Approximating the Euclidean distance using non-periodic neighbourhood sequences

András Hajdu [a,1] Lajos Hajdu [b,2]

{hajdua|hajdul}@math.klte.hu

[a] *Institute of Informatics, University of Debrecen, H-4010 Debrecen, P.O.Box 12.*

[b] *Number Theory Research Group of the Hungarian Academy of Sciences, and Institute of Mathematics, University of Debrecen, H-4010 Debrecen, P.O.Box 12.*

Abstract

In this paper we discuss some possibilities of approximating the Euclidean distance in $\mathbb{Z}^2$ by the help of digital metrics induced by neighbourhood sequences. Contrary to the earlier approaches, we use general (non-periodic) neighbourhood sequences which allows us to derive more precise results. We determine those metrics which can be regarded as the best approximations to the Euclidean distance in some sense. We compare our results with earlier studies of Das [3] and Mukherjee et al. [11].

Key words:
Digital Geometry, Neighbourhood Sequences, Euclidean Distance, Digital Metric
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1 Introduction

In 2-dimensional digital applications it is often very useful to have an appropriate (digital) distance function on $\mathbb{Z}^2$. Thus the investigation of digital distance functions and metrics becomes more and more important. See e.g. the survey paper [10] of Melter for an account, and the papers [2,3,5–7,12,14] and the references given there for earlier results and the present state. One of the

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most essential tasks is to give a convenient digital metric, which approximates the Euclidean metric L_2 on $\mathbb{Z}^2$ well.

In [13] Rosenfeld and Pfaltz introduced the digital metrics d_4 and d_8 in $\mathbb{Z}^2$, based on cityblock and chessboard motions, respectively. Cityblock motion allows movements only in horizontal and vertical directions, while chessboard motion in diagonal ones, as well. The distance of two points is the number of steps required to reach either point from the other. To obtain a better approximation of L_2, Rosenfeld and Pfaltz recommended the alternate use of cityblock and chessboard motions, which defines the d_{oct} distance.

By allowing arbitrary periodic mixture of these motions, Das et al. [4] introduced the concept of periodic neighbourhood sequences. In [8] Fazekas et al. defined the notion of general (not necessarily periodic) neighbourhood sequences. The main advantage of these sequences over the classical motions is that they provide more flexibility in moving on the plane. Making use of this property, Das [3] and Mukherjee et al. [11] determined distance functions that provide good approximations of the Euclidean distance in a certain sense. However, in these papers only periodic neighbourhood sequences were used.

In this paper we perform an approximation of the 2D Euclidean distance by distance functions $d(A)$ based on general (non-periodic) neighbourhood sequences A. In contrast with results obtained using periodic sequences (see e.g. [1,3,6,11] and the references given there), in this way we can give the actually best approximating sequence, instead of a finite part of it. This allows us to formulate more precise statements, and obtain better results. In fact we distinguish two types of approximation. On one hand, we investigate the general situation, i.e. the problem of finding a digital metric $d(A)$ which approximates L_2 best. On the other hand, we consider the problem of approximating L_2 from below, separately. In the latter case the problem is to find the digital metric $d(A)$ minorating L_2 on $\mathbb{Z}^2$, which approximates L_2 best.

To measure the error of approximation, we compare the disks of radii k with $k \in \mathbb{N}$ of L_2, and of the distance functions $d(A)$. A similar but slightly different error function was used by Das and Chatterji [6], and Mukherjee et al. [11] for periodic neighbourhood sequences. Interestingly, the best approximating sequences we obtain are (mostly) Beatty sequences, thus they can be constructed very easily. For each type of approximation under consideration we give neighbourhood sequences such that the corresponding distance functions are metrics on $\mathbb{Z}^2$. Thus we get good approximation of the Euclidean distance by digital metrics. In particular, we determine the digital metric $d(A)$, best approximating L_2 from below "uniformly", i.e. independently of the sense of approximation (see Problem 3 and Theorem 6).

The structure of the paper is as follows. In the second section we introduce our notation. In the third section we formulate three problems, which summarize our aims in a precise form. In the fourth section we solve these problems, by giving the best approximating neighbourhood sequences. In the last section we compare the sequences obtained with those recommended by Das in [3], and Mukherjee et al. in [11]. As we use a different method to measure the error of the approximation than the authors in [3] and [11], we choose a third type of error function to this purpose.

2 Basic concepts and notation

First we recall some definitions and notation from [4] and [8].

Let q be a point in $\mathbb{Z}^2$. The i-th coordinate of q is denoted by $\mathrm{Pr}_i(q)$ ($i = 1, 2$). Let $M \in \{0, 1, 2\}$. The points $q, r \in \mathbb{Z}^2$ are called M-neighbours, if the following two conditions hold:

- $|\,\mathrm{Pr}_i(q) - \mathrm{Pr}_i(r)| \le 1 \quad (i = 1, 2)$,
- $|\,\mathrm{Pr}_1(q) - \mathrm{Pr}_1(r)| + |\,\mathrm{Pr}_2(q) - \mathrm{Pr}_2(r)| \le M$.

The sequence $A = (a(i))_{i=1}^{\infty}$, where $a(i) \in \{1, 2\}$ for all $i \in \mathbb{N}$, is called a 2-dimensional (shortly 2D) neighbourhood sequence. If for some $l \in \mathbb{N}$, $a(i+l) = a(i)$ ($i \in \mathbb{N}$), then A is periodic with period l. In this case we briefly write $A = (a(1), a(2), \ldots, a(l))$. The set of the 2D-neighbourhood sequences will be denoted by S_2.

