COMMON FACTORS IN SERIES OF CONSECUTIVE TERMS OF ASSOCIATED LUCAS AND LEHMER SEQUENCES

L. HAJDU[1] AND M. SZIKSZAI[2]

Abstract. For a sequence of arbitrary integers $B = (B_n)_{n=0}^{\infty}$ let G_B denote the smallest number such that for every $k \geq G_B$ one can find k consecutive terms of B with the property that none of these terms is coprime to all the others. If G_B exists we say that B is a Pillai sequence. This paper links up with our recent works by giving a full characterization of this property for associated Lucas and Lehmer sequences. The more general T-Pillai property is also considered.

1. Introduction

Pillai [14] asked the following question. Is it true that in every set of $k \geq 2$ consecutive integers one can always find an element which is coprime to all the others among them? Pillai [14], showed that the answer to the above question is positive for $2 \leq k \leq 16$, but negative if $17 \leq k \leq 430$. Later Brauer [3] extended Pillai's result by proving that the latter is valid for $k \geq 431$ as well. The problem has already been generalized in many ways, for example by replacing the coprimality property with severely relaxed requirements. For results in this direction we mention the papers of Caro [4], Hajdu and Saradha [5] and the references given there.

The content of this article is strongly related to another type of generalization, namely if we replace consecutive integers with consecutive terms of an integer sequence. So let $B = (B_n)_{n=0}^{\infty}$ be a sequence of arbitrary integers and write g_B for the smallest number k with the property that there exists k consecutive terms of B such that none of these terms is coprime to all the others. Similarly, we define the quantity G_B as the least value such that for every $k \geq G_B$ one can find k consecutive terms of B satisfying the latter property. We will say that B is a Pillai sequence whenever G_B exists. By the classical results of Pillai and Brauer we have that $g_{\mathbb{N}} = G_{\mathbb{N}} = 17$, hence $\mathbb{N}$ is a Pillai sequence.

Ohtomo and Tamari [13] investigated the case when B is an arithmetic progression. They proved that every such sequence is a Pillai sequence. In our recent works [6, 7] we considered the problem in recurrence sequences, showing that apart from some degenerate cases every linear and elliptic divisibility sequence is a Pillai sequence. We could also give the exact values of the quantities g_B and G_B in the case of Lucas and Lehmer sequences. From these results one might think that being a Pillai sequence is some kind of "automatic" property for recurrence sequences, but this is not the situation. We showed in [6] that a family of so-called associated Lucas and Lehmer sequences is not Pillai and even the quantity g_B does not exist.

[1]Research supported in part by the OTKA grants K100339 and NK101680. The publication is supported by the TÁMOP-4.2.2.C-11/1/KONV-20102-0001 project. The project has been supported by the European Union, co-financed by the European Social Fund.

[2]This research was supported by the European Union and the State of Hungary, co-financed by the European Social Fund in the framework of TÁMOP-4.2.4.A/2-11/1-2012-0001 'National Excellence Program'.

Associated Lucas and Lehmer sequences are integer-valued linear recurrence sequences of order two and four, respectively. A well-known example of the former is the sequence of Lucas numbers. Note that these sequences are also known as Lucas and Lehmer sequences of the second kind. The arithmetic and divisibility properties of such sequences have been investigated by many authors from many aspects. Here we just mention a few of the most important and interesting results.

One of the main questions is about the greatest common divisors of terms of associated Lucas or Lehmer sequences. A nice theorem of McDaniel [11], gives us a precise formula for calculating the greatest common divisor of two terms in such a sequence. McDaniel [12], also investigated the problems of factorizing Lucas numbers as product of two integers differing by a constant, and representing them as difference of two squares. Building on earlier works, mostly on that of Somer [16], Smyth [15] described the sets of terms of associated Lucas sequences which are divisible by their indices. Smyth even showed that the numbers in these sets can be represented as products of so-called basic numbers. Luca and Shorey [8, 9, 10] investigated diophantine equations with products of terms of associated Lucas sequences, where the indices are chosen under certain assumptions.

