x_4^4 - 2 = 0$ are $2, -1$. Since $x_4^4 = 2$ gives Lie algebra, we get

$$X = \begin{bmatrix} 1 & 0 & 0 \\ x_2^3 & x_3^3 & 0 \\ x_2^4 & x_3^4 & -1 \end{bmatrix}, \quad x_3^3 \neq 0.$$

For $x_4^3 \neq 0$ we obtain the equations $x_3^3 = 1 - x_4^4$ and $x_3^4 = \frac{1}{x_4^3}(2 + x_4^4 - x_4^4 x_4^4)$ and we get

$$X = \begin{bmatrix} 1 & 0 & 0 \\ x_2^3 & 1 - x_4^4 & x_4^3 \\ x_2^4 & \frac{1}{x_4^3}(2 + x_4^4 - x_4^4 x_4^4) & x_4^4 \end{bmatrix}, \quad x_4^3 \neq 0.$$

In the case (b) we obtain

$$x_3^2 \neq 0,\ x_4^3 = 0,\ x_4^4 = x_2^2 + x_3^3 = -1,\ x_2^3 = \frac{1}{x_3^2}(x_2^2 - 1)(x_3^3 - 1) = \frac{-x_2^2 x_2^2 - x_2 + 2}{x_3^2},$$

it follows

$$X = \begin{bmatrix} x_2^2 & x_3^2 & 0 \\ -\frac{1}{x_3^2}(x_2^2 x_2^2 + x_2 - 2) & -(x_2^2 + 1) & 0 \\ x_2^4 & x_3^4 & -1 \end{bmatrix}, \quad x_3^2 \neq 0,$$

hence the proof is finished. □

7 Binary Lie $\mathcal{M}^5(\mathfrak{n})$-algebras

As a consequence of Proposition 6.3 we obtain

Proposition 7.1. Every binary Lie $\mathcal{M}^5(\mathfrak{n})$-algebra $\mathfrak{c}(X)$ belongs to one of the following disjoint families defined by two equivalent conditions:

$Bl^{(1)}$: *a)* the subspaces $\mathfrak{c}''$ and $\mathcal{C}(\mathfrak{c}')$ are ideals in $\mathfrak{c}(X)$,

$$b)\ X = \begin{bmatrix} x_2^2 & 0 & 0 \\ x_2^3 & x_3^3 & 0 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix},\ x_2^2 = 1 \text{ or } x_2^2 = x_4^4 - 1 \neq 0; \text{ and } x_2^2 x_3^3 \neq 0,$$

$Bl^{(2)}$: *a)* the subspace $\mathcal{C}(\mathfrak{c}')$ is ideal in $\mathfrak{c}$, but $\mathfrak{c}''$ is not,

$$b)\ X = \begin{bmatrix} 1 & 0 & 0 \\ x_2^3 & x_3^3 & x_4^3 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix}, \quad x_4^3 \neq 0,$$

$Bl^{(3)}$: *a)* the subspace $\mathfrak{c}''$ is ideal in $\mathfrak{c}$, but $\mathcal{C}(\mathfrak{c}')$ is not,

$$b)\ X = \begin{bmatrix} x_2^2 & x_3^2 & 0 \\ \frac{1}{x_3^2}(x_2^2 - 1)(x_3^3 - 1) & x_3^3 & 0 \\ x_2^4 & x_3^4 & x_2^2 + x_3^3 \end{bmatrix}, \quad (x_2^2 + x_3^3 - 1)x_3^2 \neq 0.$$

Conditions a) for the ideal property show that isomorphic binary Lie $\mathcal{M}^5(\mathfrak{n})$-algebras belong to the same family. We now examine the binary Lie $\mathcal{M}^5(\mathfrak{n})$-algebra families in detail.

7.1 $Bl^{(1)}$-family

By Propositions 6.3 and 7.1, a binary Lie $\mathcal{M}^5(\mathfrak{n})$-algebra $\mathfrak{c}(X)$ belongs to the $Bl^{(1)}$-family if and only if X is a lower triangular matrix satisfying $x_2^2 x_3^3 \neq 0$, moreover, $x_2^2 = 1$ if $\mathfrak{c}(X)$ is a non-Lie algebra and $x_2^2 = x_4^4 - 1 \neq 0$ if $\mathfrak{c}(X)$ is a Lie algebra.

Proposition 7.2. Any binary Lie $\mathcal{M}^5(\mathfrak{n})$-algebra $\mathfrak{c}(X)$ in the $Bl^{(1)}$-family is isomorphic to the algebra $\mathfrak{c}(\hat{X})$, where $\hat{X} = \begin{bmatrix} x_2^2 & 0 & 0 \\ \varepsilon & x_3^3 & 0 \\ 0 & \delta & x_4^4 \end{bmatrix}$ or $\hat{X} = \begin{bmatrix} \frac{1}{x_2^2} & 0 & 0 \\ \varepsilon & \frac{x_3^3}{x_2^2} & 0 \\ 0 & \delta & \frac{x_4^4}{x_2^2} \end{bmatrix}$, $\varepsilon, \delta \in \{0, 1\}$. The isomorphism class of $\mathfrak{c}(X)$ contains the algebras determined by the matrices:

(i) $\begin{bmatrix} x_2^2 & 0 & 0 \\ 0 & x_3^3 & 0 \\ 0 & 0 & x_4^4 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{x_2^2} & 0 & 0 \\ 0 & \frac{x_3^3}{x_2^2} & 0 \\ 0 & 0 & \frac{x_4^4}{x_2^2} \end{bmatrix}$, if in each pair of conditions ($x_2^3 = 0$, $x_2^2 \neq x_3^3$) and ($x_3^4 = 0$, $x_3^3 \neq x_4^4$) one of the conditions is fulfilled,

(ii) $\begin{bmatrix} x_2^2 & 0 & 0 \\ 0 & x_3^3 & 0 \\ 0 & 1 & x_3^3 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{x_2^2} & 0 & 0 \\ 0 & \frac{x_3^3}{x_2^2} & 0 \\ 0 & 1 & \frac{x_3^3}{x_2^2} \end{bmatrix}$, if one of the conditions ($x_2^3 = 0$, $x_2^2 \neq x_3^3$) and both conditions ($x_3^4 \neq 0$, $x_3^3 = x_4^4$) are fulfilled,

(iii) $\begin{bmatrix} x_2^2 & 0 & 0 \\ 1 & x_2^2 & 0 \\ 0 & 0 & x_4^4 \end{bmatrix}$, if both conditions ($x_2^3 \neq 0$, $x_2^2 = x_3^3$) and one of the conditions ($x_3^4 = 0$, $x_3^3 \neq x_4^4$) are fulfilled,

(iv) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, if the conditions $x_2^3 \neq 0$, $x_3^4 \neq 0$ are fulfilled.

Proof. Assume that $x_2^2 = 1$, $x_4^4 \neq 2$, $\hat{x}_2^2 = 1$, $\hat{x}_4^4 \neq 2$, or $x_2^2 = x_4^4 - 1 \neq 0$, $\hat{x}_2^2 = \hat{x}_4^4 - 1 \neq 0$, corresponding to the cases of non-Lie binary Lie algebras, respectively, Lie algebras. We consider the isomorphism conditions given by the matrix equation (19) in Theorem 6.1 for the matrix A of the partial isomorphism (cf. Lemma 4.1 (b)). The equations given by the first two terms of the last row yields

$$u^0 \begin{bmatrix} \delta p^3 + (\hat{x}_4^4 - \frac{1}{u^0})p^4 \\ \delta q^3 + (\hat{x}_4^4 - \frac{1}{u^0}x_2^2)q^4 - \frac{r^4}{u^0}x_2^3 - (p^1q^2 - p^2q^1)\frac{1}{u_0}x_2^4 \end{bmatrix} = u^2 \begin{bmatrix} p^1 \\ q^1 \end{bmatrix} - u^1 \begin{bmatrix} p^2 \\ q^2 \end{bmatrix}.$$

Clearly, any value of the vector $u^0 \begin{bmatrix} \delta p^3 + (\hat{x}_4^4 - \frac{1}{u^0})p^4 \\ \delta q^3 + (\hat{x}_4^4 - \frac{1}{u^0}x_2^2)q^4 - \frac{r^4}{u^0}x_2^3 - (p^1q^2 - p^2q^1)\frac{1}{u_0}x_2^4 \end{bmatrix}$ can be expressed as a linear combination of the linearly independent vectors $\begin{bmatrix} p^1 \\ q^1 \end{bmatrix}$ and $\begin{bmatrix} p^2 \\ q^2 \end{bmatrix}$ with freely specified coefficients u^2 and u^1. Hence these equations do not give any restriction for the solution of the remaining equations.
Now, we assume $p^1 \neq 0$, hence $u^0 = 1$. The last two diagonal elements imply $\hat{x}_3^3 = x_3^3$, $\hat{x}_4^4 = x_4^4$, consequently $\hat{x}_2^2 = x_2^2$. It follows from the equation $\begin{bmatrix} p^1 & q^1x_2^2 \\ p^2 & q^2x_2^2 \end{bmatrix} = \begin{bmatrix} p^1 & q^1 \\ x_2^2p^2 & x_2^2q^2 \end{bmatrix}$ that for $x_2^2 \neq 1$ one has $p^2 = q^1 = 0$. The additional equations are

$$\delta = \frac{x_3^4p^1q^2 - (x_4^4 - x_3^3)r^4}{r^3}, \quad \varepsilon = \frac{x_2^3r^3 + (x_2^2 - x_3^3)q^3}{q^2}, \quad (x_3^3 - 1)p^3 = 0,$$

where $p^1q^2 \neq 0$ and $r^3 \neq 0$ can take any value $\neq 0$, hence we can assume $\varepsilon, \delta \in \{0, 1\}$. We obtain from the first equation that $\delta \neq 0$ for $x_3^4 \neq 0$ and $x_4^4 = x_3^3$, otherwise $\delta = 0$ can be achieved by choosing suitable parameters r^4, p^1, q^2. From the second equation it follows that $\varepsilon \neq 0$ for $x_2^3 \neq 0$ and $x_2^2 = x_3^3$, otherwise $\varepsilon = 0$ is possible with suitable parameters q^2, q^3, r^3. The only remaining equation $(x_3^3 - 1)p^3 = 0$ is always solvable, e.g. by $p^3 = 0$. Hence we get solution with $p^1 \neq 0$ giving the matrix $\hat{l}_0$ of the form $\begin{bmatrix} x_2^2 & 0 & 0 \\ \varepsilon & x_3^3 & 0 \\ 0 & \delta & x_4^4 \end{bmatrix}$, $\varepsilon, \delta \in \{0, 1\}$. If $\varepsilon = \delta = 1$ then $x_2^2 = x_3^3 = x_4^4$, hence we obtain $x_2^2 = x_3^3 = x_4^4 = 1$ in the case of $Bl^{(1)}$-family, according to Corollary 7.1. It follows that the assertions (i), (ii), (iii) and (iv) are true.
Now, we consider the case if $p^1 = 0$ and $p^2q^1 \neq 0$. We get from the first two rows that