Let $q, r \in \mathbb{Z}^2$ and $A \in S_2$. The point sequence $q = q_0, q_1, \ldots, q_m = r$, where q_{i-1} and q_i are $a(i)$-neighbours in $\mathbb{Z}^2$ ($1 \le i \le m$), is called an A-path of length m from q to r. The A-distance $d(q, r; A)$ of q and r is defined as the length of the shortest A-path(s) between them. We shortly write $d(A)$ for the distance function generated by the neighbourhood sequence A. In general $d(A)$ is not a metric on $\mathbb{Z}^2$, however, by a theorem of Nagy [12] we can check this property. Nagy's result describes the general nD case, but we formulate it only for 2D. Note that this assertion was proved by Das et al. [4] for periodic neighbourhood sequences.

Theorem 1 (see [12]) *Let $A \in S_2$. Then $d(A)$ is a metric on $\mathbb{Z}^2$ if and only if for any $s, t \in \mathbb{N}$*

$$\sum_{i=1}^{s} a(i) \le \sum_{i=t}^{s+t-1} a(i).$$

Let $q, r \in \mathbb{Z}^2$. As usual, the L_p ($p > 0$) distance of q and r is defined by

$$L_p(q, r) = (|\,\mathrm{Pr}_1(q) - \mathrm{Pr}_1(r)|^p + |\,\mathrm{Pr}_2(q) - \mathrm{Pr}_2(r)|^p)^{\frac{1}{p}},$$

and

$$L_\infty(q,r) = \max\{|\operatorname{Pr}_1(q) - \operatorname{Pr}_1(r)|,\ |\operatorname{Pr}_2(q) - \operatorname{Pr}_2(r)|\}.$$

We have $L_{p_1} \leq L_{p_2}$ for every $q, r \in \mathbb{Z}^2$, provided that $p_1 \geq p_2$.

Obviously,

$$L_1(q,r) = d_4(q,r) = d(q,r;(1)) \text{ and } L_\infty(q,r) = d_8(q,r) = d(q,r;(2)).$$

The constant periodic neighbourhood sequences (1) and (2), respectively, spread in the slowest and fastest way in $\mathbb{Z}^2$ among the 2D-neighbourhood sequences. So for every $A \in S_2$ we have $L_1(q,r) \leq d(q,r;A) \leq L_\infty(q,r)$. It is a natural problem to find the neighbourhood sequences, whose distance functions approximates the 2D Euclidean distance L_2 best in some sense. On the other hand, as we already mentioned in the introduction, such sequences have important practical applications as well. To handle this problem, we compare the regions occupied by a sequence $A \in S_2$ with the Euclidean disks. As the neighbourhood sequences spread in $\mathbb{Z}^2$ in a translation invariant way, we may choose the origin $\mathbf{0} \in \mathbb{Z}^2$ as the starting point. For illustration, Figure 1 shows that the metric d_{oct} (generated by $(1,2)$) is "closer" to L_2, than d_4 or d_8.

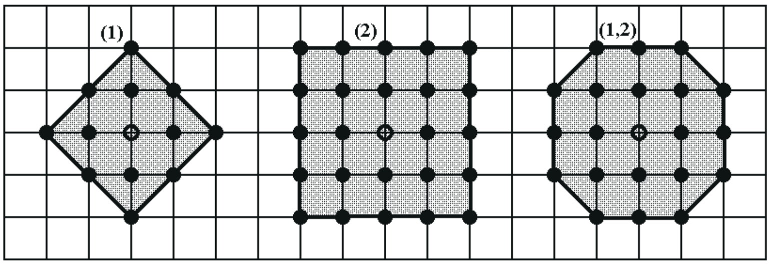


Fig. 1. The regions occupied by the neighbourhood sequences (1), (2) and $(1,2)$ after two steps.

Let $A \in S_2$. For every $k \in \mathbb{N}$, let

$$A_k = \{q \in \mathbb{Z}^2 : d(\mathbf{0}, q; A) \leq k\}$$

denote the region occupied by A after k steps, and write $H(A_k)$ for the convex hull of A_k in $\mathbb{R}^2$. Observe that $H(A_k)$ in general is an octagon which is symmetric to the coordinate axes and to the lines $y = x$ and $y = -x$ in the $[x, y]$ plane. Let

$$O_k = \{q \in \mathbb{Z}^2 : L_2(\mathbf{0}, q) \leq k\}$$

and

$$G_k = \{q \in \mathbb{R}^2 : L_2(\mathbf{0}, q) \leq k\}$$

be the disks of radius k in $\mathbb{Z}^2$ and $\mathbb{R}^2$, respectively. The sets A_k and O_k will be called the k-disks of the distances $d(A)$ and L_2, respectively.

We will often use the number of 1 and 2 values occurring among the first k elements of a neighbourhood sequence A. So for every $k \in \mathbb{N}$ put

$$\mathbf{1}_A(k) = |\{a(i) \,:\, a(i) = 1,\ 1 \leq i \leq k\}|$$

and

$$\mathbf{2}_A(k) = |\{a(i) \,:\, a(i) = 2,\ 1 \leq i \leq k\}|,$$

where $A = (a(i))_{i=1}^{\infty}$. For convenience, write $\mathbf{1}_A(0) = \mathbf{2}_A(0) = 0$. Note that $\mathbf{1}_A(k) + \mathbf{2}_A(k) = k$ $(k \in \mathbb{N})$.

For any $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the largest integer which is less than or equal to x, and $\lceil x \rceil$ the smallest integer which is greater than or equal to x. Let $\alpha \in \mathbb{R}$ with $0 \leq \alpha \leq 1$, and let $A = (a(i))_{i=1}^{\infty}$, $B = (b(i))_{i=1}^{\infty}$ be sequences of 1-s and 2-s, defined by

$$a(i) = \lfloor i\alpha \rfloor - \lfloor (i-1)\alpha \rfloor + 1, \quad b(i) = \lceil i\alpha \rceil - \lceil (i-1)\alpha \rceil + 1 \ \ (i \in \mathbb{N}).$$

The sequences A, B are called Beatty sequences on the letters 1, 2. Clearly, for every $k \in \mathbb{N}$ we have

$$\mathbf{2}_A(k) = \lfloor k\alpha \rfloor \ \text{ and } \ \mathbf{2}_B(k) = \lceil k\alpha \rceil.$$

Conversely, these equalities define Beatty sequences which are uniquely determined. We refer to [9] for the basic properties of Beatty sequences and their generalizations.

