ON THE DISTRIBUTION OF POLYNOMIALS WITH BOUNDED HEIGHT

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Dedicated to Professor Andrzej Schinzel on the occasion of his 80th birthday.

Abstract. We provide an asymptotic expression for the probability that a randomly chosen polynomial with given degree, having integral coefficients bounded by some B, has a prescribed signature. We also give certain related formulas and numerical results along this line. Our theorems are closely related to earlier results of Akiyama and Pethő, and also yield extensions of recent results of Dubickas and Sha.

1. Introduction

Let d be a positive integer, $B \geq 1$ a real number. Denote by $\mathcal{H}_d(B)$ the set of $(d+1)$-dimensional vectors $(p_0, \ldots, p_d)$ satisfying $|p_i| \leq B$ $(0 \leq i \leq d), p_d \neq 0$. In the case $B = 1$ we write simply $\mathcal{H}_d$ instead of $\mathcal{H}_d(1)$.

Given a polynomial $P \in \mathbb{R}[X]$, the non-real roots of P appear in complex conjugate pairs. Thus $d = r + 2s$, where r denotes the number of real roots and s the number of non-real pairs of roots of P. As we shall work with arbitrary but fixed d and then r is uniquely determined by s, we call s the signature of P. Identifying the vector $(p_0, \ldots, p_d) \in \mathbb{R}^{d+1}$ and the polynomial $p_d X^d + p_{d-1} X^{d-1} + \cdots + p_0$ the set $\mathcal{H}_d(B)$ splits naturally into $\lfloor d/2 \rfloor + 1$ disjoint subsets according to the signature. In the sequel $\mathcal{H}_d(s, B)$ denotes the subset of $\mathcal{H}_d(B)$ whose elements have signature s. If $B = 1$, in place of $\mathcal{H}_d(s, 1)$ we shall simply write $\mathcal{H}_d(s)$. Plainly, $\mathcal{H}_d(s, B)$ is a bounded set in $\mathbb{R}^{d+1}$ for any $B > 0$, and we will prove that it is Lebesgue measurable. For the Lebesgue measure (which we shall often simply call volume) of $A \subset \mathbb{R}^n$ we write $\lambda_n(A)$ or $\lambda(A)$, if the dimension n is obvious.

Following Dubickas and Sha [4] denote by $\mathcal{D}_d^*(s, B)$[1] the set of polynomials $f(X) = p_d X^d + p_{d-1} X^{d-1} + \cdots + p_0 \in \mathbb{Z}[X]$ with $p_d \neq 0$, $|p_i| \leq B$ $(i =$

Date: March 26, 2017.
2010 *Mathematics Subject Classification.* 11C08.
Key words and phrases. Polynomials, height, signature, distribution.
*Research supported in part by the NKFI grant NK104208.
**Research supported in part by the NKFI grant K115479.
***Research supported in part by the NKFI grants NK104208 and K115479.

[1] In fact Dubickas and Sha [4] called (r, s) the signature of P and used the notation $\mathcal{D}_d^*(r, s, B)$ instead of $\mathcal{D}_d^*(s, B)$. As we frequently cite the papers of Akiyama and Pethő [1] and [2], where only s was used for the signature and sets of polynomials were denoted according to this convention, we follow their notation.

$0, \dots, d)$ and such that f has signature s. That is, $\mathcal{D}_d^*(s,B) = \mathcal{H}_d(s,B) \cap \mathbb{Z}^{d+1}$. Denote by $D_d^*(s,B)$ the number of elements of $\mathcal{D}_d^*(s,B)$. They proved

$$B^{d+1} \ll D_d^*(s,B) \ll B^{d+1} \tag{1}$$

by using a lower bound for the number of integer polynomials approximating appropriately a real polynomial of degree d and signature s. They wrote: "It would be of interest to obtain an asymptotic formula as (1.1) in our setting as well." That is, they ask for an asymptotic formula for $D_d^*(s,B)$.

In this paper we improve considerably (1) by providing an asymptotic formula for $D_d^*(s,B)$, thus we fulfill their request. It is important to mention that Akiyama and Pethő [1, 2] considered a similar problem, when instead of the absolute values of the coefficients of the polynomials, the absolute values of the roots of the polynomials are assumed to be bounded. Our method works for other height functions, too. For its application it is sufficient to prove that the boundary of the set of polynomials of height at most B is a smooth function. Moreover, one needs that the volume of the sets of polynomials with given signature of degree d and of height B is $\gg B^d$, see an example in the last section.

Beside this, we prove asymptotic upper and lower estimates for the number $D_d(s,B)$ of elements of $\mathcal{D}_d(s,B)$, where $\mathcal{D}_d(s,B)$ is the subset of elements of $\mathcal{D}_d^*(s,B)$ with $p_d = 1$. It turns out that as one would expect, the magnitude of $D_d(s,B)$ is B^d. However, in this case we are unable to provide a more precise statement.

