ON THE EQUATION $A!B! = C!$

L. HAJDU, Á. PAPP, AND T. SZAKÁCS

Abstract. We consider the equation in the title in positive integers A, B, C. We give an explicit upper bound for C in terms of the difference $k := B - A$. Further, we show that for $k \leq 10^6$ this equation has only one (long known) non-trivial solution, given by $6!7! = 10!$.

1. Introduction and the main result

The question of finding all products of factorials yielding a factorial is a long standing problem, studied by many authors. Here we only mention a few related results; for a survey of the topic see e.g. Guy [8], section B23. Consider the equation

$$n! = \prod_{i=1}^{r} a_i! \tag{1}$$

with $r \geq 2$ in positive integers $n, a_1, \dots, a_r$, with $a_1 \geq \dots \geq a_r > 1$. Observe that this equation has infinitely many solutions given by

$$n = a_2! \dots a_r!, \ a_1 = n - 1, \text{ with } a_2, \dots, a_r \text{ arbitrary.}$$

For example, we have $6! = 5!3!$ or $12! = 11!3!2!$. Such solutions are called trivial. Obviously, equation (1) has infinitely many trivial solutions. On the other hand, according to a conjecture of Surányi, the only non-trivial solution to (1) with $r = 2$ is $10! = 7!6!$, while a conjecture of Hickerson predicts that the only non-trivial solutions to (1) are given by $9! = 7!3!3!2!$, $10! = 7!6! = 7!5!3!$, $16! = 14!5!2!$ (see e.g. Erdős [4], pp. 27-28). These conjectures have been checked for $n \leq 10^6$ by Caldwell [3]. Erdős [4] (see Theorem 2) proved that writing $P(m)$ for

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the largest prime factor of the positive integer m (with the convention $P(1) = 1$), the assertion

$$P(n(n+1)) > 4 \log n \tag{2}$$

would imply that equation (1) has only finitely many non-trivial solutions - however, (2) is far from being established. (See also [7], p. 70.) Luca [10] proved that assuming the *abc*-conjecture, (1) has only finitely may solutions. This result (beside obtaining other related theorems) has been made more explicit by Luca, Saradha and Shorey [11].

We also mention that after multiplying both sides of (1) by $n!$, we get an equation of the form

$$n! \prod_{i=1}^{r} a_i! = y^2.$$

This equation also attracted a lot of attention. For related results, here we only mention a classical paper of Erdős and Graham [6] together with the recent paper of Luca, Saradha and Shorey [11], and the references there.

In this paper we consider the case $r = 2$, and rewrite equation (1) as

$$A!B! = C! \tag{3}$$

with positive integers A, B, C satisfying $C \geq B \geq A > 1$.

As we noted already, the problem of finding all solutions to equation (3) is still open. Beside the results mentioned so far, we recall a theorem of Erdős [5] saying that in all solutions of (3) with C large enough, we have $C - B \leq 5 \log \log C$. This result has been recently sharpened by Bath and Ramachandra [1] to $C - B \leq ((1 + \varepsilon)/\log 2) \log \log C$ for $C > C_\varepsilon$, with arbitrary $\varepsilon > 0$. Recalling the result of Luca, under the *abc*-conjecture we have $C - B = 1$ for C large enough. Note that, however, if we would assume that say $C - B = 2$, equation (3) would still remain very hard to solve. In this direction, we only refer to a paper of Luca [9] and the references there.

In this paper, our purpose is to show the finiteness of the solutions to (3) with $k := B - A$ bounded. Our main result provides an explicit upper bound for C in terms of k. Certainly, this immediately implies that for any fixed k, (3) has only finitely many solutions. Further, we show that the only non-trivial solution to (3) with $k \leq 10^6$ is the well-known one mentioned earlier.

Theorem 1.1. *Writing $k = B - A$, for all non-trivial solutions of equation* (3) *different from $(A, B, C) = (6, 7, 10)$ we have $C < 5k$. Further, if $k \leq 10^6$, then the only non-trivial solution to* (3) *is given by $(A, B, C) = (6, 7, 10)$.*

2. Proof

To prove the theorem, we need some lemmas. The first one provides explicit lower and upper bounds for the prime counting function $\pi(x)$.

Lemma 2.1. *We have*

(i) $\frac{x}{\log x}\left(1+\frac{1}{2\log x}\right) < \pi(x)$ *for* $x \geq 59$,

(ii) $\pi(x) < \frac{x}{\log x}\left(1+\frac{3}{2\log x}\right)$ *for* $x \geq 1$.

Proof. Parts (i) and (ii) are formulas (3.1) and (3.2) in Rosser and Schoenfeld [14], respectively. □

We shall also need an explicit estimate for the number of primes in an interval.

Lemma 2.2. *Let* $M > 0$ *and* $N > 1$*. Then we have*

$$\pi(M+N) - \pi(M) \leq 2\pi(N).$$

Proof. The statement is formula (1.12) in Montgomery and Vaughan [12]. □

We shall also need bounds for the n-th prime p_n.

Lemma 2.3. *We have*

(i) $n(\log n + \log\log n - 3/2) < p_n$ *for* $n \geq 2$,

(ii) $p_n < n(\log n + \log\log n - 1/2)$ *for* $n \geq 20$.

Proof. Parts (i) and (ii) are formulas (3.10) and (3.11) in [14], respectively. □

We need a simple variant of the Stirling-formula, too.