3 Three approximation problems

To decide how a digital distance $d(A)$ approximates the Euclidean distance L_2 on $\mathbb{Z}^2$, we compare the k-disks A_k and O_k. A natural approach could be to choose the number of integer points in the symmetric difference $A_k \bigtriangledown O_k$ as an error function. However, there is no exact formula for the number of integer points inside O_k. So we follow a slightly different method which is a variant of that used in [6] and [11]. Namely, we compare the sets $H(A_k)$ and G_k, and choose A to minimize the area of $H(A_k) \bigtriangledown G_k$. Of course, it can be done only separately for each k. However, surprisingly it turns out that for every $k \in \mathbb{N}$ the very same A can be chosen to minimize this area. So this neighbourhood sequence A can be regarded as the one that approximates L_2 best (in the above sense).

According to the these principles, we investigate the function

$$TE_A(k) = \text{Area}(H(A_k) \bigtriangledown G_k),$$

called the total error of the approximation at the k-th step. We also use the relative error at the k-th step, defined as

$$RE_A(k) = \frac{TE_A(k)}{k^2\pi},$$

and the limit relative error (if it exists)

$$RE_A = \lim_{k\to\infty} RE_A(k).$$

We note that the authors in [6] and [11], instead of measuring the area of the symmetric difference $H(A_k) \bigtriangledown G_k$, simply took the difference $|\mathrm{Area}(H(A_k)) - \mathrm{Area}(G_k)|$ as an error function. Clearly, our approach is more sensitive to the "matching" of $H(A_k)$ and G_k. It also should be mentioned that in [6] and [11] additional error functions, such as perimeter error and shape error were considered, as well.

We perform several types of approximation. Our aims can be summarized in the following problems. The first problem concerns the general case.

Problem 1. *Find a neighbourhood sequence $A^{(1)} \in S_2$ (if exists) such that for every $B \in S_2$ and $k \in \mathbb{N}$*

$$\mathrm{Area}(H(A_k^{(1)}) \bigtriangledown G_k) \leq \mathrm{Area}(H(B_k) \bigtriangledown G_k).$$

We consider separately the case when the octagons $H(A_k)$ cover G_k for every $k \in \mathbb{N}$, that is the corresponding function $d(A)$ minorates L_2.

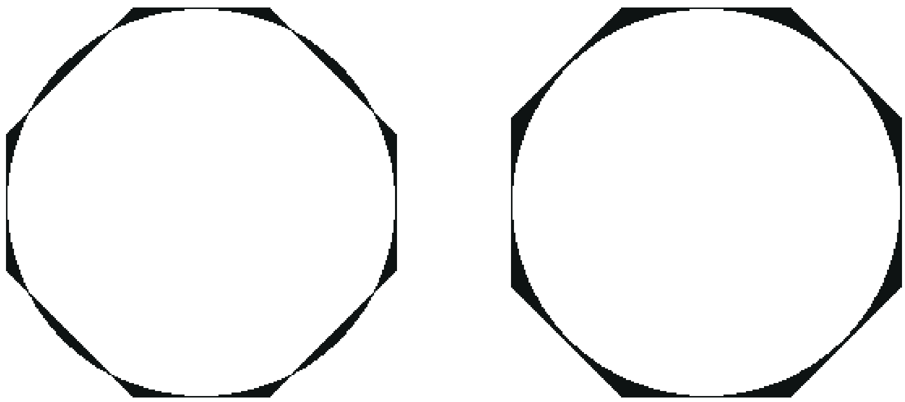
Problem 2. *Find a neighbourhood sequence $A^{(2)} \in S_2$ (if exists) such that $H(A_k^{(2)}) \supseteq G_k$ for every $k \in \mathbb{N}$, and for every $B \in S_2$, $H(B_k) \supseteq G_k$ implies that*

$$\mathrm{Area}(H(A_k^{(2)}) \setminus G_k) \leq \mathrm{Area}(H(B_k) \setminus G_k).$$

Note that it does not make sense to consider a problem with $H(A_k) \subseteq G_k$. Indeed, observe that $H(A_k)$ is contained in G_k if and only if the first k elements of A are all 1-s. This is the reason why we do not take up the problem of majorating L_2 by digital metrics $d(A)$.

Figure 2 illustrates Problems 1 and 2.

We will construct two neighbourhood sequences, satisfying the requirements of Problems 1 and 2, respectively. Moreover, we will give a sequence such that



(a) (b)

Fig. 2. The error of approximation (a) in the general case (b) when $H(A_k) \supseteq G_k$.

the corresponding distance function is a *metric*, and it can be considered as the digital metric which approximates L_2 best in the sense of Problem 1.

We also investigate the following "discrete" version of Problem 2. Note that Problem 1 does not have a similar variant.

Problem 3. *Find a neighbourhood sequence $A^{(3)} \in S_2$ (if exists) such that $A_k^{(3)} \supseteq O_k$ for every $k \in \mathbb{N}$, and if $B \in S_2$ with $B_k \supseteq O_k$, then $B_k \supseteq A_k^{(3)}$.*

Observe that the sequence $A^{(3)}$ has the nice property that the corresponding distance function $d\left(A^{(3)}\right)$ is "uniformly" the best one to approximate L_2 from below. That is, for any $B \in S_2$, if

$$d(q,r;B) \le L_2(q,r) \quad \text{for any } q,r \in \mathbb{Z}^2, \text{ then}$$

$$d(q,r;B) \le d\left(q,r;A^{(3)}\right) \quad \text{for any } q,r \in \mathbb{Z}^2.$$

In Theorem 6 we will solve Problem 3, by constructing the sequence $A^{(3)}$ having the desired property. Interestingly, it will turn out that the distance function $d(A^{(3)})$ is a metric on $\mathbb{Z}^2$. To show this, the following lemma will be useful.