$u^0 = x_2^2 = \frac{1}{\hat{x}_2^2}$. It follows from the equations of the last two diagonal elements that $\hat{x}_3^3 = \frac{x_3^3}{x_2^2}$, $\hat{x}_4^4 = \frac{x_4^4}{x_2^2}$. Hence the additional equations are

$$\delta = \frac{(x_3^3 - x_4^4)r^4 - p^2q^1x_3^4}{x_2^2r^3},\ -p^2q^1 \neq 0, \quad \varepsilon = \frac{(1-x_3^3)p^3}{x_2^2p^2},\ x_2^2p^2 \neq 0$$
$$x_2^3r^3 + (x_2^2 - x_3^3)q^3 = \varepsilon q^2 x_2^2.$$

Similarly to the previous case we obtain that $\delta \neq 0$ for $x_3^4 \neq 0$ and $x_4^4 = x_3^3$, otherwise $\delta = 0$ with a suitable choice of r^4. The second equation gives $\varepsilon = 0$ with the choice $p^3 = 0$. The last equation is equivalent to $x_2^3r^3 + (x_3^3 - x_2^2)q^3 = 0$, it gives a contradiction if $x_2^3 \neq 0$ and $x_3^3 = x_2^2$, since $r^3 \neq 0$. Hence the matrix equation has a solution with $p^1 = 0$ such that the matrix $\hat{l}_0$ is of the form $\begin{bmatrix} \frac{1}{x_2^2} & 0 & 0 \\ 0 & \frac{x_3^3}{x_2^2} & 0 \\ 0 & \delta & \frac{x_4^4}{x_2^2} \end{bmatrix}$, $\delta \in \{0,1\}$, if and only if $x_2^3 = 0$ or $x_3^3 \neq x_2^2$, contained in the conditions of the assertions (i), (ii). Thus we obtain the proof of the proposition. $\square$

Definition 7.1. The binary Lie $\mathcal{M}^5(\mathfrak{n})$-algebras $\mathfrak{bl}_1^{\mathfrak{n}}(\lambda,\mu)$, $\mathfrak{bl}_2^{\mathfrak{n}}(\lambda)$, $\mathfrak{bl}_3^{\mathfrak{n}}(\mu)$, $\mathfrak{bl}_4^{\mathfrak{n}}$, $\mathfrak{bl}_5^{\mathfrak{n}}(\kappa,\lambda)$, $\mathfrak{bl}_6^{\mathfrak{n}}(\kappa)$, $\mathfrak{bl}_7^{\mathfrak{n}}(\kappa)$, determined by the matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} \mu \neq 2, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix} \lambda \neq 2, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix} \mu \neq 2, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$
$$\begin{bmatrix} \kappa & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \kappa+1 \end{bmatrix}, \quad \begin{bmatrix} \kappa & 0 & 0 \\ 0 & \kappa+1 & 0 \\ 0 & 1 & \kappa+1 \end{bmatrix} \kappa \neq -1, \quad \begin{bmatrix} \kappa & 0 & 0 \\ 1 & \kappa & 0 \\ 0 & 0 & \kappa+1 \end{bmatrix}, \tag{27}$$

respectively, where $\kappa\lambda \neq 0$, are called binary Lie $\mathcal{M}^5(\mathfrak{n})$-algebras of *normal form*. Their multiplication are given by

$\mathfrak{bl}_1^{\mathfrak{n}}(\lambda,\mu)$: $e_1e_2 = e_4$, $e_0e_1 = e_1$, $e_0e_2 = e_2$, $e_0e_3 = \lambda e_3$, $e_0e_4 = \mu e_4$,

$\mathfrak{bl}_2^{\mathfrak{n}}(\lambda)$: $e_1e_2 = e_4$, $e_0e_1 = e_1$, $e_0e_2 = e_2$, $e_0e_3 = \lambda e_3 + e_4$, $e_0e_4 = \lambda e_4$,

$\mathfrak{bl}_3^{\mathfrak{n}}(\mu)$: $e_1e_2 = e_4$, $e_0e_1 = e_1$, $e_0e_2 = e_2 + e_3$, $e_0e_3 = e_3$, $e_0e_4 = \mu e_4$,

$\mathfrak{bl}_4^{\mathfrak{n}}$: $e_1e_2 = e_4$, $e_0e_1 = e_1$, $e_0e_2 = e_2 + e_3$, $e_0e_3 = e_3 + e_4$, $e_0e_4 = e_4$,

$\mathfrak{bl}_5^{\mathfrak{n}}(\kappa,\lambda)$: $e_1e_2 = e_4$, $e_0e_1 = e_1$, $e_0e_2 = \kappa e_2$, $e_0e_3 = \lambda e_3$, $e_0e_4 = (\kappa+1)e_4$,

$\mathfrak{bl}_6^{\mathfrak{n}}(\kappa)$: $e_1e_2 = e_4$, $e_0e_1 = e_1$, $e_0e_2 = \kappa e_2$, $e_0e_3 = (\kappa+1)e_3 + e_4$, $e_0e_4 = (\kappa+1)e_4$,

$\mathfrak{bl}_7^{\mathfrak{n}}(\kappa)$: $e_1e_2 = e_4$, $e_0e_1 = e_1$, $e_0e_2 = \kappa e_2 + e_3$, $e_0e_3 = \kappa e_3$, $e_0e_4 = (\kappa+1)e_4$.

with respect to a special distinguished basis.

Remark 7.3. The matrices listed in (27) are in Jordan canonical form.

Corollary 7.4. The $\mathcal{M}^5(\mathfrak{n})$-algebras $\mathfrak{bl}_1^{\mathfrak{n}}(\lambda,\mu)$, $\mathfrak{bl}_2^{\mathfrak{n}}(\lambda)$, $\mathfrak{bl}_3^{\mathfrak{n}}(\mu)$, $\mathfrak{bl}_4^{\mathfrak{n}}$ are non-Lie $\mathcal{M}^5(\mathfrak{n})$-algebras, and $\mathfrak{bl}_5^{\mathfrak{n}}(\kappa,\lambda)$, $\mathfrak{bl}_6^{\mathfrak{n}}(\kappa)$, $\mathfrak{bl}_7^{\mathfrak{n}}(\kappa)$ are Lie $\mathcal{M}^5(\mathfrak{n})$-algebras.

From Proposition 7.2 follows

Proposition 7.5. Any binary Lie $\mathcal{M}^5(\mathfrak{n})$-algebra in the $Bl^{(1)}$-family is isomorphic to an algebra of *normal form.* The algebras of normal form $\mathfrak{bl}_1^{\mathfrak{n}}(\lambda,\mu)$, $\mathfrak{bl}_2^{\mathfrak{n}}(\lambda)$, $\mathfrak{bl}_3^{\mathfrak{n}}(\mu)$, $\mathfrak{bl}_4^{\mathfrak{n}}$, $\mathfrak{bl}_5^{\mathfrak{n}}(1,\lambda)$, $\mathfrak{bl}_6^{\mathfrak{n}}(1)$, $\mathfrak{bl}_7^{\mathfrak{n}}(\kappa)$ are uniquely determined in the isomorphism class. Isomorphic binary Lie $\mathcal{M}^5(\mathfrak{n})$-algebras of normal form in the $Bl^{(1)}$-family are pairs of Lie algebras given by the relations

$$\mathfrak{bl}_5^{\mathfrak{n}}(\kappa,\lambda) \cong \mathfrak{bl}_5^{\mathfrak{n}}(\frac{1}{\kappa},\frac{\lambda}{\kappa}), \quad \mathfrak{bl}_6^{\mathfrak{n}}(\kappa) \cong \mathfrak{bl}_6^{\mathfrak{n}}(\frac{1}{\kappa}), \quad \text{where} \quad \kappa \neq 1.$$

Proof. According to Proposition 7.2 the binary Lie $\mathcal{M}^5(\mathfrak{n})$-algebras satisfy

a) if $x_4^4 \neq x_3^3$ and $x_2^2 \neq x_3^3$, then $\varepsilon = \delta = 0$,

b) if $x_4^4 = x_3^3$ and $x_2^2 \neq x_3^3$, then $\varepsilon = 0$, $\delta \in \{0,1\}$,

c) if $x_4^4 \neq x_3^3$ and $x_2^2 = x_3^3$, then $\delta = 0$, $\varepsilon \in \{0,1\}$,

d) if $x_2^2 = x_3^3 = x_4^4 = 1$ and $\delta \in \{0,1\}$, $\quad \varepsilon \in \{0,1\}$.

We denote $e_0e_2 = \kappa e_2$, $e_0e_3 = \lambda e_3$, $e_0e_4 = \mu e_4$, where $\kappa = 1$ or $\kappa = \mu - 1$. It follows that for any κ, λ, μ there is the possibility $\varepsilon = \delta = 0$, hence we get $\mathfrak{bl}_1^{\mathfrak{n}}(\lambda,\mu)$ and $\mathfrak{bl}_5^{\mathfrak{n}}(\kappa,\lambda)$. We have $\varepsilon = 0$, $\delta = 1$ only if $\lambda = \mu$ giving $\mathfrak{bl}_2^{\mathfrak{n}}(\lambda)$ and $\mathfrak{bl}_6^{\mathfrak{n}}(\kappa)$. The case $\varepsilon = 1$, $\delta = 0$ is possible only if $\kappa = \lambda$ hence we get $\mathfrak{bl}_3^{\mathfrak{n}}(\mu)$ and $\mathfrak{bl}_7^{\mathfrak{n}}(\kappa)$. The case $\varepsilon = \delta = 1$ gives the multiplication $\mathfrak{bl}_4^{\mathfrak{n}}$. $\square$

We get according to Proposition 6.3.:

Lemma 7.6. The non-Lie Malcev algebras among the non-Lie binary Lie algebras in the $Bl^{(1)}$-family are $\mathfrak{bl}_1^{\mathfrak{n}}(\lambda,-1)$, $\lambda \in \mathbb{K}$, $\mathfrak{bl}_2^{\mathfrak{n}}(-1)$, $\mathfrak{bl}_3^{\mathfrak{n}}(-1)$, which are isomorphic to the algebras $\mathfrak{m}_5(\lambda)$, $\lambda \in \mathbb{K}$, $\mathfrak{m}_3$, $\mathfrak{m}_4$, respectively, given in the list (7).