We also give a formula for $\lambda(\mathcal{H}_d(s,B))$ for any d, s and B, involving integrals. Our formulas are similar to those obtained by Akiyama and Pethő [1, 2]. Akiyama and Pethő could handle the integrals occurring there by Selberg integrals, and gave the precise volumes of the corresponding sets for small values of d. In our case, unfortunately we cannot handle the integrals theoretically, except certain 'small' cases. To get some numerical results we apply the Monte Carlo method to approximate the occurring integrals for $d \le 15$.

The structure of the paper is the following. In the next section we give our theoretical results. Then we prove our theorems. In the fourth section our numerical results are given for $d \le 15$. Finally, we indicate some open problems.

2. New results

Our main result is the following.

Theorem 2.1. *We have*

$$D_d^*(s,B) = \lambda_{d+1}(\mathcal{H}_d(s))B^{d+1} + O(B^d).$$

Moreover, $\lambda_{d+1}(\mathcal{H}_d(s)) > 0$ *for all* d *and* s.

In our proof we follow closely the ideas of Akiyama and Pethő [2]. However instead of using a classical result of Davenport [3], we prove our result by arguing that the box counting dimension of the boundary of $\mathcal{H}_d(s,B)$ is one less then that of the set itself.

Our next theorem concerns $D_d(s,B)$. However, in this case, similarly to the result of Dubickas and Sha [4] for $D_d^*(s,B)$, we can only describe the magnitude of $D_d(s,B)$, without achieving an asymptotic formula.

Theorem 2.2. *We have*

$$B^d \ll D_d(s,B) \ll B^d,$$

where the implied constants depend only on d.

We note that following the proof of Theorem 2.2, one can see that in fact the implied constants can be given in a refined way, depending also on s. Since we have $s \leq d$, we chose to give the above simpler formulation.

Our further aim is to derive a formula for the volume of $\mathcal{H}_d(s,B)$. For this purpose we need some preparation. Denote by $S_j(x_1,\ldots,x_d)$ $(j=1,\ldots,d)$ the j-th elementary symmetric polynomial of $x_1,\ldots,x_d$, that is

$$S_j(x_1,\ldots,x_d) = \sum_{1\leq i_1<\cdots<i_j\leq d} x_{i_1}\cdots x_{i_j}.$$

For later use we define $S_0(x_1,\ldots,x_d)=1$. For $B>0$ let $H_d(s,B)$ denote the set of such d-dimensional real vectors $(x_1,\ldots,x_r,y_1,\ldots,y_s,z_1,\ldots,z_s)$ which satisfy the inequalities

$$-B \leq S_j(x_1,\ldots,x_r,y_1+iz_1,y_1-iz_1,\ldots,y_s+iz_s,y_s-iz_s) \leq B \ (1\leq j\leq d)$$

and $z_j \neq 0$ $(j=1,\ldots,s)$, where $i=\sqrt{-1}$. If $B=1$, we simply write $H_d(s)$ for this set.

Obviously, we have $(p_0,\ldots,p_d)\in\mathcal{H}_d(s,B)$ if and only if $|p_d|\leq B$ with $p_d\neq 0$, and the vector $(x_1,\ldots,x_r,y_1,\ldots,y_s,z_1,\ldots,z_s)$ belongs to $H_d(s,B/|p_d|)$, where $x_1,\ldots,x_r,y_1\pm z_1 i,y_s\pm z_s i$ are the roots of $p_dX^d+\cdots+p_0$.

Denote by $\mathrm{Res}(P(X),Q(X))$ the resultant of $P(X),Q(X)\in\mathbb{R}[X]$. For any possible s and positive real number B put

$$\mathcal{H}_d^*(s,B) := \{(p_0,\ldots,p_{d-1})\in\mathbb{R}^d \ : \ (p_0,\ldots,p_{d-1},1)\in\mathcal{H}_d(s,B)\},$$

and

$$\mathcal{H}_d^+(s,B) := \{(p_0,\ldots,p_d)\in\mathcal{H}_d(s,B) \ : \ p_d>0\}.$$

When $B=1$, the above sets are simply denoted by $\mathcal{H}_d^*(s)$ and $\mathcal{H}_d^+(s)$, respectively.

By the above notion, we have the following theorem.