Lemma 2 *Let $\alpha \in \mathbb{R}$ with $0 \le \alpha \le 1$, and let $A \in S_2$ be the unique sequence with $\mathbf{2}_A(k) = \lfloor k\alpha \rfloor$ for every $k \in \mathbb{N}$. Then $d(A)$ is a metric.*

PROOF. Suppose to the contrary that $d(A)$ is not a metric. Then using Theorem 1, for some $n, N \in \mathbb{N}$ with $n < N$

$$\sum_{i=1}^{n} a(i) > \sum_{i=N-n+1}^{N} a(i) \tag{1}$$

holds. Clearly, we may suppose that $N-n+1>n$. We rewrite (1) as

$$\mathbf{2}_A(n) > \mathbf{2}_A(N) - \mathbf{2}_A(N-n). \tag{2}$$

As all the numbers in (2) are integers, we obtain

$$\mathbf{2}_A(N-n) \geq \mathbf{2}_A(N) + 1 - \mathbf{2}_A(n). \tag{3}$$

Observe that by the definition of A, for every $k \in \mathbb{N}$ we have

$$\mathbf{2}_A(k) \leq k\alpha < \mathbf{2}_A(k) + 1. \tag{4}$$

Combining (3) and (4), we get

$$(N-n)\alpha \geq \mathbf{2}_A(N-n) \geq \mathbf{2}_A(N) + 1 - \mathbf{2}_A(n) > N\alpha - n\alpha$$

which is a contradiction. Hence the lemma follows. □

4 The solution of the approximation problems

In this section we construct "extremal" sequences described in Problems 1, 2 and 3. We start with Problem 2, as it is the simplest to handle.

4.1 Approximating L_2 from below

In this subsection we consider only neighbourhood sequences A with $H(A_k) \supseteq G_k$ for all $k \in \mathbb{N}$. As we have already mentioned, it means that the corresponding distance function $d(A)$ minorates L_2. The next result gives a solution to Problem 2.

Theorem 3 *Let $A^{(2)} = (a^{(2)}(i))_{i=1}^{\infty}$ be the unique 2D-neighbourhood sequence defined by $\mathbf{2}_{A^{(2)}}(k) = \lceil k(\sqrt{2}-1)\rceil$ ($k \in \mathbb{N}$), that is*

$$a^{(2)}(i) = \lceil i(\sqrt{2}-1)\rceil - \lceil (i-1)(\sqrt{2}-1)\rceil + 1 \qquad (i \in \mathbb{N}).$$

Then $H(A_k^{(2)}) \supseteq G_k$ for any $k \in \mathbb{N}$, and $B \in S_2$, $H(B_k) \supseteq G_k$ implies that

$$\mathrm{Area}(H(A_k^{(2)}) \setminus G_k) \leq \mathrm{Area}(H(B_k) \setminus G_k).$$

PROOF. Let k be a fixed positive integer, and let $B \in S_2$ be arbitrary such that $H(B_k)$ contains G_k. Clearly, the vertices of $H(B_k)$ with positive coordinates are $(k, \mathbf{2}_B(k))$ and $(\mathbf{2}_B(k), k)$. Hence $H(B_k) \supseteq G_k$ implies that the line $x + y = k + \mathbf{2}_B(k)$ has at most one point in common with the circle $x^2 + y^2 = k^2$ in the $[x, y]$-plane. A simple calculation gives that it is equivalent to

$$(k - \mathbf{2}_B(k))^2 - 2\,(\mathbf{2}_B(k))^2 \leq 0, \tag{5}$$

that is $\mathbf{2}_B(k)/k \geq \sqrt{2} - 1$.

One can easily check (see also [1,6,11]) that $\mathrm{Area}(H(B_k)) = 4k^2 - 2(k - \mathbf{2}_B(k))^2$, whence the total error of the approximation is

$$TE_B(k) = 4k^2 - 2(k - \mathbf{2}_B(k))^2 - \pi k^2. \tag{6}$$

Clearly, $TE_B(k)$ is minimal, when $\mathbf{2}_B(k)$ is minimal. Taking into consideration that $\mathbf{2}_B(k)/k \geq \sqrt{2} - 1$ and $\mathbf{2}_B(k)$ is an integer, we obtain that $TE_B(k)$ takes its minimum when $\mathbf{2}_B(k) = \lceil k(\sqrt{2} - 1) \rceil$. Hence the theorem follows. □

Figure 3 shows how the octagons $H(A_k^{(2)})$ of the neighbourhood sequence $A^{(2)}$ defined in Theorem 3 approximate G_k for $k = 2, 5, 7, 9, 12$. The dark regions show the error of the approximation.

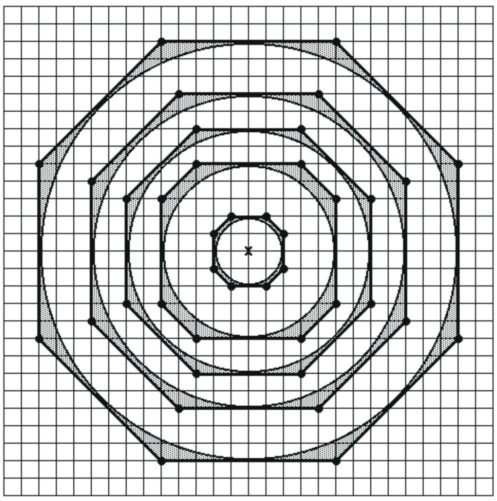


Fig. 3. Approximating G_k by $H(A_k^{(2)})$ for $k = 2, 5, 7, 9, 12$.