In this paper, we continue the investigation of Pillai sequences, that started in [6, 7], giving a full characterization of this property in associated Lucas and Lehmer sequences. Our main result shows that for such a sequence being Pillai depends on the parities of the coefficients in their defining recurrences only. We treat the non-degenerate and degenerate sequences in separate theorems. In each case, when the sequence $\hat{v}$ proves to be Pillai, upper bounds for $g_{\hat{v}}$ and $G_{\hat{v}}$ are provided. A result on the particular sequence L of Lucas numbers is also given. This shows that while this sequence is not Pillai, the quantity g_L exists.

We also give an insight into the more general T-Pillai property which is obtained by replacing the coprimality condition with the so-called T-coprimality, where the "unwanted" gcd-s are from a set T of integers. In the case of Pillai sequences the set of "unwanted" gcd-s is $T = \{1\}$. On this topic two results will be presented, one for the associated Lucas and another one for the associated Lehmer sequences.

2. Basic definitions and notations

In this section, we introduce the necessary background containing a brief overview of the sequences we shall work with and our terminology concerning the T-Pillai property as well.

2.1. **Associated Lucas and Lehmer sequences.** Let M and N be integers, such that $N \neq 0$. A sequence $v = (v_n)_{n=0}^{\infty}$ is called an associated Lucas sequence corresponding to the parameters (M, N) if $v_0 = 2,\ v_1 = M$ and for every $n \geq 0$ we have

$$v_{n+2} = Mv_{n+1} - Nv_n. \tag{2.1}$$

A well-known example of such sequences is the sequence of Lucas numbers denoted by $L = (L_n)_{n=0}^{\infty}$, where $(M, N) = (1, -1)$.

We shall also work with associated Lehmer sequences. Let M and N be as before. We say that a sequence $\tilde{v} = (\tilde{v}_n)_{n=0}^{\infty}$ is an associated Lehmer sequence corresponding to the parameters (M, N) if $\tilde{v}_0 = 2$, $\tilde{v}_1 = 1$ and for every $n \geq 0$ the terms satisfy the recurrence relation

$$\tilde{v}_{n+2} = \begin{cases} M\tilde{v}_{n+1} - N\tilde{v}_n, & \text{if } n \text{ is even,} \\ \tilde{v}_{n+1} - N\tilde{v}_n, & \text{if } n \text{ is odd.} \end{cases} \tag{2.2}$$

An associated Lucas (or associated Lehmer, resp.) sequence is said to be non-degenerate if the quotient of the roots of the polynomial

$$x^2 - Mx + N \quad (\text{or } x^2 - \sqrt{M}x + N, \text{ resp.})$$

is not a root of unity. Otherwise, we call it degenerate.

2.2. **T-Pillai sequences.** Let T be a set of arbitrary integers. We say that two integers p and q are T-coprime if $\gcd(p, q) \in T$. We would like to use this notion as a generalization of coprimality, hence always suppose that $1 \in T$. As one can easily see the case $T = \{1\}$ gives back the canonical definition of coprimality.

Replacing coprimality with T-coprimality in the definition of Pillai sequences leads us to the notion of T-Pillai sequences. Let B be a sequence of arbitrary integers as before and let $g_B(T)$ denote the smallest number k such that there exists k consecutive terms of B with the property that none of these is T-coprime to all the others. Similarly, write $G_B(T)$ for the least value such that for every $k \geq G_B(T)$ there exists k consecutive terms of B with the latter property. To be consistent with the terminology presented in Section 1, we use g_B and G_B for the case $T = \{1\}$ and we call B a Pillai sequence if G_B exists.

3. New results

We begin with a simple but important remark about the case when the corresponding parameters are not coprime.

