ON GENERALIZATIONS OF PROBLEMS OF RECAMAN AND POMERANCE

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ABSTRACT. Answering a question of Balasubramanian, we find all primes p for which there exist p consecutive primes forming a complete residue system (mod p). On the other hand, under the prime ℓ-tuple conjecture we show that for any $k \geq 2$, there exist infinitely many sets of $\varphi(k)$ consecutive primes forming reduced residue classes (mod k). The problems considered are generalizations of those of Recaman and Pomerance, respectively.

1. INTRODUCTION

Let $2 = p_1 < p_2 < \cdots$ denote the sequence of all primes. Let k and l be positive integers with $\gcd(k, l) = 1$. Denote by $p(k, l)$ the least prime $p \equiv l \pmod{k}$. We write $P(k)$ for the maximal value of $p(k, l)$ for all l.

A prime p is called a Recaman prime, if the *first* p primes form a complete residue system (mod p). Pomerance [11] showed that there are only finitely many Recaman primes. Recently, Hajdu and Saradha [4] proved that the only Recaman prime is $p = 2$. An integer $k \geq 2$ is called a P-integer, if the *first* $\varphi(k)$ primes coprime to k form a reduced residue system (mod k). Pomerance [11] proved that there exist only finitely many P-integers. Under certain conditions, Hajdu and Saradha [4] and [13] determined all P-integers. Hajdu, Saradha and Tijdeman [5] proved that if k is a P integer, then $k \leq 10^{3500}$, and that if the Riemann Hypothesis is true, then the only P-integers are given by $k = 2, 4, 6, 12, 18, 30$. Finally, this was unconditionally verified by Yang and Togbé [14].

After the talk of the first author in the DMANT 2015 meeting, Balasubramanian proposed the variaton of the above problems where the

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first k (resp. $\varphi(k)$) primes are replaced by any block of k (resp. $\varphi(k)$) consecutive primes.

To be more precise, we introduce some new definitions. An integer k is called a *B-prime* if there exist k *consecutive* primes forming a complete residue system (mod k). Further, an integer k is called a *B-integer*, if there exist $\varphi(k)$ *consecutive* primes forming a reduced residue system (mod k).

Note that the Recaman prime 2 is a B-prime also. Further the P-integers $2, 4, 6, 12, 18, 30$ are also B-integers. When a prime k is a B-prime, we have

$$P(k) \leq p_{\pi(k)+k-1}. \tag{1}$$

From well known estimates in Prime Number Theory, it is clear that $p_{\pi(k)+k-1} \ll k \log k$. In fact, the implicit constant lies between 1 and 1.04 for $k \geq 10^{93}$. This leads us to make a more general definition as follows. We say that a prime k is a *shifted P_α-prime* if there exist k primes not exceeding $\alpha k \log k$ forming a complete residue system. Finally, an integer k is called a *shifted P_α-integer* if there exist $\varphi(k)$ primes not exceeding $\alpha k \log k$ forming a reduced residue system (mod k).

In this paper, we show that the only B-primes are 2,3,7 and there is no shifted P_α-prime with $\alpha = 1.1954$. Pomerance [11, Theorem 2] showed that if k is any positive integer, then

$$P(k) \geq (e^\gamma + o(1))\varphi(k) \log k$$

where φ denotes the Euler totient function, and $\gamma = 0.577\ldots$ is Euler's constant. In particular when k is a prime, this gives

$$P(k) \geq (e^\gamma + o(1))k \log k.$$

Here the implied constant is not explicit and may be very small. By Theorem 2.2 below, we see that

$$P(k) > 1.1954k \log k$$

for all primes k. It appears that one needs to take $k > 10^{10^{10}}$, in order to get

$$P(k) \geq e^\gamma k \log k$$

by the method in this paper.

Finding upper bound for $P(k)$ is a well known problem. Linnik [8] showed that

$$P(k) \leq ck^L$$

where c and L are effectively computable constants. There is a huge literature on finding the best constant L.