7.2 $Bl^{(2)}$-family

Proposition 7.7. Any binary Lie algebra $\mathfrak{c}(X)$ in the $Bl^{(2)}$-family is isomorphic to the algebra $\hat{\mathfrak{c}}(\hat{X})$ determined by $\hat{X} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x_3^3 + x_4^4 & 1 \\ 0 & x_3^4x_4^3 - x_4^4x_3^3 & 0 \end{bmatrix}$.

Proof. We consider the isomorphism conditions given by the matrix equation (19) for the matrix A of the partial isomorphism (cf. Lemma 4.1 (b)). The equation $u^0 = 1$ follows from $\begin{bmatrix} p^1 & q^1 \\ p^2 & q^2 \end{bmatrix} = \begin{bmatrix} u^0p^1 & u^0q^1 \\ u^0p^2 & u^0q^2 \end{bmatrix}$. The last column gives the equations $r^3 = \frac{1}{x_4^3}(p^1q^2 - p^2q^1)$, $r^4 = -\frac{x_4^4}{x_4^3}(p^1q^2 - p^2q^1)$. With these replacements we get from the third column

$$\hat{x}_3^3 = x_3^3 + x_4^4, \quad \hat{x}_3^4 = x_3^4x_4^3 - x_4^4x_3^3. \tag{28}$$

The last two rows give the system of equations

$$(x_3^3 + x_4^4 - 1)\begin{bmatrix} p^3 \\ q^3 \end{bmatrix} + \begin{bmatrix} p^4 \\ q^4 \end{bmatrix} = \begin{bmatrix} 0 \\ -r^3 x_2^3 \end{bmatrix},$$
$$(x_3^4 x_4^3 - x_4^4 x_3^3)\begin{bmatrix} p^3 \\ q^3 \end{bmatrix} - \begin{bmatrix} p^4 \\ q^4 \end{bmatrix} = u^2 \begin{bmatrix} p^1 \\ q^1 \end{bmatrix} - u^1 \begin{bmatrix} p^2 \\ q^2 \end{bmatrix} + \begin{bmatrix} 0 \\ r^4 x_2^3 + x_4^2(p^1 q^2 - p^2 q^1) \end{bmatrix}.$$

for p^i, q^i, $i = 2, 3, 4$. Putting $\begin{bmatrix} p^4 \\ q^4 \end{bmatrix} = -x_2^3 \begin{bmatrix} 0 \\ r^3 \end{bmatrix}$, $\begin{bmatrix} p^3 \\ q^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we obtain

$$u^2 \begin{bmatrix} p^1 \\ q^1 \end{bmatrix} - u^1 \begin{bmatrix} p^2 \\ q^2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{x_2^3 - x_4^2 x_4^3}{x_4^3}(p^1 q^2 - p^2 q^1) \end{bmatrix}.$$

If $p^1 \neq 0$, then we get the solution $u^2 = \frac{p^2}{p^1} u^1$, $u^1 = -\frac{x_2^3 - x_4^2 x_4^3}{x_4^3} p^1$. If $p^2 \neq 0$, then we receive the solution $u^1 = \frac{p^1}{p^2} u^2$, $u^2 = -\frac{x_2^3 - x_4^2 x_4^3}{x_4^3} p^2$. Hence there exist solutions of the matrix equation and for any solution one has (28). This proves the claim. □

Definition 7.2. The $\mathcal{M}^5(\mathfrak{n})$-algebra $\mathfrak{c}(X)$ determined by $X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & \nu & 0 \end{bmatrix}$ is called binary Lie algebra of *normal form* and will be denoted by $\mathfrak{bl}_8^{\mathfrak{n}}(\lambda, \nu)$. Its multiplication of $\mathfrak{bl}_8^{\mathfrak{n}}(\lambda, \nu)$ is given by

$$e_1 e_2 = e_4, \quad e_0 e_1 = e_1, \quad e_0 e_2 = e_2, \quad e_0 e_3 = \lambda e_3 + \mu e_4, \quad e_0 e_4 = e_3.$$

From the proof of the previous proposition we get

Corollary 7.8. Any binary Lie algebra $\mathfrak{c}$ in the $Bl^{(2)}$-family is isomorphic to a unique $\mathcal{M}^5(\mathfrak{n})$-algebra $\mathfrak{bl}_8^{\mathfrak{n}}(\lambda, \nu)$ of normal form.

According to the assertion 3) (ii) in Proposition 6.3 we have

Lemma 7.9. The binary Lie algebras in the family $Bl^{(2)}$ are not Lie algebras. Moreover, a binary Lie algebra in the $Bl^{(2)}$-family is a Malcev algebra if and only if its normal form is the algebra $\mathfrak{bl}_8^{\mathfrak{n}}(1, 2)$, which is isomorphic to the algebra $\mathfrak{m}_1$ in the list (7).

7.3 $Bl^{(3)}$-family

Proposition 7.10. Any binary Lie algebra $\mathfrak{c}(X)$ in the $Bl^{(3)}$-family is isomorphic to the $\mathcal{M}^5(\mathfrak{n})$-algebra $\hat{\mathfrak{c}}(\hat{X})$ determined by

$$\hat{X} = \begin{bmatrix} \hat{x}_2^2 & 1 & 0 \\ 1 - \hat{x}_2^2 & 0 & 0 \\ 0 & 0 & \hat{x}_2^2 \end{bmatrix} = \begin{bmatrix} x_2^2 + x_3^3 & 1 & 0 \\ 1 - x_2^2 - x_3^3 & 0 & 0 \\ 0 & 0 & x_2^2 + x_3^3 \end{bmatrix}.$$

Proof. We consider the matrix equation describing the isomorphism (19) of the $\mathcal{M}^5(\mathfrak{n})$-algebras using the forms of $\hat{l}_0$ and l_0. The first row and the last two columns yield

$$q^1 = 0,\ u^0 = 1,\ \hat{x}_4^4 = x_2^2 + x_3^3,\ q^2 = \frac{r^3}{x_3^2},\ q^3 = -x_3^3 q^2,\ q^4 = [x_2^2 r^4 - (p^1 q^2 - p^2 q^1) x_3^4] \frac{1}{x_3^2}.$$

Replacing the values of q^i, $i = 2, 3, 4$, into the second column we obtain first $\hat{x}_3^2 = 1$, $\hat{x}_2^2 = x_2^2 + x_3^3$. Thereafter using $x_2^3 = \frac{1}{x_3^2}(x_2^2 - 1)(x_3^3 - 1)$ we get $\hat{x}_2^3 = 1 - x_2^2 - x_3^3$. At the end we obtain the equation

$$q^4 x_2^2 + r^4 x_2^3 + p^1 q^2 x_2^4 = (x_2^2 + x_3^3) q^4 + u^1 q^2.$$

Since $q^2 \neq 0$ this equation can be solved by choosing a suitable value of u^1. This proves the assertion. □

Definition 7.3. The $\mathcal{M}^5(\mathfrak{n})$-algebra $\mathfrak{c}(X)$ determined by $X = \begin{bmatrix} \kappa & 1 & 0 \\ 1-\kappa & 0 & 0 \\ 0 & 0 & \kappa \end{bmatrix}$, $\kappa \neq 1$ is called binary Lie algebra of *normal form* and denoted by $\mathfrak{bl}_9^{\mathfrak{n}}(\kappa)$. Its multiplication is given by

$$e_1e_2 = e_4, \quad e_0e_1 = e_1, \quad e_0e_2 = \kappa e_2 + (1-\kappa)e_3, \quad e_0e_3 = e_2, \quad e_0e_4 = \kappa e_4.$$

From the proof of the previous proposition we get

Corollary 7.11. Any binary Lie algebra $\mathfrak{c}$ in the $Bl^{(3)}$-family is isomorphic to a unique $\mathcal{M}^5(\mathfrak{n})$-algebra $\mathfrak{bl}_9^{\mathfrak{n}}(\kappa)$ of normal form.

Lemma 7.12. The $\mathcal{M}^5(\mathfrak{n})$-algebras in the $Bl^{(3)}$-family are not Lie algebras. Moreover, $\mathfrak{c}$ is a Malcev algebra in the $Bl^{(3)}$ if and only if its normal form is $\mathfrak{bl}_9^{\mathfrak{n}}(-1)$, which is isomorphic to $\mathfrak{m}_2$ in the list (7).

8 Binary Lie $\mathcal{M}^5(\mathfrak{f})$-algebras

According to the definitions formulated in Section 4, an $\mathcal{M}^5(\mathfrak{f})$-algebra $\mathfrak{c}$ is an anti-commutative semidirect sum $\mathfrak{c} = \mathfrak{l}_2 \oplus_l \mathfrak{i}$ of the 2-dimensional non-abelian Lie algebra $\mathfrak{l}_2$ and a 3-dimensional abelian ideal $\mathfrak{i} = \mathbb{K}e_2 \oplus \mathbb{K}e_3 \oplus \mathbb{K}e_4$, where the matrix of endomorphism l_1 is $Y = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.
There are no non-Lie Malcev $\mathcal{M}^5(\mathfrak{f})$-algebras, since all solvable 5-dimensional Malcev algebras are $\mathcal{M}^5(\mathfrak{n})$-algebras.

Proposition 8.1. An $\mathcal{M}^5(\mathfrak{f})$-algebra $\mathfrak{c}(X) = \mathfrak{l}_2 \oplus_l \mathfrak{i}$ is a

(i) Lie algebra if and only if $X = \begin{bmatrix} x_3^3 - 1 & 0 & 0 \\ x_2^3 & x_3^3 & 0 \\ x_2^4 & x_2^3 & x_3^3 + 1 \end{bmatrix}$, $x_3^3 \neq 1$,

(ii) non-Lie binary Lie algebra if and only if $X = \begin{bmatrix} 1 & 0 & 0 \\ x_2^3 & x_3^3 & 0 \\ x_2^4 & x_3^4 & x_3^3 + 1 \end{bmatrix}$, $x_3^3 \neq 2$ or $x_2^3 \neq x_3^4$.