Remark 4 *The octagons $H(A_k^{(2)})$ are almost regular. (For regularity we should have $\mathbf{2}_A(k) = k(\sqrt{2} - 1)$, which is impossible.) Obviously, the ratio of the inclined and horizontal (or vertical) sides of $H(A_k^{(2)})$ tends to 1 as $k \to \infty$. It was already noted by Rosenfeld and Pfaltz [13] that the "best approximating" sequence should have such property. A detailed study in this direction with pe-*

riodic sequences under slightly different circumstances was performed by Das and Chatterji [6].

Remark 5 *For the k-th total error of the approximation of L_2 with $d(A^{(2)})$ we get*

$$TE_{A^{(2)}}(k) = (4-\pi)k^2 - 2\left(k - \mathbf{2}_{A^{(2)}}(k)\right)^2.$$

Thus for the k-th relative error and for the relative error we obtain

$$RE_{A^{(2)}}(k) = \frac{4-\pi}{\pi} - \frac{2}{\pi}\left(1 - \frac{\mathbf{2}_{A^{(2)}}(k)}{k}\right)^2$$

and

$$RE_{A^{(2)}} = \frac{8(\sqrt{2}-1)-\pi}{\pi} = 0.054786175...$$

By the following theorem we solve Problem 3.

Theorem 6 *Let $A^{(3)} \in S_2$, $A^{(3)} = (a^{(3)}(i))_{i=1}^{\infty}$ be the unique sequene defined by $\mathbf{2}_{A^{(3)}}(k) = \lfloor k(\sqrt{2}-1)\rfloor$ $(k \in \mathbb{N})$, that is*

$$a^{(3)}(i) = \lfloor i(\sqrt{2}-1)\rfloor - \lfloor (i-1)(\sqrt{2}-1)\rfloor + 1 \quad (i \in \mathbb{N}).$$

Then for every $k \in \mathbb{N}$, $O_k \subseteq A_k^{(3)}$. Moreover, if $B \in S_2$ such that $O_k \subseteq B_k$ for some $k \in \mathbb{N}$, then $A_k^{(3)} \subseteq B_k$.

PROOF. Let $k \in \mathbb{N}$ be fixed, and suppose that $O_k \not\subseteq A_k^{(3)}$. Let $A^{(2)}$ be the neighbourhood sequence defined in Theorem 3. Then, as $A_k^{(2)} \supseteq O_k$, there exists an integer point q of O_k, which is also in $A_k^{(2)} \setminus A_k^{(3)}$. Since $\mathbf{2}_{A^{(2)}}(k) = \mathbf{2}_{A^{(3)}}(k) + 1$, q must be on the border of $A_k^{(2)}$.

Moreover, by $H(A_k^{(2)}) \supseteq G_k$, the only possibility is that q is the tangent point of G_k and one of the inclined sides of $H(A_k^{(2)})$. Using symmetry, this implies that q belongs to one of the lines $y = x$ or $y = -x$. However, the points being both on the perimeter of G_k and on one of these lines, do not have integer coordinates. This contradiction shows that $O_k \subseteq A_k^{(3)}$ for every $k \in \mathbb{N}$.

To prove the second statement, we show that for any $k \in \mathbb{N}$, every inclined side of $A_k^{(3)}$ contains an integer point from O_k. To this purpose, observe that as $\mathbf{2}_{A^{(3)}}(k)/k < \sqrt{2}-1$, we have

$$\left(k - \mathbf{2}_{A^{(3)}}(k)\right)^2 - 2\left(\mathbf{2}_{A^{(3)}}(k)\right)^2 > 0. \tag{7}$$

Comparing (7) with formula (5), we get that every inclined side of $H(A_k^{(3)})$ has two points in common with the perimeter of G_k. A simple calculation yields that the intersection points with positive coordinates are

$$\left(\frac{k+\mathbf{2}_{A^{(3)}}(k)+Q}{2}, \frac{k+\mathbf{2}_{A^{(3)}}(k)-Q}{2}\right)$$

and

$$\left(\frac{k+\mathbf{2}_{A^{(3)}}(k)-Q}{2}, \frac{k+\mathbf{2}_{A^{(3)}}(k)+Q}{2}\right),$$

where

$$Q=\sqrt{(k-\mathbf{2}_{A^{(3)}}(k))^2-2(\mathbf{2}_{A^{(3)}}(k))^2}.$$

For the distance D of these points we obtain

$$D=\sqrt{2[(k-\mathbf{2}_{A^{(3)}}(k))^2-2(\mathbf{2}_{A^{(3)}}(k))^2]}.$$

As k and $\mathbf{2}_{A^{(3)}}(k)$ are integers, by (7) we infer that

$$\left(k-\mathbf{2}_{A^{(3)}}(k)\right)^2-2\left(\mathbf{2}_{A^{(3)}}(k)\right)^2\geq 1$$

which yields $D\geq\sqrt{2}$. As the line containing the above intersection points is given by $x+y=k+\mathbf{2}_{A^{(3)}}(k)$ on the $[x,y]$ plane, $D\geq\sqrt{2}$ implies that there is at least one integer point on the corresponding inclined side of $H(A_k^{(3)})$, which also belongs to G_k. This shows that if $B\in S_2$ with $B_k\supseteq O_k$, then $\mathbf{2}_B(k)\geq\mathbf{2}_{A^{(3)}}(k)$ must be valid. Thus $B_k\supseteq A_k^{(3)}$, and the theorem follows. □

Remark 7 *By Lemma 2, $d(A^{(3)})$ is a metric on $\mathbb{Z}^2$. That is, among the digital metrics corresponding to neighbourhood sequences, $d(A^{(3)})$ is the best one to approximate L_2 from below in $\mathbb{Z}^2$. More precisely, the following two properties hold:*

- *for all $x,y\in\mathbb{Z}^2$ we have $d(x,y;A^{(3)})\leq L_2(x,y)$,*
- *if $d(A)$ is any metric corresponding to some $A\in S_2$ and*

$$d(x,y;A)\leq L_2(x,y)\quad for\ all\quad x,y\in\mathbb{Z}^2,$$

then

$$d(x,y;A)\leq d(x,y;A^{(3)})\quad for\ all\quad x,y\in\mathbb{Z}^2.$$

4.2 The general case

In this subsection we solve Problem 1. For this purpose the following lemma will be useful.