In 1992, Heath-Brown [6] had shown that L can be taken as 5.5. This has been improved to 5 by Xylouris [16] (see Theorem 2.1, p. 12) in 2011. A conjecture of Chowla [1] says that L is $1+\epsilon$ for arbitrary $\epsilon > 0$. Observe that as α increases, the set of shifted P_α-primes (or integers) becomes larger and larger. Under Chowla's conjecture, we see that α (as a function of k) must be of the order k^ϵ so that all primes (or integers) k may become shifted P_α-primes (or integers). On the other hand, if k is a B-integer, then we need to find $\varphi(k)$ *consecutive* primes coprime to k. Assuming the prime ℓ-tuple conjecture of Hardy and Littlewood, we deduce that every integer k is a B-integer, and in fact one can choose appropriate blocks of $\varphi(k)$ consecutive primes in infinitely many ways. We note that for $k = 2, 3, 4, 6$ this assertion easily follows unconditionally.

2. Results

Theorem 2.1. *The only B-primes are given by $2, 3, 7$.*

Theorem 2.2. *There is no shifted P_α-prime with $\alpha = 1.1954$.*

The above two results are contained in the following theorem.

Theorem 2.3. *Let k be a prime with the property that there exist k primes not exceeding max$(p_{\pi(k)+k-1}, 1.1954k \log k)$ which form a complete residue system. Then $k \in \{2, 3, 7, 11\}$.*

To get the assertions of Theorems 2.1 and 2.2 we first deduce that

$$\max(p_{\pi(k)+k-1}, 1.1954k \log k) = \begin{cases} p_{\pi(k)+k-1}, & \text{if } k < 6691068 \\ 1.1954k \log k, & \text{otherwise.} \end{cases} \tag{2}$$

Further we find that $2, 3, 7$ are B-primes since

$$\{2, 3\}, \{3, 5, 7\}, \{7, 11, 13, 17, 19, 23, 29\}$$

form complete residue systems, respectively. Also $2, 3, 7$ are not shifted P_α-primes with $\alpha = 1.1954$ since $\pi(1.1954k \log(k)) < k$ in these cases. Further, 11 is not a B-prime, since no set of 11 consecutive primes forms a complete residue system (mod 11).

Using the argument in the proof of [4, Theorem 2], one may obtain the following result which we state without proof.

Let α be a fixed positive number. Suppose k is a shifted P_α-integer with the least prime factor of k exceeding $\log(k)$. *Then there exists an effectively computable number $c(\alpha)$ depending only on α such that $k < c(\alpha)$.*

The above result leads us to speculate if there are only finitely many B-integers. We show below that the contrary is true under the prime ℓ-tuple conjecture of Hardy and Littlewood. In fact, assuming the conjecture we deduce that every integer k is a B-integer, and one can choose appropriate blocks of $\varphi(k)$ consecutive primes in infinitely many ways. We note that for $k = 2, 3, 4, 6$ this assertion easily follows unconditionally.

Before formulating our next theorem, we recall the prime ℓ-tuple conjecture. A finite set A of integers is called admissible, if for any prime p, no subset of A forms a complete residue system (mod p).

Conjecture 2.1 (The prime ℓ-tuple conjecture)**.**

Let $\{a_1, \dots, a_\ell\}$ be an admissible set of integers. Then there exist infinitely many positive integers n such that $n + a_1, \dots, n + a_\ell$ are all primes.

Remark. By a recent, deep result of Maynard [9] we know that for each ℓ, the above conjecture holds for a positive proportion of admissible ℓ-tuples.

Theorem 2.4. *Suppose that the prime ℓ-tuple conjecture is true. Then for every integer $k \geq 2$ one can find infinitely many sets of $\varphi(k)$ consecutive primes forming a reduced residue system* (mod k)*.*

Remark. In fact, in the proof of Theorem 2.4 we need the numbers $n + a_1, \dots, n + a_\ell$ occurring in the prime ℓ-tuple conjecture to be *consecutive* primes. In case of $\ell = 2$, by deep and celebrated results of Zhang [17] and Pintz [10] we know this to be true for infinitely many admissible sets $\{a_1, a_2\}$, even with $a_1 = 0$. In case of general ℓ, such a variant is known to follow from the following quantitative version of the prime ℓ-tuple conjecture, also made by Hardy and Littlewood. Let $A_0 = \{a_1, \dots, a_\ell\}$ be an admissible set with $a_1 < a_2 < \dots < a_\ell$. Put

$$I_0 = \{n \in \mathbb{N} : a_1 \leq n \leq a_\ell\} \text{ and } A_0' = I_0 \setminus A_0.$$