Proof. Identifying the maps l_0 and l_1 with their matrices we get

$$
\begin{aligned}
l_1^2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; l_1 l_0 = \begin{bmatrix} 0 & 0 & 0 \\ x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \end{bmatrix}, \; l_0 l_1 = \begin{bmatrix} x_3^2 & x_4^2 & 0 \\ x_3^3 & x_4^3 & 0 \\ x_3^4 & x_4^4 & 0 \end{bmatrix}, \; l_0 l_1^2 = \begin{bmatrix} x_4^2 & 0 & 0 \\ x_4^3 & 0 & 0 \\ x_4^4 & 0 & 0 \end{bmatrix}, \\
l_0 l_1 l_0 &= \begin{bmatrix} x_3^2 x_2^2 + x_4^2 x_2^3 & x_3^2 x_3^2 + x_4^2 x_3^3 & x_3^2 x_4^2 + x_4^2 x_4^3 \\ x_3^3 x_2^2 + x_4^3 x_2^3 & x_3^3 x_3^2 + x_4^3 x_3^3 & x_3^3 x_4^2 + x_4^3 x_4^3 \\ x_3^4 x_2^2 + x_4^4 x_2^3 & x_3^4 x_3^2 + x_4^4 x_3^3 & x_3^4 x_4^2 + x_4^4 x_4^3 \end{bmatrix}, \quad l_1 l_0 l_1 = \begin{bmatrix} 0 & 0 & 0 \\ x_3^2 & x_4^2 & 0 \\ x_3^3 & x_4^3 & 0 \end{bmatrix}, \\
l_0^2 &= \begin{bmatrix} x_2^2 x_2^2 + x_3^2 x_2^3 + x_4^2 x_2^4 & x_2^2 x_3^2 + x_3^2 x_3^3 + x_4^2 x_3^4 & x_2^2 x_4^2 + x_3^2 x_4^3 + x_4^2 x_4^4 \\ x_2^3 x_2^2 + x_3^3 x_2^3 + x_4^3 x_2^4 & x_2^3 x_3^2 + x_3^3 x_3^3 + x_4^3 x_3^4 & x_2^3 x_4^2 + x_3^3 x_4^3 + x_4^3 x_4^4 \\ x_2^4 x_2^2 + x_3^4 x_2^3 + x_4^4 x_2^4 & x_2^4 x_3^2 + x_3^4 x_3^3 + x_4^4 x_3^4 & x_2^4 x_4^2 + x_3^4 x_4^3 + x_4^4 x_4^4 \end{bmatrix}, \\
l_1 l_0^2 &= \begin{bmatrix} 0 & 0 & 0 \\ x_2^2 x_2^2 + x_3^2 x_2^3 + x_4^2 x_2^4 & x_2^2 x_3^2 + x_3^2 x_3^3 + x_4^2 x_3^4 & x_2^2 x_4^2 + x_3^2 x_4^3 + x_4^2 x_4^4 \\ x_2^3 x_2^2 + x_3^3 x_2^3 + x_4^3 x_2^4 & x_2^3 x_3^2 + x_3^3 x_3^3 + x_4^3 x_3^4 & x_2^3 x_4^2 + x_3^3 x_4^3 + x_4^3 x_4^4 \end{bmatrix}.
\end{aligned} \tag{29}
$$

According to Theorem 10.5 (c) $\mathfrak{c}$ is a Lie algebra if and only if $l_1 = l_0 l_1 - l_1 l_0$. This gives the matrix equation

$$
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -x_3^2 & -x_4^2 & 0 \\ 1 + x_2^2 - x_3^3 & x_3^2 - x_4^3 & x_4^2 \\ x_2^3 - x_3^4 & 1 + x_3^3 - x_4^4 & x_4^3 \end{bmatrix},
$$

hence $l_0 = \begin{bmatrix} x_3^3 - 1 & 0 & 0 \\ x_2^3 & x_3^3 & 0 \\ x_2^4 & x_2^3 & x_3^3 + 1 \end{bmatrix}$, proving the first assertion.

Theorem 10.5 (a) yields that $\mathfrak{c}$ is a binary Lie algebra if and only if

$$
l_0 l_1 l_0 + l_1 - l_1 l_0^2 - l_0 l_1 = 0, \quad l_0 l_1^2 - l_1 l_0 l_1 - l_1^2 = 0, \tag{30}
$$

Using (29) equations (30) implies that $\mathfrak{c}(X)$ is a binary Lie algebra if and only if one has

$$
\begin{aligned}
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} &= \begin{bmatrix} x_3^2 x_2^2 + x_4^2 x_2^3 & x_3^2 x_3^2 + x_4^2 x_3^3 & x_3^2 x_4^2 + x_4^2 x_4^3 \\ x_3^3 x_2^2 + x_4^3 x_2^3 + 1 & x_3^3 x_3^2 + x_4^3 x_3^3 & x_3^3 x_4^2 + x_4^3 x_4^3 \\ x_3^4 x_2^2 + x_4^4 x_2^3 & x_3^4 x_3^2 + x_4^4 x_3^3 + 1 & x_3^4 x_4^2 + x_4^4 x_4^3 \end{bmatrix} - \\
&- \begin{bmatrix} 0 & 0 & 0 \\ x_2^2 x_2^2 + x_3^2 x_2^3 + x_4^2 x_2^4 & x_2^2 x_3^2 + x_3^2 x_3^3 + x_4^2 x_3^4 & x_2^2 x_4^2 + x_3^2 x_4^3 + x_4^2 x_4^4 \\ x_2^3 x_2^2 + x_3^3 x_2^3 + x_4^3 x_2^4 & x_2^3 x_3^2 + x_3^3 x_3^3 + x_4^3 x_3^4 & x_2^3 x_4^2 + x_3^3 x_4^3 + x_4^3 x_4^4 \end{bmatrix} - \begin{bmatrix} x_3^2 & x_4^2 & 0 \\ x_3^3 & x_4^3 & 0 \\ x_3^4 & x_4^4 & 0 \end{bmatrix},
\end{aligned} \tag{31}
$$

and

$$
\begin{bmatrix} x_4^2 & 0 & 0 \\ x_4^3 & 0 & 0 \\ x_4^4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ x_3^2 & x_4^2 & 0 \\ x_3^3 + 1 & x_4^3 & 0 \end{bmatrix}.
$$

It follows that $X = \begin{bmatrix} x_2^2 & 0 & 0 \\ x_2^3 & x_3^3 & 0 \\ x_2^4 & x_3^4 & x_3^3 + 1 \end{bmatrix}$ satisfies (31) if and only if

$$
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x_3^3 x_2^2 + 1 & 0 \\ x_3^4 x_2^2 + (x_3^3 + 1) x_2^3 & (x_3^3 + 1) x_3^3 + 1 \end{bmatrix} - \begin{bmatrix} x_2^2 x_2^2 & 0 \\ x_2^3 x_2^2 + x_3^3 x_2^3 & x_3^3 x_3^3 \end{bmatrix} - \begin{bmatrix} x_3^3 & 0 \\ x_3^4 & x_3^3 + 1 \end{bmatrix}.
$$

Hence we get

$$(x_3^3 - x_2^2 - 1)(x_2^2 - 1) = 0, \quad (x_3^4 - x_2^3)(x_2^2 - 1) = 0.$$

We obtain two possibilities: 1) $x_2^2 = 1$ or 2) $x_2^2 = x_3^3 - 1$, $x_3^4 = x_2^3$, hence assertion (ii) is true. □

Corollary 8.2. If $\mathfrak{c}(X)$ is a binary Lie $\mathcal{M}^5(\mathfrak{f})$-algebra, then X is lower triangular matrix.

The partial isomorphisms of $\mathcal{M}^5(\mathfrak{f})$-algebras are described in Lemma 4.1 (c). According to Theorem 4.4 (c) we have

Theorem 8.3. Let $\{e_0, e_1, e_2, e_3, e_4\}$ and $\{\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$ be distinguished bases of the $\mathcal{M}^5(\mathfrak{f})$-algebras $\mathfrak{c} = \mathfrak{c}(X)$ and $\hat{\mathfrak{c}} = \hat{\mathfrak{c}}(\hat{X})$ corresponding to lower triangular matrices X and $\hat{X}$. A partial isomorphism $\varphi : \mathfrak{c} \to \hat{\mathfrak{c}}$ determined by (14) is an isomorphism if and only if

$$\begin{aligned} &\begin{bmatrix} p^1 & 0 & 0 & 0 \\ p^2 & q^2 x_2^2 & 0 & 0 \\ p^3 & q^3 x_2^2 + p^1 q^2 x_2^3 & p^1 q^2 x_3^3 & 0 \\ p^4 & q^4 x_2^2 + p^1 q^3 x_2^3 + (p^1)^2 q^2 x_2^4 & p^1 q^3 x_3^3 + (p^1)^2 q^2 x_3^4 & (p^1)^2 q^2 x_4^4 \end{bmatrix} = \\ &= \begin{bmatrix} p^1 & 0 & 0 & 0 \\ \hat{x}_2^2 p^2 & \hat{x}_2^2 q^2 & 0 & 0 \\ \hat{x}_j^3 p^j + u^1 p^2 - u^2 p^1 & \hat{x}_j^3 q^j + u^1 q^2 & \hat{x}_3^3 p^1 q^2 & 0 \\ \hat{x}_j^4 p^j + u^1 p^3 - u^3 p^1 & \hat{x}_j^4 q^j + u^1 q^3 & \hat{x}_3^4 p^1 q^2 + \hat{x}_4^4 p^1 q^3 + u^1 p^1 q^2 & \hat{x}_4^4 (p^1)^2 q^2 \end{bmatrix}. \end{aligned} \tag{32}$$

Comparing the diagonal element in this matrix equation we get:

Corollary 8.4. If X and $\hat{X}$ are lower triangular matrices such that the corresponding $\mathcal{M}^5(\mathfrak{f})$-algebras $\mathfrak{c}(X)$ and $\hat{\mathfrak{c}}(\hat{X})$ are isomorphic, then X and $\hat{X}$ have the same diagonal elements.