Lemma 8 *Define the function* $E : [0,1] \to \mathbb{R}$ *by*

$$E(y) = \begin{cases} 2y - y^2, & \text{if } y \geq \sqrt{2} - 1, \\ 2\arccos(y(y+2)) - \\ \quad 2(y+1)\sqrt{1-2y-y^2} - y^2 + 2y, & \text{otherwise.} \end{cases}$$

Then $E(y)$ *has a global minimum at* $y_0 = \frac{2\sqrt{6}-3}{5}$. *Moreover,* $E(y)$ *is strictly monotone decreasing in* $[0, y_0]$ *and strictly monotone increasing in* $[y_0, 1]$.

PROOF. Observe that $E(y)$ is continuous on $[0,1]$. Taking the derivative of E on the intervals $[0, \sqrt{2}-1]$ and $[\sqrt{2}-1, 1]$ separately, by a simple calculation the lemma follows from elementary calculus. □

Theorem 9 *Let the neighbourhood sequence* $A^{(1)} = (a^{(1)}(i))_{i=1}^{\infty}$ *be defined by*

$$a^{(1)}(i) = \begin{cases} 1, & \text{if } E\left(\frac{\mathbf{2}_{A^{(1)}}(i-1)}{i}\right) < E\left(\frac{\mathbf{2}_{A^{(1)}}(i-1)+1}{i}\right), \\ 2, & \text{otherwise,} \end{cases}$$

where E *is introduced in the previous lemma. Then for any* $B \in S_2$ *and* $k \in \mathbb{N}$,

$$\text{Area}(H(A_k^{(1)}) \bigtriangledown G_k) \leq \text{Area}(H(B_k) \bigtriangledown G_k).$$

PROOF. Let k be a fixed positive integer and $B \in S_2$. Using (6) when the inclined sides of $H(B_k)$ do not intersect G_k, and by a simple calculation in the opposite case, we obtain that the k-th total error of the approximation of L_2 by B is

$$TE_B(k) = \begin{cases} 2k^2 - \pi k^2 + 2k^2(2y - y^2), & \text{if } y \geq \sqrt{2} - 1, \\ 2k^2 - \pi k^2 + 2k^2(2\arccos(y(y+2)) - \\ \quad 2(y+1)\sqrt{1-2y-y^2} - y^2 + 2y), & \text{otherwise,} \end{cases}$$

where $y = \mathbf{2}_B(k)/k$.

Clearly, $TE_B(k)$ is minimal if and only if $E(y)$ is minimal for B. By Lemma 8

$$\min_{\substack{t \in \mathbb{Z} \\ 0 \leq t \leq k}} E\left(\frac{t}{k}\right) = \min\left\{E\left(\frac{\lfloor ky_0 \rfloor}{k}\right), E\left(\frac{\lceil ky_0 \rceil}{k}\right)\right\}, \tag{8}$$

where $y_0 = \frac{2\sqrt{6}-3}{5}$.

We prove that for every $k \in \mathbb{N}$, $\mathbf{2}_{A^{(1)}}(k) = \lfloor ky_0 \rfloor$ or $\lceil ky_0 \rceil$ according to whether

$$E\left(\frac{\lfloor ky_0 \rfloor}{k}\right) < E\left(\frac{\lceil ky_0 \rceil}{k}\right),$$

or not. By (8) this will imply that $E(y)$ is minimal for $A^{(1)}$, whence $TE_{A^{(1)}}(k) \leq TE_B(k)$ for every $B \in S_2$ and $k \in \mathbb{N}$.

We proceed by induction on k. For $k = 1$ the statement is obvious: $a^{(1)}(1) = 2$ and $E(1) < E(0)$. Suppose that for some k we have $\mathbf{2}_{A^{(1)}}(k) = \lfloor ky_0 \rfloor$; the case when $\mathbf{2}_{A^{(1)}}(k) = \lceil ky_0 \rceil$ is similar. Then, by the induction hypothesis,

$$E\left(\frac{\lfloor ky_0 \rfloor}{k}\right) < E\left(\frac{\lceil ky_0 \rceil}{k}\right) = E\left(\frac{\lfloor ky_0 \rfloor + 1}{k}\right). \tag{9}$$

We distinguish two cases.

(i) Assume first that $\lfloor ky_0 \rfloor = \lfloor (k+1)y_0 \rfloor < \lceil (k+1)y_0 \rceil = \lceil ky_0 \rceil$.