Remark 3.1. *Let $\hat{v} = (\hat{v}_n)_{n=0}^{\infty}$ be an associated Lucas or Lehmer sequence corresponding to the parameters (M, N). Observe that if the parameters are not coprime then $\gcd(M, N) \mid \hat{v}_n$ for every $n \geq 3$. Hence $\hat{v}$ is a Pillai sequence and $g_{\hat{v}} = G_{\hat{v}} = 2$.*

In view of this observation, from this point on we suppose that $\gcd(M, N) = 1$, without any further mentioning.

3.1. **Results on the Pillai property.** Our first result gives a characterization of the Pillai property in non-degenerate associated Lucas and Lehmer sequences. Interestingly, the fact whether they are Pillai or not depends on the parities of the two corresponding parameters only. Moreover, an explicit upper bound for the g and G values is also given.

Theorem 3.1. *Let $\hat{v}$ be a non-degenerate associated Lucas or Lehmer sequence corresponding to the parameters (M, N). Then $\hat{v}$ is Pillai if and only if M is even and N is odd. Further, if $G_{\hat{v}}$ exists and $\hat{v}$ is an associated Lucas sequence, then*

$$g_{\hat{v}} = G_{\hat{v}} = 2,$$

and if $\hat{v}$ is an associated Lehmer sequence, then

$$g_{\hat{v}} \leq G_{\hat{v}} \leq 1543.$$

The following statement concerns the well-known sequence of Lucas numbers. It turns out that it is not Pillai, but the corresponding g_L quantity exists and can be calculated precisely.

Theorem 3.2. *The sequence of Lucas numbers $L = (L_n)_{n=0}^{\infty}$ is not a Pillai sequence, but g_L exists and $g_L = 171$.*

To complete the investigation of the Pillai property in associated sequences we shall handle the degenerate sequences as well. The following theorem involves two separate statements, one for degenerate associated Lucas and another one for degenerate associated Lehmer sequences.

Theorem 3.3. *Let v be a degenerate associated Lucas sequence corresponding to the parameters (M, N). Then v is Pillai if and only if $(M, N) = (\pm 2, 1)$ or $(0, \pm 1)$ and in these cases $g_v = G_v = 2$.*

Now let $\tilde{v}$ be a degenerate associated Lehmer sequence corresponding to the parameters (M, N). Then $\tilde{v}$ is Pillai if and only if $(M, N) = (0, \pm 1)$ and $g_{\tilde{v}} = G_{\tilde{v}} = 171$ in both cases.

3.2. **Results on the T-Pillai property.** In this part, we investigate the more general T-Pillai property. Recall the assumption that $1 \in T$ always, hence if an integer sequence is not Pillai then it cannot be T-Pillai. Our first theorem gives a characterization of the T-Pillai property in non-degenerate associated Lucas sequences.

Theorem 3.4. *Let v be a non-degenerate associated Lucas sequence corresponding to the parameters (M, N) such that M is even and N is odd, and let T be an arbitrary set of integers with $1 \in T$. Then v is a T-Pillai sequence if and only if $2 \notin T$. Further, if v is a T-Pillai sequence, then*

$$g_v(T) = G_v(T) = 2.$$

Note that the above statement is not valid if we replace v by a non-degenerate associated Lehmer sequence. It can be easily checked with the extreme example $T = \mathbb{N} \setminus \{2\}$. Our last theorem shows that the T-Pillai property in non-degenerate associated Lehmer sequences can still be described under an extra assumption on T.

Theorem 3.5. *Let $\tilde{v}$ be a non-degenerate associated Lehmer sequence corresponding to the parameters (M, N) such that M is even and N is odd, S be a finite set of arbitrary primes with s elements and T be an arbitrary set of integers whose prime divisors are in S. Then $\tilde{v}$ is a T-Pillai sequence if and only if $2 \notin T$. Further,*

$$g_{\tilde{v}} \leq G_{\tilde{v}} \leq 20(s + 15)\log(s + 15)$$

Observe that Theorem 3.4 and 3.5 do not say anything about the cases apart from M is even and N is odd. However, we have the assumption $1 \in T$, that is, a sequence which is not Pillai cannot be T-Pillai. Thus, we consider these cases trivial.