Lemma 8.5. A basis $\{e_0, e_1, e_2, e_3, e_4\}$ of an $\mathcal{M}^5(\mathfrak{f})$-algebra $\mathfrak{c} = \mathfrak{c}(X)$ is a special distinguished basis if and only if

(a) $\{e_1, e_2, e_3, e_4\}$ is a basis of the nilpotent ideal $\mathfrak{c}'$ such that $e_4 \in \mathcal{C}(\mathfrak{c}')$ and $e_1 e_2 = e_3$, $e_1 e_3 = e_4$,

(b) the matrix $\mathbf{X}$ of the map $\lambda_0 = L_{e_0}|_{\mathfrak{c}'} : \mathfrak{c}' \to \mathfrak{c}'$ is given by

$$\mathbf{X} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0}^t & X \end{bmatrix}, \quad X = \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix}, \quad \text{where} \quad \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \tag{33}$$

with respect to the basis $\{e_2, e_3, e_4\}$ of $\mathfrak{i}$.

Proof. The form (33) of the matrix $\mathbf{X}$ means that

$$e_0 e_1 = e_1, \quad \text{and} \quad \mathbf{X}(\mathbb{K}e_2 + \mathbb{K}e_3 + \mathbb{K}e_4) \subseteq (\mathbb{K}e_2 + \mathbb{K}e_3 + \mathbb{K}e_4).$$

The multiplications in $\mathfrak{c}'$ are given by $e_1 e_2 = e_3$ and $e_1 e_3 = e_4$, hence the the centralizer $\mathcal{C}_{\mathfrak{c}'}(\mathfrak{c}'')$ of $\mathfrak{c}''$ in $\mathfrak{c}'$ is an abelian ideal. Since the first row of X is non-vanishing, $\mathfrak{c}'$ is isomorphic to $\mathfrak{f}$. Consequently, the conditions (a) and (b) imply that $\{e_0, e_1, e_2, e_3, e_4\}$ is a distinguished basis. The converse statement is clear. □

9 Normal forms of binary Lie $\mathcal{M}^5(\mathfrak{f})$-algebras

We know from Proposition 8.1 that if $\mathfrak{c}(X)$ is a binary Lie $\mathcal{M}^5(\mathfrak{f})$-algebra then X is a lower triangular matrix, and Corollary 8.4 shows that the matrices corresponding to isomorphic $\mathcal{M}^5(\mathfrak{f})$-algebras have the same diagonal elements.

Definition 9.1. An $\mathcal{M}^5(\mathfrak{f})$-algebra $\mathfrak{c}(X)$ with lower triangular matrix X is called *diagonalizable* if $\mathfrak{c}(X)$ is isomorphic to the $\mathcal{M}^5(\mathfrak{f})$-algebra $\mathfrak{c}(\hat{X})$, where $\hat{X} = \begin{bmatrix} x_2^2 & 0 & 0 \\ 0 & x_3^3 & 0 \\ 0 & 0 & x_4^4 \end{bmatrix}$. This $\mathcal{M}^5(\mathfrak{f})$-algebra $\mathfrak{c}(\hat{X})$ is called *diagonal* $\mathcal{M}^5(\mathfrak{f})$-algebra.

Proposition 8.1 yields

Lemma 9.1. For any diagonal binary Lie $\mathcal{M}^5(\mathfrak{f})$-algebra $\mathfrak{c}(X)$ the matrix X has one of the following form

(i) Lie algebra if and only if $X = \begin{bmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda+1 \end{bmatrix}$, $\lambda \neq 1$,

(ii) non-Lie binary Lie algebra if and only if $X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda+1 \end{bmatrix}$, $\lambda \neq 2$.

Now, we investigate diagonalizable binary Lie $\mathcal{M}^5(\mathfrak{f})$-algebras determined by the matrices given in Proposition 8.1 (i) and (ii), which correspond to Lie algebras and non-Lie binary Lie algebras, respectively.

Proposition 9.2. An $\mathcal{M}^5(\mathfrak{f})$-algebra $\mathfrak{c}(X)$ with lower triangular matrix X is diagonalizable if and only if the system of equations

$$(x_2^2 + x_4^4 - 2x_3^3)s = (x_3^4 - x_2^3)p^1, \quad (x_3^3 - x_2^2)s^2 + (x_2^2 - x_4^4)t + x_2^4(p^1)^2 = 0 \tag{34}$$

is solvable for some $s, t \in \mathbb{K}$.

Proof. According to Corollary 8.4, if an $\mathcal{M}^5(\mathfrak{f})$-algebra $\mathfrak{c}(X)$ with lower triangular matrix X is isomorphic to a diagonal $\mathcal{M}^5(\mathfrak{f})$-algebra $\mathfrak{c}(\hat{X})$ then X and $\hat{X}$ have the same diagonal elements. We get from the matrix equation (32)

$$\begin{bmatrix} p^2 & q^2x_2^2 & 0 \\ p^3 & q^3x_2^2 + p^1q^2x_2^3 & p^1q^2x_3^3 \\ p^4 & q^4x_2^2 + p^1q^3x_2^3 + (p^1)^2q^2x_2^4 & p^1q^3x_3^3 + (p^1)^2q^2x_3^4 \end{bmatrix} =$$

$$= \begin{bmatrix} x_2^2p^2 & x_2^2q^2 & 0 \\ x_3^3p^3 + u^1p^2 - u^2p^1 & x_3^3q^3 + u^1q^2 & x_3^3p^1q^2 \\ x_4^4p^4 + u^1p^3 - u^3p^1 & x_4^4q^4 + u^1q^3 & x_4^4p^1q^3 + u^1p^1q^2 \end{bmatrix}.$$

Hence $(x_2^2-1)p^2=0$ and

$$\begin{aligned} &u^1p^2+(x_3^3-1)p^3-u^2p^1=0, \qquad -x_2^3p^1q^2+(x_3^3-x_2^2)q^3+u^1q^2=0,\\ &u^1p^3+(x_4^4-1)p^4-u^3p^1=0, \quad -p^1q^3x_2^3-(p^1)^2q^2x_2^4+u^1q^3+(x_4^4-x_2^2)q^4=0\\ &-x_3^4p^1q^2+(x_4^4-x_3^3)q^3+u^1q^2=0. \end{aligned}$$

The parameters u^1,u^2,u^3 can be expressed by

$$u^1=(x_2^2-x_3^3)\frac{q^3}{q^2}+x_2^3p^1, \quad u^2=u^1\frac{p^2}{p^1}+(x_3^3-1)\frac{p^3}{p^1}, \quad u^3=u^1\frac{p^3}{p^1}+(x_4^4-1)\frac{p^4}{p^1}.$$

We get the additional equations

$$(x_2^2+x_4^4-2x_3^3)q^3=(x_3^4-x_2^3)p^1q^2, \quad (x_2^2-x_4^4)q^4=(u^1-x_2^3p^1)q^3-(p^1)^2q^2x_2^4. \tag{35}$$

Replacing the expression of u^1 into the second equation of (35) we get

$$(x_2^2-x_4^4)q^2q^4+(x_3^3-x_2^2)(q^3)^2=-x_2^4(q^2)^2(p^1)^2. \tag{36}$$

Equations (35) and (36) do not contain p^2, hence we can assume that $(x_2^2-1)p^2=0$ is fulfilled. Denoting $s=\frac{q^3}{q^2}$, $t=\frac{q^4}{q^2}$, the assertion follows from the first equation of (35) and from (36). □

Proposition 9.3. A binary Lie $\mathcal{M}^5(\mathfrak{f})$-algebra $\mathfrak{c}(X)$ is non-diagonalizable if and only if the matrix X has one of the following form

$$\text{(a)} \quad X=\begin{bmatrix}1&0&0\\ \beta&2&0\\ \delta&\gamma&3\end{bmatrix}, \ \beta\neq\gamma, \qquad \text{(b)} \quad X=\begin{bmatrix}1&0&0\\ \beta&0&0\\ \delta&\gamma&1\end{bmatrix}, \ (\gamma-\beta)^2\neq 4\delta.$$

Proof. If $\mathfrak{c}(X)$ is a Lie $\mathcal{M}^5(\mathfrak{f})$-algebra then $X=\begin{bmatrix}x_3^3-1&0&0\\ x_2^3&x_3^3&0\\ x_2^4&x_2^3&x_3^3+1\end{bmatrix}$, $x_3^3\neq 1$, according to Proposition 8.1. In this case the equations (34) reduce to $s^2-2t=-x_2^4(p^1)^2$ having a solution t for every value of s, hence $\mathfrak{c}(X)$ diagonalizable. If $\mathfrak{c}(X)$ is not diagonalizable then $\mathfrak{c}(X)$ is a non-Lie binary Lie $\mathcal{M}^5(\mathfrak{f})$-algebra with $X=\begin{bmatrix}1&0&0\\ x_2^3&x_3^3&0\\ x_2^4&x_3^4&x_3^3+1\end{bmatrix}$, $x_3^3\neq 2$ or $x_2^3\neq x_3^4$. We get from (34) the equations

$$(2-x_3^3)s=(x_3^4-x_2^3)p^1, \quad (x_3^3-1)s^2-x_3^3t+x_2^4(p^1)^2=0.$$

The first one is not solvable if and only if $x_3^3=2$ and $x_2^3\neq x_3^4$, giving condition (a). If $x_3^3\neq 2$ or $x_2^3=x_3^4$ then the first equation clearly has solution, namely if $x_3^3\neq 2$ then $s=\frac{(x_3^4-x_2^3)p^1}{2-x_3^3}$. Substituting a solution s into the second equation we get $x_3^3t=(x_3^3-1)s^2+x_2^4(p^1)^2$, which is not solvable for t exactly if $x_3^3=0$ and $-s^2+x_2^4(p^1)^2=\left(-\frac{(x_3^4-x_2^3)^2}{4}+x_2^4\right)(p^1)^2\neq 0$. Denoting $\beta=x_2^3$, $\gamma=x_3^4$, $\delta=x_2^4$ we get the condition $(\gamma-\beta)^2\neq 4\delta$. □

Now, we find canonical forms of the matrices corresponding to non-diagonalizable binary Lie $\mathcal{M}^5(\mathfrak{f})$-algebras, given in Proposition 9.3.