Now if

$$E\left(\frac{\lfloor ky_0 \rfloor}{k+1}\right) < E\left(\frac{\lfloor ky_0 \rfloor + 1}{k+1}\right), \tag{10}$$

then by definition $a^{(1)}(k+1) = 1$, whence

$$\mathbf{2}_{A^{(1)}}(k+1) = \mathbf{2}_{A^{(1)}}(k) = \lfloor ky_0 \rfloor = \lfloor (k+1)y_0 \rfloor.$$

Since we can write (10) as

$$E\left(\frac{\lfloor (k+1)y_0 \rfloor}{k+1}\right) < E\left(\frac{\lceil (k+1)y_0 \rceil}{k+1}\right),$$

the statement is also true for $k+1$. On the other hand, if

$$E\left(\frac{\lfloor ky_0 \rfloor}{k+1}\right) \geq E\left(\frac{\lfloor ky_0 \rfloor + 1}{k+1}\right), \tag{11}$$

then we obtain $a^{(1)}(k+1) = 2$. By $\mathbf{2}_{A^{(1)}}(k) = \lfloor ky_0 \rfloor$, this yields

$$\mathbf{2}_{A^{(1)}}(k+1) = \mathbf{2}_{A^{(1)}}(k) + 1 = \lceil ky_0 \rceil = \lceil (k+1)y_0 \rceil.$$

Now the statement for $k+1$ follows from rewriting (11) as

$$E\left(\frac{\lceil (k+1)y_0 \rceil}{k+1}\right) \leq E\left(\frac{\lfloor (k+1)y_0 \rfloor}{k+1}\right).$$

(ii) Assume now that $\lfloor ky_0 \rfloor < \lceil ky_0 \rceil = \lfloor (k+1)y_0 \rfloor < \lceil (k+1)y_0 \rceil$.

In this case we have

$$\frac{\lfloor ky_0\rfloor}{k+1} < \frac{\lfloor ky_0\rfloor}{k} < \frac{\lfloor (k+1)y_0\rfloor}{k+1} < y_0 < \frac{\lceil ky_0\rceil}{k} \leq \frac{\lceil (k+1)y_0\rceil}{k+1}. \tag{12}$$

Combining (9) and (12) with Lemma 8, we immediately obtain

$$E\left(\frac{\lfloor (k+1)y_0\rfloor}{k+1}\right) < E\left(\frac{\lceil (k+1)y_0\rceil}{k+1}\right).$$

Moreover, (12) yields

$$E\left(\frac{\lfloor ky_0\rfloor}{k+1}\right) > E\left(\frac{\lfloor (k+1)y_0\rfloor}{k+1}\right) = E\left(\frac{\lfloor ky_0\rfloor+1}{k+1}\right).$$

Hence by definition, $a^{(1)}(k+1) = 2$ and

$$\mathbf{2}_{A^{(1)}}(k+1) = \mathbf{2}_{A^{(1)}}(k) + 1 = \lfloor ky_0\rfloor + 1 = \lfloor (k+1)y_0\rfloor.$$

Thus the statement follows for $k+1$ also in this case.

Observe that as $0 < y_0 < 1/2$, the above two cases cover all the possibilities. Thus the sequence $A^{(1)}$ minimizes $E(y)$. This implies that $TE_{A^{(1)}}(k)$ is minimal for every $k \in \mathbb{N}$, and the theorem follows. $\square$

Remark 10 *We have* $A^{(1)} = (2,1,1,1,2,1,2,1,1,2,1,1,\ldots)$*. For the* k*-th total and relative errors of* $A^{(1)}$ *we obtain*

$$TE_{A^{(1)}}(k) = k^2(2-\pi+2E(y))$$

and

$$RE_{A^{(1)}}(k) = \frac{2-\pi+2E(y)}{\pi},$$

where $y = \mathbf{2}_{A^{(1)}}(k)/k$*, and* $E(y)$ *is defined in Lemma 8. By* $\lim_{k\to\infty} y = \frac{2\sqrt{6}-3}{5}$ *we get*

$$RE_{A^{(1)}} = \frac{2-\pi}{\pi} + \frac{2}{\pi}\left(2\arccos\left(\frac{3+8\sqrt{6}}{25}\right) + \frac{4\sqrt{6}-11}{5}\right) = 0.046525347\ldots$$

Figure 4 illustrates how $H(A_k^{(1)})$ approximates G_k for $k = 2,5,7,9,12$. The dark regions show the error of the approximation.

Clearly, $d(A^{(1)})$ is not a metric on $\mathbb{Z}^2$. Another unpleasant feature of $A^{(1)}$ is that it is not easy to generate: to obtain its k-th element, we have to calculate the first $k-1$ elements previously. Now we give two sequences which are easy to construct, and for every $k \in \mathbb{N}$, one of them is also the "best" to approximate G_k.

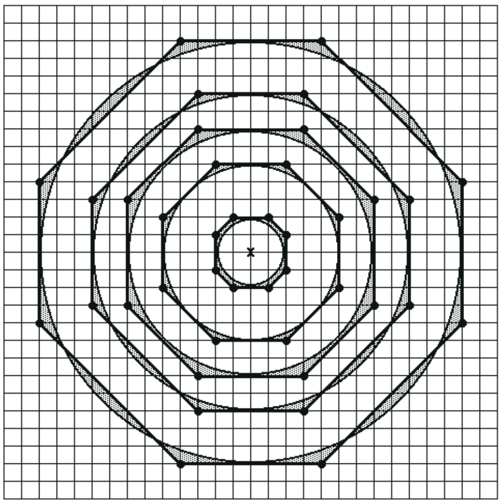


Fig. 4. Approximating G_k by $H(A_k^{(1)})$ for $k = 2, 5, 7, 9, 12$.