Finally, we note that since there are only finitely many degenerate associated Lucas and Lehmer sequences, for such sequences the T-Pillai property can be easily checked for any given set T (see also the proof of Theorem 3.3).

4. Proofs

4.1. **Some lemmas.** The first important result we mention is due to McDaniel [11] who has described the gcd-s of terms in associated Lucas and Lehmer sequences. Before giving the precise statement we need to introduce a new notation. If q is an integer then denote by $\nu_2(q)$ the exponent of 2 in the prime factorization of q.

Lemma 4.1 (Main Theorem in [11])**.** *Let $\hat{v}$ be an associated Lucas or Lehmer sequence corresponding to the parameters (M, N). Then we have*

$$\gcd(\hat{v}_i, \hat{v}_j) = \begin{cases} \hat{v}_{\gcd(i,j)}, & \text{if } \nu_2(i) = \nu_2(j); \\ 1 \text{ or } 2, & \text{otherwise.} \end{cases}$$

Our second lemma is on the multiplicity of terms in non-degenerate associated Lucas and Lehmer sequences with value ± 1. It is an immediate consequence of the celebrated result of Bilu, Hanrot and Voutier [1] concerning the primitive prime divisors Lucas and Lehmer sequences.

(M,N)	n
$(\pm 1+3N,N), N\neq \pm 1$	$1,3$
$((L_k+5F_k)/2,F_k), k\neq 0,1,2$	$1,5$
$(22,9)$	$1,7$

Table 1. Values of (M,N,n) such that $\hat{v}_n=\pm 1$

Lemma 4.2. *Let $\tilde{v}=(\tilde{v}_n)_{n=0}^{\infty}$ be a non-degenerate associated Lehmer sequence corresponding to the parameters (M,N) with even M and odd N. Then all the triples (M,N,n), where $\tilde{v}_n=\pm 1$ are given in Table 1. Here $L=(L_n)_{n=0}^{\infty}$ and $F=(F_n)_{n=0}^{\infty}$ stand for the sequence of Lucas and Fibonacci numbers, respectively.*

Proof. First, observe that from the parities of the parameters and (2.2) every term with an even indice is even too. Hence we can confine our attention to odd indices only. A well-known relation between the ordinary Lehmer sequence $\tilde{u}=(\tilde{u}_n)_{n=0}^{\infty}$ and the associated Lehmer sequence $\tilde{v}=(\tilde{v}_n)_{n=0}^{\infty}$ both corresponding to the parameters (M,N) that

$$\tilde{v}_n=\frac{\tilde{u}_{2n}}{\tilde{u}_n}.$$

That is, whenever $\tilde{u}_{2n}$ have a primitive prime divisor, then $\tilde{v}_n$ also has one. Hence, we can use the tables in [1] and with some more, but simple calculations, including the solution of a Thue equationby Magma [2], we obtain Table 1. □

We shall also need some tools concerning the Pillai and T-Pillai properties.

Lemma 4.3. *Suppose that T is a finite set of odd positive integers and let $O=(O_n)_{n=0}^{\infty}=(2n+1)_{n=0}^{\infty}$ be the sequence of odd positive integers. Put*

$$T'=\{2^{\alpha}\cdot t : t\in T,\ \alpha=0,1\}.$$

Suppose that

$$|T'(X)|\leq \frac{X}{10\log X}$$

holds for every $X>X_1$, where

$$T'(X)=\{t\in T' : t\leq X\}.$$

Then we have

$$G_O(T)\leq \max(425,2X_1+1)$$

Proof. This result is an immediate consequence of Theorem 2.10 and Corollary 2.11 in [5] □

The last lemma gives a bit more insight into some details of the so-called prime covering of the sequence of odd psoitive integers.