For every prime p let v_p be the number of residue classes (mod p) met by A_0. Clearly, for all p we have $1 \leq v_p \leq p - 1$. Put

$$\delta_{A_0} := \prod_{p \text{ prime}} \frac{1 - \frac{v_p}{p}}{\left(1 - \frac{1}{p}\right)^\ell}.$$

Note that here the product on the right hand side is convergent for any admissible set. Further if $A_0 \subseteq B$, then $\delta_{A_0} \geq \delta_B$. Let

$$S = \{n \in \mathbb{N} : n + a_1, \cdots, n + a_\ell \text{ are } \textit{all} \text{ primes}\}$$

and

$$S(X) = \{n \in S : n \leq X\}.$$

Then the quantitative version of the prime ℓ-tuple conjecture of Hardy and Littlewood asserts that

$$|S(X)| = (\delta_{A_0} + o(1))\frac{X}{(\log X)^\ell}.$$

Now we explain how this implies that there are infinitely many integers n for which $n + a_1, \cdots, n + a_\ell$ are *all consecutive* primes. Let

$$S_1 = \{n \in S : n + a_1, \cdots, n + a_\ell \text{ are } \textit{not consecutive} \text{ primes}\}.$$

It is enough to show that

$$|S_1(X)| = o\left(\frac{X}{(\log X)^\ell}\right).$$

If $n \in S_1$, then there exists $a \in A_0'$ such that $n + a$ is prime. Also $A_0^{(a)} := A_0 \cup \{a\}$ is an admissible set. For $a \in A_0'$, let

$$S_1^{(a)} = \{n \in S_1 : n + a_1, \cdots, n + a_\ell, n + a \text{ are all primes}\}.$$

Then

$$S_1 = \bigcup_{a \in A_0'} S_1^{(a)}.$$

Thus

$$|S_1(X)| \leq \sum_{a \in A_0'} (\delta_{A_0^{(a)}} + o(1))\frac{X}{(\log X)^{\ell+1}}$$

$$\leq (\delta_{A_0} + o(1))(a_\ell - a_1)\frac{X}{(\log X)^{\ell+1}} = o\left(\frac{X}{(\log X)^\ell}\right)$$

for $X \to \infty$ as desired. However, in the proof of Theorem 2.4 we avoid the use of the quantitative version of the conjecture. In fact, we apply an elementary argument showing that the prime ℓ-tuple conjecture itself implies the existence of infinitely many n such that the numbers $n + a_1, \ldots, n + a_\ell$ are *consecutive* primes.

As a simple corollary of Theorem 2.4, we obtain

Corollary 2.1. *Suppose that the prime ℓ-tuple conjecture is true. Then every integer $k \geq 2$ is a B-integer.*

Remark. It is obvious that 2 is a B-integer. Since for $k = 3, 4, 6$ there are only two coprime residue classes, and both classes contain infinitely many primes, there must be infinitely many "switches" between these classes in pairs of consecutive primes. Hence $k = 3, 4, 6$ are (unconditionally) also B-integers.

In view of the above remarks and theorems, we propose the following

Conjecture 2.2. *Every integer $k \geq 2$ is a B-integer.*

3. Lemmas

The proof of Theorem 2.3 follows similar line of arguments as the proof of [4, Theorem 2]. We record here three lemmas necessary for the proof. The first lemma is from Rosser and Schoenfeld [12].

Lemma 3.1. *Let p_n denote the n-th prime. Then*
(i) $p_n > n(\log(n) + \log_2(n) - \frac{3}{2})$ *for* $n > 1$;
(ii) $p_n < n(\log(n) + \log_2(n))$ *for* $n \geq 6$.

Here and henceforth, $\log_2(n)$ denotes $\log\log(n)$ for any real number $n > 1$. For $n \geq 1$ the Jacobsthal function $g(n)$ is defined as the smallest integer such that any sequence of $g(n)$ consecutive integers contains an element which is coprime to n. This function has been studied by many authors, and good lower as well as upper bounds are known (see e.g. [7], [15], [11], [3] and [2] for some results and history). Further, the exact values of $g(n)$ when n is the product of the first $h < 50$ primes is given in [3, Table 1].