Lemma 9.4. The $\mathcal{M}^5(\mathfrak{f})$-algebras $\mathfrak{c}(X)$ and $\hat{\mathfrak{c}}(\hat{X})$, determined by the matrices

$$X = \begin{bmatrix} 1 & 0 & 0 \\ \beta & 2 & 0 \\ \delta & \gamma & 3 \end{bmatrix}, \ \beta \neq \gamma \quad \text{and} \quad \hat{X} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix}, \tag{37}$$

are isomorphic non-diagonalizable binary Lie algebras.

Proof. A partial isomorphism $\varphi : \mathfrak{c} \to \hat{\mathfrak{c}}$ described in Lemma 4.1. (c) is an isomorphism if and only if it satisfies the equation (32) with the given matrices (37). We obtain the system of equations

$$\begin{aligned} &p^3 + (u^1+1)p^2 - u^2p^1 = 0, \quad q^3 = -(1+u^1-p^1\beta)q^2 = -(u^1-1-p^1\gamma)q^2 \\ &2p^4 + (u^1-1)p^3 - u^3p^1 = 0, \quad 2q^4 + (u^1-1-p^1\beta)q^3 = (p^1)^2q^2\delta, \end{aligned} \tag{38}$$

consequently,

$$p^1 = \frac{2}{\beta-\gamma}, \quad q^3 = -(u^1-1-p^1\gamma)q^2,$$

$$p^3 = -(u^1+1)p^2 + u^2p^1, \quad p^4 = -\frac{1}{2}((u^1-1)p^3 - u^3p^1), \quad q^4 = \frac{1}{2}((p^1)^2q^2\delta - (u^1-1-p^1\beta)q^3).$$

Replacing the formulas obtained for p^1 and q^3 into the other three equations we get a partial isomorphism satisfying the system (38), proving the assertion. $\square$

Let $\mathbb{K}^{\circ 2} \subset \mathbb{K}^\circ$ be the subgroup of squares in the multiplicative group $\mathbb{K}^\circ$ of the field $\mathbb{K}$.

Lemma 9.5. Let $\sigma : \mathbb{K}^\circ/\mathbb{K}^{\circ 2} \to \mathbb{K}^\circ$ be an arbitrary section of the factor group $\mathbb{K}^\circ/\mathbb{K}^{\circ 2}$ in the group $\mathbb{K}^\circ$. The $\mathcal{M}^5(\mathfrak{f})$-algebras $\mathfrak{c}(X)$ and $\hat{\mathfrak{c}}(\hat{X})$, determined by the matrices

$$X = \begin{bmatrix} 1 & 0 & 0 \\ \beta & 0 & 0 \\ \delta & \gamma & 1 \end{bmatrix}, \ (\gamma-\beta)^2 \neq 4\delta, \quad \text{and} \quad \hat{X} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma(X) & 0 & 1 \end{bmatrix}, \tag{39}$$

where $\sigma(X) = \sigma\left((\delta - \frac{1}{4}(\gamma-\beta)^2)\mathbb{K}^{\circ 2}\right)$, are isomorphic non-diagonalizable binary Lie algebras. Moreover, the $\mathcal{M}^5(\mathfrak{f})$-algebras determined by the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ s & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ t & 0 & 1 \end{bmatrix}, \ s, t \in \mathbb{K}^\circ$$

are isomorphic if and only if s and t belong to the same coset of $\mathbb{K}^{\circ 2}$ in $\mathbb{K}^\circ$.

Proof. A partial isomorphism $\varphi : \mathfrak{c} \to \hat{\mathfrak{c}}$ is an isomorphism if and only if the matrices (39) satisfy

$$\begin{bmatrix} p^3 & q^3 + p^1 q^2 \beta & 0 \\ p^4 & q^4 + p^1 q^2 \beta + (p^1)^2 q^2 \delta & (p^1)^2 q^2 \gamma \end{bmatrix} =$$
$$= \begin{bmatrix} u^1 p^2 - u^2 p^1 & u^1 q^2 & 0 \\ \sigma p^2 + p^4 + u^1 p^3 - u^3 p^1 & \sigma q^2 + q^4 + u^1 q^3 & p^1 q^3 + u^1 p^1 q^2 \end{bmatrix},$$

giving the equations

$$p^3 = u^1 p^2 - u^2 p^1, \quad q^3 = (u^1 - p^1 \beta) q^2 = (p^1 \gamma - u^1) q^2,$$
$$\sigma p^2 + u^1 p^3 = u^3 p^1, \quad \sigma q^2 = q^2 (p^1 \beta + (p^1)^2 \delta) - u^1 q^3.$$

Equivalently, we obtain

$$u^1 = \frac{1}{2} p^1 (\beta + \gamma), \quad p^3 = \left(\frac{1}{2} (\beta + \gamma) p^2 - u^2 \right) p^1, \quad q^3 = \frac{1}{2} (\gamma - \beta) p^1 q^2,$$
$$u^3 = \frac{p^2}{p^1} \sigma + \frac{1}{2} (\beta + \gamma) p^3, \quad \sigma = (p^1)^2 (\delta - \frac{1}{4} (\gamma - \beta)^2). \tag{40}$$

Putting $(p^1)^2 = \frac{4\sigma(X)}{4\delta - (\gamma - \beta)^2}$ we get that the equations (40) have solutions if and only if

$$\sigma(X) = \sigma \left((\delta - \frac{1}{4} (\gamma - \beta)^2) \mathbb{K}^{\circ 2} \right) \in \left(\delta - \frac{1}{4} (\gamma - \beta)^2 \right) \mathbb{K}^{\circ 2},$$

i.e. $\delta - \frac{1}{4}(\gamma - \beta)^2$ and $\sigma(X) = \sigma\left((\delta - \frac{1}{4}(\gamma - \beta)^2)\mathbb{K}^{\circ 2}\right)$ belong to the same coset of $\mathbb{K}^{\circ 2}$ in $\mathbb{K}^{\circ}$. It follows that the second claim is true. Hence the assertions are proved. □

Definition 9.2. The binary Lie $\mathcal{M}^5(\mathfrak{f})$-algebras of *normal form* are the algebras $\mathfrak{bl}_1^{\mathfrak{f}}(\lambda)$, $\mathfrak{bl}_2^{\mathfrak{f}}$, $\mathfrak{bl}_3^{\mathfrak{f}}(\delta)$, $\mathfrak{bl}_4^{\mathfrak{f}}(\lambda)$ determined by the matrices:

$$\mathfrak{bl}_1^{\mathfrak{f}}(\lambda) : \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda + 1 \end{bmatrix}, \ \lambda \neq 2, \qquad \mathfrak{bl}_2^{\mathfrak{f}} : \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix},$$
$$\mathfrak{bl}_3^{\mathfrak{f}}(\delta) : \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \delta & 0 & 1 \end{bmatrix}, \ \delta \in \mathbb{K}^{\circ}, \qquad \mathfrak{bl}_4^{\mathfrak{f}}(\lambda) : \quad \begin{bmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda + 1 \end{bmatrix}, \ \lambda \neq 1.$$

given by the multiplications

$\mathfrak{bl}_1^{\mathfrak{f}}(\lambda)$: $e_1 e_2 = e_3, \ e_1 e_3 = e_4, \ e_0 e_1 = e_1, \ e_0 e_2 = e_2, \ e_0 e_3 = \lambda e_3, \ e_0 e_4 = (\lambda + 1) e_4,$

$\mathfrak{bl}_2^{\mathfrak{f}}$: $e_1 e_2 = e_3, \ e_1 e_3 = e_4, \ e_0 e_1 = e_1, \ e_0 e_2 = e_2 + e_3, \ e_0 e_3 = 2e_3 - e_4, \ e_0 e_4 = 3e_4,$

$\mathfrak{bl}_3^{\mathfrak{f}}(\delta)$: $e_1 e_2 = e_3, \ e_1 e_3 = e_4, \ e_0 e_1 = e_1, \ e_0 e_2 = e_2 + \delta e_4, \ e_0 e_4 = e_4,$

$\mathfrak{bl}_4^{\mathfrak{f}}(\lambda)$: $e_1 e_2 = e_3, \ e_1 e_3 = e_4, \ e_0 e_1 = e_1, \ e_0 e_2 = (\lambda - 1) e_2, \ e_0 e_3 = \lambda e_3, \ e_0 e_4 = (\lambda + 1) e_4.$

with respect to a special distinguished basis.

It follows from Lemmas 9.1, 9.4, 9.5:

Proposition 9.6. Any binary Lie $\mathcal{M}^5(\mathfrak{f})$-algebra is isomorphic to a binary Lie $\mathcal{M}^5(\mathfrak{f})$-algebra of normal form, where $\mathfrak{bl}_1^{\mathfrak{f}}(\lambda)$, $\mathfrak{bl}_2^{\mathfrak{f}}$, $\mathfrak{bl}_3^{\mathfrak{f}}(\delta)$ are non-Lie, and $\mathfrak{bl}_4^{\mathfrak{f}}(\lambda)$ are Lie $\mathcal{M}^5(\mathfrak{f})$-algebras. The normal forms $\mathfrak{bl}_1^{\mathfrak{f}}(\lambda)$, $\mathfrak{bl}_2^{\mathfrak{f}}$, $\mathfrak{bl}_4^{\mathfrak{f}}(\lambda)$ are uniquely determined in their isomorphism class. The algebras $\mathfrak{bl}_3^{\mathfrak{f}}(\delta_1)$ and $\mathfrak{bl}_3^{\mathfrak{f}}(\delta_2)$ are isomorphic if and only if δ_1 and δ_2 belong to the same coset of $\mathbb{K}^{\circ 2}$ in $\mathbb{K}^{\circ}$.