Lemma 4.4. *The sequence of odd integers $O=(O_n)_{n=0}^{\infty}=(2n+1)_{n=0}^{\infty}$ is a Pillai sequence and $g_O=G_O=86$. Further, suppose that for some even non-negative integer t the 86 consecutive odd integers $t+1,\dots,t+171$ have the property that none of them is coprime to all the others. Then for any $i\in\{2,4,\dots,170\}$ there exists an even non-negative t' such that none of $t'+1,\dots,t'+171$ is coprime to all the others, $t'+j\equiv t+j\ (p)$ for all primes p with $2<p<86$, and $\nu_2(t'+i)>\nu_2(t'+j)$ for every $i\neq j\in\{1,2,\dots,171\}$.*

Proof. The fact that O is a Pillai sequence is a consequence of Theorem 1 in [13]. The values $G_O = g_O = 86$ are given by Theorem 2.8. of [5]. To prove the second part of the statement, one can extend the construction given in the proof of Theorem 2.8. of [5]. Namely, the appropriate values for t are constructed by the help of a linear system of congruences, where the moduli are the odd primes < 86. Extending this system by a congruence modulo an appriopriate power of 2 (say 2^7), one can obtain infinitely many values for t' satisfying the required properties. □

4.2. **Proofs of the theorems.**

Proof of Theorem 3.1. We split the proof into three parts corresponding to the parities of the parameters involved. Note that the case when M and N are both even is excluded by the condition $\gcd(M, N) = 1$.

Case 1./a) Let M be even and N be odd and suppose that $\hat{v}$ is a non-degenerate associated Lucas sequence. The defining relation (2.1) gives that every term is even in $\hat{v}$ which immediately implies that $\hat{v}$ is a Pillai sequence with $g_{\hat{v}} = G_{\hat{v}} = 2$.

Case 1./b) Now let M and N be the same as before and suppose that $\hat{v}$ is a non-degenerate associated Lehmer sequence. Using Lemma 4.2 we obtain that the set

$$T = \{i : \hat{v}_i = \pm 1\},$$

is exactly one of the followings.

$$\{1\},\ \{1,3\},\ \{1,5\},\ \{1,7\}.$$

Consider $k \geq 2$ consecutive terms of $\hat{v}$, say $\hat{v}_{t+1},\ \ldots,\ \hat{v}_{t+k}$. Observe that the recurrence relation (2.2) gives us that every second term of $\hat{v}$ is even, hence by Lemma 4.1 it is sufficient to show that for every odd $i \in \{t+1, \ldots, t+k\}$ there exists an odd $j \in \{t+1, \ldots, t+k\} \setminus \{i\}$ such that $\gcd(i, j) \notin T$. That is, if the sequence $O = (O_n)_{n=0}^{\infty} = (2n+1)_{n=0}^{\infty}$ of positive odd integers forms a T-Pillai sequence then the same is true for $\hat{v}$. The possibilities for T and Lemma 4.3 with some simple calculation yield

$$g_{\hat{v}} \leq G_{\hat{v}} \leq 1543.$$

Case 2.) Suppose that M and N are both odd and $\hat{v}$ is a non-degenerate associated Lucas or Lehmer sequence. Let $k = 2^{\alpha}$ for some $\alpha > 0$ and consider k consecutive terms of $\hat{v}$, say $\hat{v}_{t+1}, \ldots, \hat{v}_{t+k}$. By the choice of k there exist two indices $i, j \in \{t+1, \ldots, t+k\}$ such that $\nu_2(i) > \nu_2(j) > \nu_2(l)$ hold for every $l \in \{t+1, \ldots, t+k\} \setminus \{i, j\}$. Observe that by the defining relations (2.1) and (2.2) we have that precisely every third term of $\hat{v}$ is even. Obviously, i and j differ by $2^{\alpha-1}$, that is, cannot be divisible by 3 at the same time. Without losing generality we can suppose that $3 \nmid i$. Then Lemma 4.1 gives us that $\gcd(\hat{v}_i, \hat{v}_l) = 1$ for every $i \neq l \in \{t+1, \ldots, t+k\}$. Thus $\hat{v}$ is not a Pillai sequence in this case.