It was observed by Jacobsthal that for integers k with $\ell(k) > \log(k)$ we have $g(k) = \omega(k) + 1$ where $\ell(k)$ is the least prime divisor of k, and $\omega(k)$ is the number of distinct prime divisors of k. In particular this is true if k is a prime i.e., $g(k) = 2$ in this case. Further, $g(k) \geq \omega(k) + 1$ is obviously valid for any k. We shall use these assertions throughout the paper without any further reference. The following lemma is Proposition 1.1 of Hagedorn [3].

Lemma 3.2. *We have*

$$g\left(\prod_{i=1}^{h} p_i\right) \geq 2p_{h-1} \quad \textit{for} \quad h > 2.$$

The next result due to Pomerance [11] is an important ingredient in this problem.

Lemma 3.3. *Let k and m be integers with $0 < m \leq \frac{k}{1+g(k)}$ and $\gcd(m, k) = 1$. Then $P(k) > (g(m) - 1)k$.*

4. Proofs

Proof of Theorem 2.3. We restrict to k prime so that $g(k) = 2$. First take $k \geq 10^{93}$. By (2),

$$\max(p_{\pi(k)+k-1}, 1.1954k \log k) = 1.1954k \log k.$$

Put

$$h = \left\lfloor \frac{0.9688 \log(k)}{\log_2(k)} \right\rfloor + 1.$$

Then

$$h < \frac{0.9946 \log(k)}{\log_2(k)}$$

giving

$$\log(h) < \log_2(k) - \log_3(k) \quad \text{and} \quad \log_2(h) < \log_3(k).$$

This by Lemma 3.1 (ii) implies

$$p_h < 0.9946 \log(k) < \log(k).$$

Let m be the product of the first h primes coprime to k. Since $p_h < \log(k) < k$, we see that m is indeed the product of all the first h primes. Hence

$$m < p_h^h < e^{0.9946 \log(k)} < \frac{k}{3}.$$

Thus by Lemmas 3.2 and 3.3, we have

$$P(k) > (g(m) - 1)k \geq (2p_{h-1} - 1)k.$$

Now

$$h - 1 \geq 0.9688 \frac{\log(k)}{\log_2(k)} - 1 > 0.943 \frac{\log(k)}{\log_2(k)}.$$

Hence by Lemma 3.1 (i)

$$p_{h-1} \geq X \left(\log(X) + \log_2(X) - \frac{3}{2} \right)$$

where $X = 0.943 \frac{\log(k)}{\log_2(k)}$. Let

$$F(k) = 2X \left(\log(X) + \log_2(X) - \frac{3}{2} - \frac{1}{2X} \right) k - 1.1954k \log(k).$$

Then $F(k) = k \log(k) f(k)$ with

$$f(k) := \frac{1.886}{\log_2(k)} \left(\log(X) + \log_2(X) - \frac{3}{2} - \frac{1}{2X} \right) - 1.1954.$$

Observe that $f(k)$ is an increasing function of k and hence $f(k) \geq f(10^{93})$, since $k \geq 10^{93}$. As $f(10^{93}) \geq 0.0005$, we find that $F(k) > 0$ which implies that $P(k) > 1.1954k \log k$. Hence k is not a P_α-prime with $\alpha = 1.1954$. This proves the theorem for $k \geq 10^{93}$.

Next consider $6691068 \leq k < 10^{93}$. By (2),

$$\max(p_{\pi(k)+k-1}, 1.1954k \log k) = 1.1954k \log(k).$$

Suppose $k \in [10^{43}, 10^{93})$. The largest integer h such that $p_h < \log(10^{43})$ is 25. Taking

$$m = \prod_{j=1}^{25} p_j,$$

we find that $\gcd(m,k) = 1$ and

$$m < \frac{10^{43}}{3} \leq \frac{k}{g(k)+1}.$$

From [3, Table 1], $g(m) = 258$. Hence by Lemma 3.3,

$$P(k) > 257k > 1.1954 \times 93 \log(10)k > 1.1954 \times k \log(k).$$

This proves the proposition for $k \in [10^{43}, 10^{93})$. Let $k \in [10^a, 10^b)$. In Table 1, we give the values of $(a,b), h$, the exact value of $g(m)$ from [3, Table 1] where $m = \prod_{i=1}^{h} p_i$ so that

$$p_h < \log(10^a), P(k) > 1.1954k \log(k).$$

Then the assertion of the theorem follows for k in this interval. Thus

h	7	8	9	11	14	18
$g(m)$	26	34	40	58	90	132
(a,b)	(8,9)	(9,10)	(10,14)	(14,19)	(19,27)	(27,43)

Table 1. Values of h, $g(m)$ and (a,b).

we conclude that $k < 10^8$. Further, we take $k \in [6691068, 10^8)$ with $h = 7, g(m) = 26$ to get the assertion of the theorem.