Since for real field $\mathbb{R}$ one has $\mathbb{R}^{\circ 2} = \{t \in \mathbb{R} : \ t > 0\}$ we get

Corollary 9.7. Any real $\mathcal{M}^5(\mathfrak{f})$-algebra is isomorphic to a unique $\mathcal{M}^5(\mathfrak{f})$-algebra of normal form

$$\mathfrak{bl}_1^{\mathfrak{f}}(\lambda), \quad \mathfrak{bl}_2^{\mathfrak{f}}, \quad \mathfrak{bl}_3^{\mathfrak{f}}(\delta),\ \delta = \pm 1, \quad \mathfrak{bl}_4^{\mathfrak{f}}(\lambda).$$

10 Summary

According to Proposition 5.3, Corollary 7.4, Proposition 7.5, Lemma 7.6, Corollary 7.8, Lemma 7.9, Corollary 7.11, Lemma 7.12 and Proposition 9.6, we get the classification of non-Lie binary Lie $\mathcal{M}^5(\mathfrak{a})$-algebras:

Theorem 10.1. The non-Lie $\mathcal{M}^5$-algebras are $\mathcal{M}^5(\mathfrak{n})$- or $\mathcal{M}^5(\mathfrak{f})$-algebras, in particular, the non-Lie Malcev $\mathcal{M}^5$-algebras are $\mathcal{M}^5(\mathfrak{n})$-algebras.
Any non-Lie $\mathcal{M}^5$-algebra is isomorphic to an algebra of normal form given in

	$\mathcal{M}^5(\mathfrak{n})$-algebra	$\mathcal{M}^5(\mathfrak{f})$-algebra
binary Lie	$\mathfrak{bl}_1^{\mathfrak{n}}(\lambda,\mu)$, $\mathfrak{bl}_2^{\mathfrak{n}}(\lambda)$, $\mathfrak{bl}_3^{\mathfrak{n}}(\mu)$, $\mathfrak{bl}_4^{\mathfrak{n}}$, $\mathfrak{bl}_8^{\mathfrak{n}}(\lambda,\nu)$, $\mathfrak{bl}_9^{\mathfrak{n}}(\kappa)$	$\mathfrak{bl}_1^{\mathfrak{f}}(\lambda)$, $\mathfrak{bl}_2^{\mathfrak{f}}$, $\mathfrak{bl}_3^{\mathfrak{f}}(\delta)$
Malcev	$\mathfrak{bl}_1^{\mathfrak{n}}(\lambda,-1)$, $\mathfrak{bl}_2^{\mathfrak{n}}(-1)$, $\mathfrak{bl}_3^{\mathfrak{n}}(-1)$, $\mathfrak{bl}_8^{\mathfrak{n}}(1,2)$, $\mathfrak{bl}_9^{\mathfrak{n}}(-1)$	

where $\kappa, \lambda, \mu, \nu \in \mathbb{K}, \delta \in \mathbb{K}^{\circ}$. The non-Lie $\mathcal{M}^5$-algebras of normal form are uniquely determined in their isomorphism class up to the isomorphism

$$\mathfrak{bl}_3^{\mathfrak{f}}(\delta_1) \cong \mathfrak{bl}_3^{\mathfrak{f}}(\delta_2), \quad \text{if} \quad \delta_1\mathbb{K}^{\circ 2} = \delta_2\mathbb{K}^{\circ 2}.$$

From Propositions 5.3, 5.5, Corollary 7.4, Proposition 7.5 and Proposition 9.6, we obtain the classification of Lie $\mathcal{M}^5$-algebras:

Theorem 10.2. Any Lie $\mathcal{M}^5$-algebra is isomorphic to an algebra of normal form given by

$\mathcal{M}^5(\mathfrak{a})$	$\mathcal{M}^5(\mathfrak{n})$	$\mathcal{M}^5(\mathfrak{f})$
$\mathfrak{bl}_1^{\mathfrak{a}}(\kappa,\lambda,\mu)$, $\mathfrak{bl}_2^{\mathfrak{a}}(\kappa,\lambda)$, $\mathfrak{bl}_3^{\mathfrak{a}}(\kappa)$, $\mathfrak{bl}_4^{\mathfrak{a}}(\kappa,\lambda,\mu)^*$, $\mathfrak{bl}_5^{\mathfrak{a}}(\kappa,\lambda,\mu)^*$	$\mathfrak{bl}_5^{\mathfrak{n}}(\kappa,\lambda)$, $\mathfrak{bl}_6^{\mathfrak{n}}(\kappa)$, $\mathfrak{bl}_7^{\mathfrak{n}}(\kappa)$	$\mathfrak{bl}_4^{\mathfrak{f}}(\lambda)$

The $\mathcal{M}^5$-algebras of normal form are uniquely determined in their isomorphism class up to the following isomorphisms:

$$\mathfrak{bl}_1^{\mathfrak{a}}(\pi(\kappa),\pi(\lambda),\pi(\mu)) \cong \mathfrak{bl}_1^{\mathfrak{a}}(\frac{1}{\rho(\kappa)},\frac{\rho(\lambda)}{\rho(\kappa)},\frac{\rho(\mu)}{\rho(\kappa)}) \cong \mathfrak{bl}_1^{\mathfrak{a}}(\frac{\sigma(\mu)}{\sigma(\kappa)},\frac{1}{\sigma(\kappa)},\frac{\sigma(\lambda)}{\sigma(\kappa)}) \cong \mathfrak{bl}_1^{\mathfrak{a}}(\frac{\tau(\lambda)}{\tau(\kappa)},\frac{\tau(\mu)}{\tau(\kappa)},\frac{1}{\tau(\kappa)}),$$

$$\pi,\rho,\sigma,\tau \in \mathrm{Sym}(\{\kappa,\lambda,\mu\}),\ \kappa\lambda\mu \neq 0,$$

$$\mathfrak{bl}_2^{\mathfrak{a}}(\kappa,\lambda) \cong \mathfrak{bl}_2^{\mathfrak{a}}(\frac{1}{\kappa},\frac{\lambda}{\kappa}),\quad \kappa \neq 1,$$

$$\mathfrak{bl}_4^{\mathfrak{a}}(\kappa,\lambda,\mu)^* \cong \mathfrak{bl}_4^{\mathfrak{a}}(\frac{1}{\kappa},\frac{\lambda}{\kappa^2},\frac{\mu}{\kappa})^*,\quad \kappa \neq 1,$$

$$\mathfrak{bl}_5^{\mathfrak{n}}(\kappa,\lambda) \cong \mathfrak{bl}_5^{\mathfrak{n}}(\frac{1}{\kappa},\frac{\lambda}{\kappa}),\quad \kappa \neq 1,$$

$$\mathfrak{bl}_6^{\mathfrak{n}}(\kappa) \cong \mathfrak{bl}_6^{\mathfrak{n}}(\frac{1}{\kappa}),\quad \kappa \neq 1.$$

In the case of the real field $\mathbb{K}=\mathbb{R}$, from Corollaries 5.6, 7.4, 7.8, 7.11, 9.7 we obtain:

Corollary 10.3. Any real $\mathcal{M}^5$-algebra is isomorphic to a unique algebra of normal form given by the list

$\mathfrak{bl}_1^{\mathfrak{a}}(\kappa,\lambda,\mu)$, $\mathfrak{bl}_2^{\mathfrak{a}}(\kappa,\lambda)$, $\mathfrak{bl}_3^{\mathfrak{a}}(\kappa)$, $\mathfrak{bl}_4^{\mathfrak{a}}(\kappa,\lambda,\mu)$, $\mathfrak{bl}_5^{\mathfrak{a}}(\kappa,\lambda,\mu)$, $\mathfrak{bl}_1^{\mathfrak{n}}(\lambda,\mu)$, $\mathfrak{bl}_2^{\mathfrak{n}}(\lambda)$, $\mathfrak{bl}_3^{\mathfrak{n}}(\mu)$, $\mathfrak{bl}_4^{\mathfrak{n}}$, $\mathfrak{bl}_5^{\mathfrak{n}}(\kappa,\lambda)$, $\mathfrak{bl}_6^{\mathfrak{n}}(\kappa)$, $\mathfrak{bl}_7^{\mathfrak{n}}(\kappa)$, $\mathfrak{bl}_8^{\mathfrak{n}}(\lambda,\nu)$, $\mathfrak{bl}_9^{\mathfrak{n}}(\kappa)$, $\mathfrak{bl}_1^{\mathfrak{f}}(\lambda)$, $\mathfrak{bl}_2^{\mathfrak{f}}$, $\mathfrak{bl}_3^{\mathfrak{f}}(1)$, $\mathfrak{bl}_3^{\mathfrak{f}}(-1)$, $\mathfrak{bl}_4^{\mathfrak{f}}(\lambda)$, satisfying the inequalities

$$\mathfrak{bl}_1^{\mathfrak{a}}(\kappa,\lambda,\mu),\quad -1 \le \mu \le \lambda \le \kappa \le 1,\ \kappa\lambda\mu \neq 0,$$

$$\mathfrak{bl}_2^{\mathfrak{a}}(\kappa,\lambda),\quad 0 < |\kappa| \le 1,\ \lambda \neq 0,$$

$$\mathfrak{bl}_4^{\mathfrak{a}}(\kappa,\lambda,\mu),\quad 0 < |\kappa| \le 1,\ \lambda \neq 0,$$

$$\mathfrak{bl}_5^{\mathfrak{n}}(\kappa,\lambda),\quad 0 < |\kappa| \le 1,$$

$$\mathfrak{bl}_6^{\mathfrak{n}}(\kappa),\quad 0 < |\kappa| \le 1.$$

APPENDIX: Anti-commutative semidirect sum

In this section we investigate binary Lie anti-commutative semidirect sums $\mathfrak{l}_2 \oplus_l \mathfrak{i}$ with respect to a bilinear map $l : \mathfrak{l}_2 \times \mathfrak{i} \to \mathfrak{i}$, where $\mathfrak{l}_2$ is the 2-dimensional non-abelian Lie algebra with multiplication $e_0e_1 = e_1$ and $\mathfrak{i}$ is an abelian algebra. We denote by $l_\xi : \mathfrak{i} \to \mathfrak{i}$ the map induced on the abelian ideal $\mathfrak{i}$ by the left multiplication $L_{(\xi,0)} : \mathfrak{l}_2 \oplus_l \mathfrak{i} \to \mathfrak{l}_2 \oplus_l \mathfrak{i}$ for given $\xi \in \mathfrak{l}_2$.