Case 3.) Suppose that M is odd and N is even and $\hat{v}$ is a non-degenerate associated Lucas or Lehmer sequence. Let $k \geq 2$ and consider k consecutive terms of $\hat{v}$, say $\hat{v}_{t+1}, \ldots, \hat{v}_{t+k}$. Then there exists an index $i \in \{t+1, \ldots, t+k\}$ such that $\nu_2(i) > \nu_2(j)$ for every $i \neq j \in \{t+1, \ldots, t+k\}$. Lemma 4.1 gives us $\gcd(\hat{v}_i, \hat{v}_j) = 1$ or 2, from which 2 is not possible because by the recurrence relations (2.1) and (2.2) we have that every term of $\hat{v}$ is odd. Hence, the sequence cannot be Pillai and even $g_{\hat{v}}$ does not exist. □

Proof of Theorem 3.2. Let $L = (L_n)_{n=0}^{\infty}$ be the sequence of Lucas numbers. The first part of the statement, namely that L is not a Pillai sequence is a simple consequence of Theorem 3.1 because now $(M, N) = (1, -1)$. We split the proof of the second part of the statement into two. First we prove the existence of 171 consecutive Lucas numbers with the property that none of them is coprime to all the others, then we show that one cannot find such a string of consecutive Lucas numbers with fewer terms.

From the first part of the statement of Lemma 4.4 we know that for some non-negative even integer t the 86 consecutive odd integers $t+1, t+3, \dots, t+171$ have the property that none of them is coprime to all the others. Further, there exists a positive integer t' such that $t'+1, t'+3, \dots, t'+171$ are 86 consecutive odd integers having the latter property and $3 \mid t'+84$ and $\nu_2(t'+84) > \nu_2(t'+j)$ for every $j \in \{1, 2, \dots, 171\} \setminus \{84\}$. Now consider the corresponding consecutive Lucas numbers of the above indices, namely $L_{t'+1}, L_{t'+2}, \dots, L_{t'+171}$. Observe that the reccurence relation (2.1) gives us that every third Lucas number is even, hence by $3 \mid t'+84$, $L_{t'+84}$ is even and cannot be coprime to all the others. As it was mentioned above for every $i \in \{t'+1, t'+3, \dots, t'+171\}$ there exists an index $j \in \{t'+1, t'+3, \dots, t'+171\} \setminus \{i\}$ such that $\gcd(i, j) \neq 1$. Note that besides $L_1 = 1$ there are no other ± 1 values in L. From Lemma 4.1 it follows that in this case $\gcd(L_i, L_j) \neq 1$. Finally for every $i \in \{t'+2, t'+4, \dots, t'+170\} \setminus \{t'+84\}$ there exists an index $j \in \{t'+2, t'+4, \dots, t'+170\} \setminus \{t'+84\}$ such that $i \neq j$ and $\nu_2(i) = \nu_2(j)$. That is, $\gcd(i, j) \geq 2$, hence $\gcd(L_i, L_j) \neq 1$ which shows that $L_{t'+1}, L_{t'+2}, \dots, L_{t'+171}$ are 171 consecutive Lucas numbers none of which is coprime to all the others. Thus, $g_L \leq 171$.