Next, we take $90107 \leq k < 6691068$. In this case, we find that $p_{\pi(k)+k-1} < 1.25k \log(k)$. Then we take $h = 6, g(m) = 22$ to exclude these values of k by Lemma 3.3.

Thus $k < 90107$. For these values of k we give a computational argument. Let k be fixed. Suppose S_k denotes the set of residues mod k of all the primes upto $p_{\pi(k)+k-1}$. If

$$|S_k| = k \tag{3}$$

then, k may be a B-prime. We check that (3) is valid only for $k = 2, 3, 7, 11$. Further 11 is not a B-prime as there is no set of 11 consecutive primes among the first 15 primes which yields a complete residue system. On the other hand, 2,3,7 give consecutive primes forming a complete residue system as mentioned in Section 2. This proves the theorem. □

Proof of Theorem 2.4. Let $k \geq 2$ be an arbitrary integer. We shall show that under the prime ℓ-tuple conjecture, k is a B-integer i.e., there exists $\varphi(k)$ consecutive primes forming a reduced residue system mod k. Let $A = \{a_1, \ldots, a_{\varphi(k)}\}$ be the set of all positive integers coprime to k with

$$1 = a_1 < \cdots < a_{\varphi(k)} < k.$$

The set A may not be an admissible set. We construct an admissible set out of A as follows. Put

$$P = \prod_{\substack{p-\text{prime} \\ p\nmid k, p\leq\varphi(k)}} p.$$

Let $B = \{b_1, \ldots, b_{\varphi(k)}\}$ be a set of positive integers such that

$$(4) \quad b_1 = a_1 = 1; \quad b_i \equiv a_i \pmod{k} \text{ and } b_i \equiv 1 \pmod{P} \text{ (for } i \geq 2).$$

Firstly, note that such b_i's exist by the Chinese Remainder Theorem. Next we show that B is an admissible set. Since $|B| = \varphi(k)$ and B contains integers coprime to k, it is enough to restrict to primes $p \leq \varphi(k)$ and $p \nmid k$. Then by (4), every $b_i \equiv 1 (\text{mod } p)$, hence B cannot have a complete residue system (mod p). By applying the prime ℓ-tuple conjecture to B, we find infinitely many $n > k$ for which

$$n + b_1, \cdots, n + b_{\varphi(k)}$$

are all primes and hence coprime to k. But these primes *may not be consecutive primes.* To ensure this, we proceed as follows. Let

$$M = \max_{b\in B} b$$

and I the set of positive integers n with $n \leq M$. Further let

$$C = \{c \in I \setminus B : B \cup \{c\} \text{ is admissible}\}.$$

Let $t = |C|$ and write $C' = I \setminus (B \cup C)$. Thus for $c' \in C', B \cup \{c'\}$ is not an admissible set. Hence there exists a prime $p \leq M$ such that $B \cup \{c'\}$ has a complete residue system (mod p).

Note that $M > k$ by (4). We construct an admissible set $S \supseteq B$, such that $S \cup \{c\}$ is not admissible for any $c \in C$. If $t = 0$ then take $S = B$. If $t \geq 1$, take primes $q_1 < \cdots < q_t$ exceeding M and put

$$Q = \prod_{p<q_1+\cdots+q_t} p.$$

Let us enumerate the elements of C as $c_1, \ldots, c_t$. Corresponding to each c_i, we construct a set $D^{(i)}$ as follows. Let $d_1^{(i)}$ satisfy

$$d_1^{(i)} > M, d_1^{(i)} \equiv 1 \left(\text{mod } \frac{Q}{q_i}\right) \text{ and } d_1^{(i)} (\text{mod } q_i) \notin B \cup \{c_i\}.$$