Corollary 11 *For $j = 1, 2$ and $i \in \mathbb{N}$ put*

$$c^{(j)}(i) = \begin{cases} j, & \text{if } i = 1, \\ \lfloor i\frac{2\sqrt{6}-3}{5}\rfloor - \lfloor (i-1)\frac{2\sqrt{6}-3}{5}\rfloor + 1, & \text{if } i > 1, \end{cases}$$

and write $C^{(1)} = (c^{(1)}(i))_{i=1}^{\infty}$ and $C^{(2)} = (c^{(2)}(i))_{i=1}^{\infty}$. Then for every $B \in S_2$ and $k \in \mathbb{N}$,

$$\min\{TE_{C^{(1)}}(k),\ TE_{C^{(2)}}(k)\} \leq TE_B(k).$$

PROOF. Observe that for every $k \in \mathbb{N}$, $\mathbf{2}_{C^{(1)}}(k) = \lfloor ky_0 \rfloor$ and $\mathbf{2}_{C^{(2)}}(k) = \lceil ky_0 \rceil$, where $y_0 = \frac{2\sqrt{6}-3}{5}$. Hence $A_k^{(1)} = C_k^{(1)}$ or $C_k^{(2)}$, and the statement follows from Theorem 9. □

Remark 12 *As $\mathbf{2}_{C^{(1)}}(k) = \lfloor ky_0 \rfloor$ for every $k \in \mathbb{N}$, by Lemma 2, $d(C^{(1)})$ is a metric on $\mathbb{Z}^2$. Thus in a sense $d(C^{(1)})$ can be considered to be the best metric (coming from a neighbourhood sequence) to approximate the Euclidean distance on $\mathbb{Z}^2$. Note that $RE_{C^{(1)}} = RE_{A^{(1)}}$.*

5 Comparing approximation results

In this section we compare our results with those of Das in [3], and Mukherjee et al. in [11]. Das [3] used an error function which measures the average difference between the Euclidean distance and the "simple metric value" generated by a neighbourhood sequence. He concluded that the periodic neighbourhood sequence $S = (1, 1, 2, 1, 2)$, which generates a "simple metric", should be used to approximate L_2. Note that for every $k \in \mathbb{N}$, $H(S_k) \not\supseteq G_k$. The authors in [11] in their approximation procedure, considered area, perimeter and shape errors at the same time, and restricted their investigations to relatively short sequences. For their purposes the use of $(1, 1, 2)$ was sufficient. We propose to use the sequence $C^{(1)}$ defined in Corollary 11 to approximate L_2. Clearly, as S is a "refinement" of $(1, 1, 2)$, its approximating properties (in our sense of approximation) are better as well. Thus we compare S and $C^{(1)}$ here, and omit the error data of $(1, 1, 2)$.

Since we used a different error function than Das in [3], we choose a third one to compare our results. We examine how the k-disks A_k approximate the k-disks G_k in the digital sense. That is, we count the k-th discrete total error

$$DTE_A(k) = |A_k \bigtriangledown O_k|,$$

being the number of grid points in the symmetric difference of A_k and O_k, where $A \in S_2$, $k \in \mathbb{N}$. The k-th discrete relative error of the approximation is defined as

$$DRE_A(k) = \frac{DTE_A(k)}{|O_k|}.$$

The following table shows the discrete relative errors of the neighbourhood sequences $S = (1, 1, 2, 1, 2)$ given in [3] and $C^{(1)}$ given in Corollary 11, both providing metrics on $\mathbb{Z}^2$.

Table 1
Discrete relative errors of distance functions generated by S and $C^{(1)}$.

k	$DRE(k)$	
	$S = (1, 1, 2, 1, 2)$	$C^{(1)} = \left(\lfloor i\frac{2\sqrt{6}-3}{5}\rfloor - \lfloor (i-1)\frac{2\sqrt{6}-3}{5}\rfloor + 1\right)_{i=1}^{\infty}$
10	0.12618297...	0.11356467...
50	0.06322498...	0.05863607...
100	0.05551135...	0.05169176...
200	0.05218540...	0.04909694...
500	0.05023499...	0.04755593...
1000	0.04955390...	0.04704176...

From Table 1 we can see that $C^{(1)}$ behaves better also in this "digital" sense.

References

[1] Aswatha Kumar, M., Mukherjee, J., Chatterji, B.N., Das, P.P.: A geometric approach to obtain best octagonal distances, *Ninth Scandinavian Conf. Image Process.*, 1995, pp. 491-498.

[2] Danielsson, P.E.: 3D octagonal metrics, *Eighth Scandinavian Conf. Image Process.*, 1993, pp. 727-736.

[3] Das, P.P.: Best simple octagonal distances in digital geometry, *J. Approx. Theory* **68** (1992), 155-174.

[4] Das, P.P., Chakrabarti P.P. and Chatterji B.N.: Distance functions in digital geometry, *Inform. Sci.* **42** (1987), 113-136.

[5] Das, P.P., Chakrabarti P.P. and Chatterji B.N.: Generalised distances in digital geometry, *Inform. Sci.* **42** (1987), 51-67.

[6] Das, P.P. and Chatterji B.N.: Octagonal distances for digital pictures, *Inform. Sci.* **50** (1990), 123-150.

[7] Das, P.P., Mukherjee J. and Chatterji B.N.: The t-cost distance in digital geometry, *Inform. Sci.* **59** (1992), 1-20.

[8] Fazekas, A., Hajdu, A. and Hajdu, L.: Lattice of generalized neighbourhood sequences in nD and ∞D, *Publ. Math. Debrecen* **60** (2002), 405-427.

[9] Lothaire, M.: Combinatorics on words. Addison-Wesley Publishing Co., Reading, Mass., 1983, xix+238 pp.

[10] Melter, R.A.: A survey on digital metrics, *Contemporary Mathematics* **119** (1991), 95-106.

[11] Mukherjee, J., Das, P.P., Aswatha Kumar, M. and Chatterji, B.N.: On approximating Euclidean metrics by digital distances in 2D and 3D, *Pattern Recognition Lett.* **21** (2000), 573-582.

[12] Nagy, B.: Distance functions based on generalised neighbourhood sequences in finite and infinite dimensional spaces, In: 5^{th} *International Conference on Applied Informatics* (Editors: E. Kovács and Z. Winkler), Eger, Hungary, 2001, pp. 183-190.

[13] Rosenfeld, A. and Pfaltz, J.L.: Distance functions on digital pictures, *Pattern Recognition* **1** (1968), 33-61.

[14] Yamashita, M. and Ibaraki, T.: Distances defined by neighbourhood sequences, *Pattern Recognition* **19** (1986), 237-246.