Now it is sufficient to show that if one takes 170 or less consecutive Lucas numbers then one of them must be coprime to all the others. Let $2 \leq k \leq 170$ and consider k consecutive Lucas numbers $L_{t+1}, \dots, L_{t+k}$. Observe that there cannot be 86 odd indices among $t+1, \dots, t+k$. Thus, by Lemma 4.4 we know that one of the odd indices, say i, is coprime to all the other odd indices. Hence, from Lemma 4.1 it follows that $\gcd(L_i, L_j) = L_1 = 1$ for every odd index $j \neq i$. Note that if $3 \mid i$ would be valid then obviously there can be at most 5 consecutive odd indices among $t+1, \dots, t+k$. On the other hand, if $3 \nmid i$ then $\gcd(i, j) = 1$ for every $j \neq i$, that is L_i is coprime to all the other terms. This means that to complete the proof we shall deal with the cases where k can be at most 11 and 3 must divide i. Using Lemma 4.1, one can easily check that in such sets of consecutive Lucas numbers there always exists one term with the property that it is coprime to all the others. □

Proof of Theorem 3.3. First let $\hat{v} = (\hat{v})_{n=0}^{\infty}$ be a degenerate associated Lucas sequence corresponding to the parameters (M, N). Recall that $\gcd(M, N) = 1$, and $N \neq 0$. Let α and β be the roots of the polynomial $x^2 - Mx + N$. By definition, we have that the quotient of these roots is a root of unity. This quotient can only be a rational or an algebraic integer of degree at most two, hence, one of the followings:

$$\pm 1, \quad \pm i, \quad \pm \varepsilon, \quad \pm \varepsilon^2,$$

where $\varepsilon = \frac{1+i\sqrt{3}}{2}$. Suppose that $\alpha/\beta = -\varepsilon$. Then $M = (1-\varepsilon)\beta$ and $N = -\varepsilon\beta^2$ which gives us $M^2 = N$. From the coprimality of M and N, we have that $(M, N) = (1, 1)$, because $(M, N) = (-1, 1)$ would imply $\alpha/\beta \neq -\varepsilon$. Checking all the possibilites, in an analogous way we obtain that the parameters of degenerate associated Lucas sequences correspond to one of the pairs

$$(M, N) = (0, \pm 1), \ (\pm 1, 1), \ (\pm 2, 1).$$

The related sequences are either periodic, or lead to trivial cases, like the sequences of positive integers, and can be handled easily; see also the papers [3, 6, 13]. The situation for degenerate associated Lehmer sequences is similar, so we omit the details. □

Proof of Theorem 3.4. In the proof of Theorem 3.1 we mentioned that if M is even and N is odd then every term of the associated Lucas sequence v is even. Consider $k \geq 2$ consecutive terms of v, say $v_{t+1}, \dots, v_{t+k}$. There exists an index $i \in \{t+1, \dots, t+k\}$ such that $\nu_2(i) > \nu_2(j)$

for every $i \neq j \in \{t+1, \ldots, t+k\}$. Using Lemma 4.1, we have that $\gcd(v_i, v_j) = 1$ or 2 for every $i \neq j \in \{t+1, \ldots, t+k\}$ from which 1 is not possible because all the terms are even. Thus, if $2 \in T$ then the sequence is not T-Pillai but if $2 \notin T$, then it is. □

Proof of Theorem 3.5. Suppose that M is even and N is odd, $\tilde{v}$ is the corresponding non-degenerate associated Lehmer sequence and T is an arbitrary set of integers, such that the elements of T do not have prime divisors outside a given finite set S of primes with s elements. First, suppose that $2 \in T$ and consider k consecutive terms of $\tilde{v}$ say $\tilde{v}_{t+1}, \ldots, \tilde{v}_{t+k}$. There exists an index $i \in \{t+1, \ldots, t+k\}$ such that $\nu_2(i) > \nu_2(j)$ for every $i \neq j \in \{t+1, \ldots, t+k\}$. Using Lemma 4.1, we have that $\gcd(\tilde{v}_i, \tilde{v}_j) = 1$ or 2. Hence $\tilde{v}_i$ is T-coprime to all the the others and $\tilde{v}$ cannot be T-Pillai.