Since $B \cup \{c_i\}$ is an admissible set, it is possible to find $d_1^{(i)}$ as above. Now consider

$$B \cup \{c_i\} \cup \{d_1^{(i)}\}.$$

If this has a complete residue system $\pmod{q_i}$, then put $D^{(i)} = \{d_1^{(i)}\}$. If not, we choose

$$d_2^{(i)} > M, d_2^{(i)} \equiv 1 \left(\bmod \frac{Q}{q_i}\right) \text{ and } d_2^{(i)} (\bmod q_i) \notin B \cup \{c_i\} \cup \{d_1^{(i)} (\bmod q_i)\}.$$

If $B \cup \{c_i\} \cup \{d_1^{(i)}, d_2^{(i)}\}$ has a complete residue system (modq_i), take $D^{(i)} = \{d_1^{(i)}, d_2^{(i)}\}$. Otherwise, we proceed to find $d_3^{(i)}$ and so on. This process has at most $q_i - 1 - \varphi(k)$ steps. Thus $D^{(i)}$ has at most $q_i - 1 - \varphi(k)$ elements with the property that

$$B \cup \{c_i\} \cup D^{(i)}$$

has a complete residue system (mod q_i) and every element of $D^{(i)}$ exceeds M. Take

$$S = B \cup D^{(1)} \cup \cdots \cup D^{(t)}.$$

We show that S is an admissible set. Firstly,

$$|S| \leq \varphi(k) + q_1 + \cdots + q_t - t(\varphi(k) + 1) < q_1 + \cdots + q_t$$

since $t \geq 1$. Hence we need to consider only primes $p < q_1 + \cdots + q_t$. Let p be such a prime with $p \neq q_i$ $(1 \leq i \leq t)$. Then by the definition of Q and the construction of the sets $D^{(i)}$, all the elements of $D^{(i)}$ are $\equiv 1 (\bmod p)$ and as $1 \in B$ we get

$$S \equiv B \pmod{p}.$$

(By the above notation we mean $\{s \pmod{p} : s \in S\} = \{b \pmod{p} : b \in B\}$). Since B is an admissible set, we see that S cannot have a complete residue system (mod p). Let now $p = q_i$ for some i with $1 \leq i \leq t$. Then

$$S \equiv B \cup D^{(i)} (\bmod p).$$

Since $c_i \notin B \cup D^{(i)}$, by $q_i > M$ and the construction of $D^{(i)}$, the set S does not contain a complete residue system (mod p). Thus S is an admissible set.

Note that for $1 \leq i \leq t$, $S \cup \{c_i\}$ is *not* an admissible set since it contains a complete residue system (mod q_i). Also for any $c' \in C'$, $S \cup \{c'\}$ is not an admissible set since $B \cup \{c'\}$ is not admissible, by definition and in this case there exists a complete residue system (mod p) for some $p \leq M$. Summarizing, S is an admissible set, but $S \cup \{c\}$ for $c \in I \setminus B$ is not an admissible set. Thus for any $c \in I \setminus B$ there exists a prime p_c such that $S \cup \{c\}$ contains a complete residue system (mod

p_c). As seen earlier, p_c can be taken as not exceeding M or equal to q_i for some i with $1 \le i \le t$. Hence $p_c \le q_t$. Now we apply the prime ℓ-tuple conjecture to the set S to find infinitely many $n > q_t$ such that

$$n + s \text{ is prime for } s \in S. \tag{5}$$

For any $c \in I \setminus B$, there exists a complete residue system (mod p_c) in $S \cup \{c\}$ and hence in $\{n+s, s \in S \cup \{c\}\}$ for any n, and in particular for those n satisfying (5). Thus $p_c|(n+s)$ for some $s \in S \cup \{c\}$. Since $n+s$ for $s \in S$ are all primes $> q_t$, this implies that $s = c$. That is, $p_c \mid n+c$, whence $n + c$ is not a prime for any $c \in I \setminus B$. This means that $n + b$ with $b \in B$ are $\varphi(k)$ consecutive primes, all coprime to k. Since by the construction of B these numbers belong to different residue classes (mod k), we get that k is a B-integer. In fact there are infinitely many sets of $\varphi(k)$ consecutive primes, coprime to k, belonging to different residue classes (mod k). □

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