Theorem 10.4. The semidirect sum $\mathfrak{l}_2 \oplus_l \mathfrak{i}$ is a

(i) binary Lie algebra if and only if

$$0 = l_\eta \cdot l_\xi \cdot l_\eta - l_\xi \cdot l_\eta \cdot l_\eta + l_{\xi\eta\cdot\eta} - l_\eta \cdot l_{\eta\xi}, \tag{41}$$

(ii) Malcev algebra if and only if

$$l_{\xi\zeta} \cdot l_\eta = -l_\zeta \cdot l_\eta \cdot l_\xi + l_{\xi\eta\cdot\zeta} - l_\xi \cdot l_{\eta\zeta} + l_\eta \cdot l_\xi \cdot l_\zeta. \tag{42}$$

Proof. Since $\mathfrak{l}_2 \oplus 0$ is a subalgebra of $\mathfrak{l}_2 \oplus_l \mathfrak{i}$ we investigate only the second components of the expressions on the left, respectively, right hand side of the Sagle identity (4). As $\mathfrak{i}$ is abelian, the left hand side of this identity is the sum of cyclic permutations of

$$l_{\xi\eta\cdot\zeta}(T) - l_\tau\big(l_{\xi\eta}(Z) - l_\zeta l_\xi(Y) + l_\zeta l_\eta(X)\big)$$

for all (ξ, X), (η, Y), $(\zeta, Z), (\tau, T) \in \mathfrak{l}_2 \oplus_l \mathfrak{i}$. The right hand side is

$$l_{\xi\zeta}\big(l_\eta(T) - l_\tau(Y)\big) - l_{\eta\tau}\big(l_\xi(Z) - l_\zeta(X)\big).$$

Hence $\mathfrak{l}_2 \oplus_l \mathfrak{i}$ satisfies the Sagle identity if one has

$$\begin{aligned} &l_{\xi\eta\cdot\zeta}(T) + l_{\eta\zeta\cdot\tau}(X) + l_{\zeta\tau\cdot\xi}(Y) + l_{\tau\xi\cdot\eta}(Z)- \\ &-l_\tau\big(l_{\xi\eta}(Z) - l_\zeta l_\xi(Y) + l_\zeta l_\eta(X)\big) - l_\xi\big(l_{\eta\zeta}(T) - l_\tau l_\eta(Z) + l_\tau l_\zeta(Y)\big)- \\ &-l_\eta\big(l_{\zeta\tau}(X) - l_\xi l_\zeta(T) + l_\xi l_\tau(Z)\big) - l_\zeta\big(l_{\tau\xi}(Y) - l_\eta l_\tau(X) + l_\eta l_\xi(T)\big) = \\ &=l_{\xi\zeta}\big(l_\eta(T) - l_\tau(Y)\big) - l_{\eta\tau}\big(l_\xi(Z) - l_\zeta(X)\big). \end{aligned} \tag{43}$$

This is satisfied if and only if for $\xi, \eta, \zeta \in \mathfrak{l}_2$ the identity (42) holds, proving assertion (*ii*). Applying Remark 2.1 we put $Y \mapsto Z$, $X \mapsto T$, $\eta \mapsto \zeta$, $\xi \mapsto \tau$ into equation (43). We get

$$l_{\xi\eta\cdot\eta}(X) + l_{\eta\xi\cdot\xi}(Y) - l_\xi l_{\xi\eta}(Y) + l_\xi l_\eta l_\xi(Y) - l_\xi l_\eta l_\eta(X) - l_\eta l_{\eta\xi}(X) + l_\eta l_\xi l_\eta(X) - l_\eta l_\xi l_\xi(Y) = 0$$

for all $(\xi, X), (\eta, Y) \in \mathfrak{l}_2 \oplus_l \mathfrak{i}$, which is equivalent to (41), hence the assertion (*i*) follows. □

Let $\xi = \xi_0 e_0 + \xi_1 e_1,\ \eta = \eta_0 e_0 + \eta_1 e_1,\ \zeta = \zeta_0 e_0 + \zeta_1 e_1 \in \mathfrak{l}_2$ be the decomposition of the vectors $\xi, \eta, \zeta \in \mathfrak{l}_2$ and denote $l_i = l_{e_i},\ i = 0, 1$.

Theorem 10.5. The semidirect sum $\mathfrak{l}_2 \oplus_l \mathfrak{i}$ is a

(a) binary Lie algebra if and only if

$$l_0 l_1 l_0 + l_1 - l_1 l_0^2 - l_0 l_1 = 0, \quad l_0 l_1^2 - l_1 l_0 l_1 - l_1^2 = 0,$$

(b) Malcev algebra if and only if it is binary Lie and

$$l_1 l_0^2 - l_0^2 l_1 + l_0 l_1 + l_1 l_0 = 0, \quad l_1^2 l_0 - l_1 l_0 l_1 + l_1^2 = 0,$$

(c) Lie algebra if and only if

$$l_1 = l_0 l_1 - l_1 l_0.$$

Proof. According to the identity (41), $\mathfrak{l}_2 \oplus_l \mathfrak{i}$ is a binary Lie algebra if and only if it satisfies

$$0 = (\xi_0\eta_0\eta_1 - \xi_1\eta_0^2)(l_0 l_1 l_0 + l_1 - l_1 l_0^2 - l_0 l_1) + (\xi_0\eta_1^2 - \xi_1\eta_0\eta_1)\,(l_0 l_1^2 - l_1 l_0 l_1 - l_1^2),$$

for any $\xi_0, \eta_0, \xi_1, \eta_1 \in \mathbb{K}$. Hence $\mathfrak{l}_2 \oplus_l \mathfrak{i}$ is a binary Lie algebra if and only if the coefficients of $\xi_0\eta_0\eta_1 - \xi_1\eta_0^2$ and $\xi_0\eta_1^2 - \xi_1\eta_0\eta_1$ equal zero, hence we get the assertion (a).
From the identity (42) characterizing Malcev algebras we obtain

$$\begin{aligned} &l_0^2 l_1 - l_0 l_1 l_0 - l_1 l_0 - l_1 = 0, \quad l_1 l_0^2 - l_0^2 l_1 + l_0 l_1 + l_1 l_0 = 0, \quad l_0 l_1 l_0 - l_1 l_0^2 - l_0 l_1 + l_1 = 0, \\ &l_0 l_1^2 - l_1^2 l_0 - 2 l_1^2 = 0, \quad l_1 l_0 l_1 - l_0 l_1^2 + l_1^2 = 0, \quad l_1^2 l_0 - l_1 l_0 l_1 + l_1^2 = 0. \end{aligned}$$

The sum of the first three and the second three identities gives 0, so we omit the first and fourth identity. We get the identities characterizing Malcev algebras, proving assertion (b).
The semidirect sum $\mathfrak{l}_2 \oplus_l \mathfrak{i}$ is a Lie algebra if and only if $l_{\xi\eta} = l_\xi \cdot l_\eta - l_\eta \cdot l_\xi$, giving $l_1 = l_0 l_1 - l_1 l_0$, hence we get assertion (c). □

11 Conflict of Interest

The authors have no conflict of interest to declare that are relevant to this article.

References

[1] M. A. Akivis, V. V. Goldberg, Local algebras of a differentiable quasigroup. Bull. Amer. Math. Soc. 43, (2006), 207–226.

[2] Á. Figula and P. T. Nagy, Classification of a family of four-dimensional anti-commutative algebras and their groups of automorphisms, Linear Algebra Applications 656, No. 1, (2023), 385–408,

[3] A. T. Gainov, Identical relations for binary Lie rings, (Russian), Uspehi Mat. Nauk (N.S.) 12, No. 3(75), (1957), 141–146.

[4] A. T. Gainov, Binary Lie algebras of lower ranks, (Russian), Algebra i Logika, 2, No. 4, (1963), 21–40.

[5] R. Ghanam, I. Strugar and G. Thompson, Matrix representations for low dimensional Lie algebras, Extracta Mathematicae 20, (2005), 151–184.

[6] V. V. Gorbatsevich, A. L. Onishchik and E. B. Vinberg, Foundations of Lie Theory; Lie Transformation Groups, Springer-Verlag, Berlin, (1997).

[7] A. N. Grishkov, On the theory of finite-dimensional binary Lie algebras, (Russian), Algebra i Logika 16, No. 5, (1977), 549–556.

[8] A. N. Grishkov, Structure and representation of binary-Lie algebras, Math. USSR Izv. 17, No. 2, (1981), 243–269.

[9] A. N. Grishkov, Three aspects of the exponential map, In A. G. Pinus, K. N. Ponomarev (eds): Algebra and Model Theory 2, Novosib. State Tech. Uni., (1999), 50–60, https://www.ime.usp.br/ grishkov/papers/dok.pdf

[10] A. N. Grishkov, M. Rasskazova and L. Sabinina, Binary Lie Algebras with Identities, in: Scientific Legacy of Professor Zbigniew Oziewicz (H. M. C. García, J. de J. Cruz Guzmán, L. H. Kauffman, H. Makaruk, eds.), Series on knots and everything, 75. World Scientific, New Jersey, (2023), 295–307.

[11] K. H. Hoffman and R. Kunze, Linear Algebra, 2nd ed. Prentice Hall, New Jersey, (1971).

[12] F. S. Kerdman, Moufang loops in the large, (Russian), Algebra i Logika, 18, (1979), 523–555.

[13] F. S. Kerdman, Schreier theorem for analytic Moufang loops. Algebra and Logic 19, (1980), 179–190.

[14] E. N. Kuzmin, Malcev algebras of dimension five over a field of characteristic zero, (Russian), Algebra i Logika, 9, (1970), 691–700.

[15] E. N. Kuzmin, The connection between Malcev algebras and analytic Moufang loops, (Russian), Algebra i Logika, 10, No. 1, (1971), 3–22.

[16] E. N. Kuzmin, Binary Lie algebras of small dimensions, Algebra and Logic 37, (1998), 181–186.

[17] A. I. Malcev, Analytic loops, (Russian), Mat. Sb. N.S. 36(78), No. 3, (1955), 569–576.

[18] G. M. Mubarakzyanov, Classification of real structures of Lie algebras of fifth order, (Russian), Izv. Vyssh. Uchebn. Zaved. Mat., 3:34, (1963), 99–106.

[19] P. T. Nagy, Extension of local loop isomorphism, Monatsh. Math. 112, (1991), 221–225.

[20] P. T. Nagy, K. Strambach, *Loops in Group Theory and Lie Theory*, Expositions in Mathematics 35, Walter de Gruyter, Berlin–New York (2002).

[21] A. R. Perry, A classification of real indecomposable solvable Lie algebras of small dimension with codimension one nilradicals, Master's thesis, Utah State University, (2007), http://arxiv.org/abs/1311.6069

[22] L. V. Sabinin, Smooth quasigroups and loops: forty-five years of incredible growth, Comment. Math. Univ. Carolin. 41, (2000), 377–400.

[23] A. A. Sagle, Malcev algebras, Trans. Amer. Math. Soc. 101, (1961), 426–458.

[24] A. A. Sagle, On anti-commutative algebras and analytic loops, Canad. J. Math. 17, (1965), 550–558.