Now suppose that $2 \notin T$. Take

$$T' = \{i : \tilde{v}_i \in T\}.$$

Recall the relation

$$\tilde{v}_n = \frac{\tilde{u}_{2n}}{\tilde{u}_n}$$

and the primitive prime divisor theorem in [1]. From these we have that T' is a finite set with at most $s + 15$ elements. Recall that the reccurence relation (2.2) means that the parity of a term is exactly the same as of its index. Taking k consecutive terms of $\tilde{v}$, say $\tilde{v}_{t+1}, \ldots, \tilde{v}_{t+k}$ and i with $\nu_2(i) > \nu_2(j)$ for every $i \neq j \in \{t+1, \ldots, t+k\}$, we have that $\gcd(\tilde{v}_i, \tilde{v}_j) = 2$ for every even $i \neq j \in \{t+1, \ldots, t+k\}$. But since $2 \notin T$ the remainder of the proof is exactly the same as in Case 1./b) in the proof of Theorem 3.1 when $\hat{v}$ is a non-degenerate associated Lehmer sequence. We only have to replace T with T' there. □

Acknowledgements

We are grateful to the referee for her/his useful comments, in particular for drawing our attention to the applicability of the primitive prime divisor theorem from [1].

References

[1] Y. Bilu, G. Hanrot, P. M. Voutier, *Existence of primitive divisors of Lucas and Lehmer numbers. With an appendix by M. Mignotte*, J. Reine Angew. Math **539** (2001), 75-122.

[2] W. Bosma, J. Cannon, C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput., **24** (1997), 235265.

[3] A. T. Brauer, *On a property of consecutive integers*, Bull. Amer. Math. Soc. **47** (1941), 328-331.

[4] Y. Caro, *On a division property of consecutive integers*, Israel J. Math. **33** (1979), 32-36.

[5] L. Hajdu, N. Saradha, *On a problem of Pillai and its generalizations*, Acta Arith. **144** (2010), 323-347.

[6] L. Hajdu, M. Szikszai, *On the GCD-s of k consecutive terms of Lucas sequences*, J. Number Theory **132** (2012), 3056-3069.

[7] L. Hajdu, M. Szikszai, *On common factors within a series of consecutive terms of an elliptic divisibility sequence*, Publ. Math. Debrecen, to appear.

[8] F. Luca, T. N. Shorey, *Diophantine equations with products of consecutive terms in Lucas sequences*, J. Number Theory **114** (2005), 541-560.

[9] F. Luca, T. N. Shorey, *Diophantine equations with products of consecutive terms in Lucas sequences, II*, Acta Arith. **133** (2008), 53-71.

[10] F. Luca, T. N. Shorey, *Product of members of Lucas sequences with indices in an interval being a power*, J. Number Theory **129** (2009), 303-315.

[11] W. L. McDaniel, *The G.C.D. in Lucas sequences and Lehmer number sequences*, Fibonacci Quart. **29** (1991), 24-29.

[12] W. L. McDaniel, *On the Factorization of Lucas Numbers*, Fibonacci Quart. **39** (2001), 206-210.

[13] M. Ohtomo, F. Tamari, *On relative prime number in a sequence of positive integers*, J. Stat. Plan. Inf. **106** (2002), 509-515.

[14] S. S. Pillai, *On M consecutive integers - I*, Proc. Indian Acad. Sci., Sect. A. **11** (1940), 6-12.
[15] C. Smyth, *The terms in Lucas sequences divisible by their indices*, J. Integer Sequences **13** (2010), Article 10.2.4.
[16] L. Somer, *Divisibility of terms in Lucas sequences of the second kind by their subscripts*, Applications of Fibonacci numbers, Vol. **5** (St. Andrews, 1992), Kluwer Acad. Publ., Dordrecht, 1993, pp. 515-525.

MSC2010: 11B39

Institute of Mathematics, University of Debrecen, Debrecen, P.O. Box 12., H-4010, Hungary
E-mail address: hajdul@science.unideb.hu

E-mail address: szikszai.marton@science.unideb.hu