Lemma 2.4. *For all* $n \geq 1$ *we have*

$$\sqrt{2\pi}\cdot n^{n+1/2}\cdot e^{-n} \leq n! \leq e\cdot n^{n+1/2}\cdot e^{-n}.$$

Proof. For $n = 1$ the assertion can be readily checked. For $n \geq 2$ the statement immediately follows from the more refined bounds

$$\sqrt{2\pi}\cdot n^{n+1/2}\cdot e^{-n+1/(12n+1)} < n! < \sqrt{2\pi}\cdot n^{n+1/2}\cdot e^{-n+1/12n}$$

given by Robbins [13]. □

Now, we have all the tools to prove our theorem.

Proof of Theorem 1.1. In view of $C > B$, the equation $A!B! = C!$ can be rewritten as

$$A! = (B+1)\cdots C. \tag{4}$$

Observe that no prime p with $C/2 < p \le C$ can appear on either side of the above equation. That is, all such primes must belong to the interval $(A, B]$. So for all solutions A, B, C we have

$$\pi(C) - \pi(C/2) \le \pi(B) - \pi(A).$$

Note that this inequality is the main observation behind our results. Using parts (i) and (ii) of Lemma 2.1 to bound the left hand side and Lemma 2.2 to bound the right hand side of the above inequality, recalling the notation $k = B - A$ we obtain

$$\frac{C}{\log C}\left(1+\frac{1}{2\log C}\right) - \frac{C/2}{\log C/2}\left(1+\frac{3}{2\log C/2}\right) < \frac{2k}{\log k}. \tag{5}$$

(Note that here we tacitly assumed that $C \ge 59$, whence $k \ge 2$. However, $C < 59$ would be a much better bound for C than the one we get by the general argument.) If contrary to what we want to prove, $C \ge 5k$ would hold, then (5) would imply

$$\frac{C}{\log C}\left(1+\frac{1}{2\log C}\right) - \frac{C/2}{\log C/2}\left(1+\frac{3}{2\log C/2}\right) < \frac{2C/5}{\log 2C/5}.$$

It is obvious that for large C, the above inequality cannot hold. A simple calculation with Magma [2] shows that this is the case whenever $C > 10^6$. However, by Caldwell's result [3] mentioned earlier, we know that the only solution to (3) with $C \le 10^6$ is given by $(A, B, C) = (6, 7, 10)$. Hence we get that apart from this solution we always have $C < 5k$, and the first part of the theorem follows.

Now we consider the second statement. Assume first that $k \le 850000$. Observe that in (4), none of $B + 1, B + 2, \dots, C$ can be a prime. Let p_{n+1} be the first prime greater than C. By Bertrand's postulate we have that $p_n > C/2$. This shows that $A < p_n \le B$ must be valid. Thus, by (3), we obtain that

$$(p_n + 1)\cdots(p_{n+1} - 1) \ge (B + 1)(B + 2)\cdots C = A! \ge (p_n - k)!\ . \tag{6}$$

Now using Lemmas 2.3, 2.4 and $k \le 850000$, a simple Magma calculation gives $n+1 \le 78200$, whence $C \le 10^6$. So in this case the theorem follows from the result of Caldwell [3].

Assume now that $850000 < k \le 10^6$. Then by what we have proved already, we get $C < 5k \le 5 \cdot 10^6$. By Caldwell's result we may also assume that $10^6 < C$. By a simple Magma program we get that the length of the longest prime-free interval inside $(10^6, 5 \cdot 10^6)$ is 153. If $C > 1000507$ (which is a prime), then $A \ge 507$. However, then

$$10^{1153} < 507! \le A! = (B + 1)(B + 2)\cdots C < (5 \cdot 10^6)^{153} < 10^{1071}$$

yields a contradiction. So we are left with the cases $1000000 < C < 1000507$. Writing $p_n < C < p_{n+1}$, based upon (6) we must have

$$(p_n + 1) \cdots (p_{n+1} - 1) \geq (p_n - 10^6)! \ .$$

Checking the few possibilities corresponding to the remaining values of C by Magma, we obtain that C must belong to one of the intervals

$$(999983, 1000003), \quad (1000003, 1000033), \quad (1000039, 1000081),$$

with the endpoints being consecutive primes. Note that though in fact $C > 10^6$, we shall also need to consider all elements of the first interval later. To exclude these cases, one can do the following. (We used Magma to perform the necessary calculations.) Consider all possibilities for a putative solution C in one of the above intervals, and check that for any possible value of $B+1$ with $B+1 \leq C$ in the same interval, writing p for the largest prime divisor of $(B+1)(B+2)\cdots C$, we can find a prime q with $q < p$ such that $q \nmid (B+1)(B+2)\cdots C$. In view of (4) this yields a contradiction. Since for any choice of C and $B+1$ one can always find such a q, the theorem follows. □

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L. Hajdu, Á. Papp
University of Debrecen,
Institute of Mathematics,
H-4010 Debrecen, P.O. Box 12.
Hungary

T. Szakács
Eszterházy Károly University of Applied Sciences,
Institute of Mathematics and Informatics,
H-3300 Eger, Leányka str. 4.
Hungary
E-mail address: hajdul@science.unideb.hu
E-mail address: papp98@gmail.com
E-mail address: szakacs.tamas@uni-eszterhazy.hu