Theorem 2.3. *Let* $R_j(X)=X^2-2y_jX+y_j^2+z_j^2$ $(j=1,\ldots,s)$. *Then*

$$\lambda_d(\mathcal{H}_d^*(s,B)) = \frac{2^s}{r!s!}\int\limits_{H_d(s,B)} |\Delta_r|\Delta_s\Delta_{r,s}\prod_{j=1}^{s}|z_j| \ dx_1\ldots dx_r dy_1 dz_1\ldots dy_s dz_s,$$

where

$$\Delta_r = \prod_{1\le j<k\le r} (x_j - x_k),$$
$$\Delta_s = \prod_{1\le j<k\le s} \mathrm{Res}(R_j(X), R_k(X)),$$
$$\Delta_{r,s} = \prod_{j=1}^{r}\prod_{k=1}^{s} R_k(x_j).$$

Furthermore, we have

$$\lambda_{d+1}(\mathcal{H}_d^+(s,B)) = B^{d+1}\int_0^1 u^d \lambda\left(\mathcal{H}_d^*\left(s,\frac{1}{u}\right)\right)du.$$

We note that by Theorem 2.1 we know that $\lambda_{d+1}(\mathcal{H}_d(s,B))$ exists for any $B>0$. Further, in view of $\lambda_{d+1}(\mathcal{H}_d(s,B)) = 2\lambda_{d+1}(\mathcal{H}_d^+(s,B))$ (see Corollary 3.1 below), the above theorem gives a formula (though implicit) for $\lambda_{d+1}(\mathcal{H}_d(s,B))$, for any $B>0$.

3. Proofs

In this section we prove our theorems. First we investigate $\mathcal{H}_d^+(s,B)$ for $B>0$. Later we also need to consider the set $\mathcal{H}_d^-(s,B)$, which is the set of vectors $\mathbf{v}$, such that $-\mathbf{v}\in\mathcal{H}_d^+(s,B)$.

Lemma 3.1. *The set $\mathcal{H}_d^+(s)$ has positive Jordan measure and its boundary is the union of finitely many algebraic surfaces.*

Proof. Following Akiyama and Pethő [2], denote by $\mathcal{E}_d^{(s)}(B)$ ($s=0,\dots,\lfloor d/2\rfloor$) the set of vectors $(p_0,\dots,p_{d-1})\in\mathbb{R}^d$ such that the corresponding polynomial $X^d+p_{d-1}X^{d-1}+\dots+p_0$ has signature s, and all of its roots lie in the disc of radius B.

Denote by $\mathcal{A}_d^{(s)}$ the set of points of $\mathcal{H}_d(s)$ with $p_d>0$. Let $\psi : \mathcal{A}_d^{(s)}\mapsto\mathbb{R}^d$ be the continuous mapping

$$\psi(p_0,\dots,p_d) = (p_0p_d^{d-1},\dots,p_{d-2}p_d,p_{d-1}).$$

The polynomial $P(X)=p_dX^d+p_{d-1}X^{d-1}+\dots+p_0$ has signature s and its coefficients satisfy the inequalities $0<p_d\le 1$, $|p_j|\le 1$, $(j=0,\dots,d-1)$. Plainly, with the substitution $p_dX=Y$ we have

$$p_d^{d-1}P(X) = Q(Y) = Y^d+p_{d-1}Y^{d-1}+p_{d-2}p_dY^{d-2}+\dots+p_0p_d^{d-1}$$

and $Q(Y)$ has the same signature as $P(X)$. Moreover, by Proposition 2.5.9. of [7], all roots of $Q(Y)$ lie in the disc of radius 2. Hence $\psi(\mathcal{A}_d^{(s)})=\mathcal{F}_d^{(s)}(p_d)$, where

$$\mathcal{F}_d^{(s)}(p_d) := \mathcal{E}_d^{(s)}(2)\cap([-p_d^{d-1},p_d^{d-1}]\times\dots\times[-p_d,p_d]\times[-1,1]).$$

By Lemma 2.1 of [2], $\mathcal{E}_d^{(s)}(B)$ is Jordan measurable for any $B > 0$, thus $\mathcal{F}_d^{(s)}(p_d)$ is Jordan measurable, as well. Denote by $F_d^{(s)}(p_d)$ its d-dimensional Jordan measure. The function $F_d^{(s)}(p_d)$ is continuous for $p_d > 0$, because $\mathcal{E}_d^{(s)}(2)$ is independent of p_d, and its boundary is by Theorem 7.1. of [1] the union of finitely many algebraic surfaces. Also, the box $[-p_d^{d-1}, p_d^{d-1}] \times \cdots \times [-p_d, p_d] \times [-1, 1]$ depends continuously on p_d. The Jacobian of the mapping ψ is $p_d^{d(d-1)/2}$, hence $\lambda_d(\mathcal{A}_d^{(s)}) = p_d^{d(d-1)/2} F_d^{(s)}(p_d)$. Thus we get

$$\begin{aligned}\lambda_{d+1}(\mathcal{H}_d^+(s)) &= \lim_{t\to 0} \int_t^1 \lambda_d(\mathcal{A}_d^{(s)}) dp_d \\ &= \lim_{t\to 0} \int_t^1 p_d^{d(d-1)/2} F_d^{(s)}(p_d) dp_d.\end{aligned}$$

As $\mathcal{H}_d^+(s)$ is bounded and $F_d^{(s)}(p_d)$ is continuous for $p_d > 0$, this integral exists.

Now we prove that $\lambda_{d+1}(\mathcal{H}_d^+(s))$ is positive. For this purpose, assume $1/2 \le p_d \le 1$ in the rest of this proof. (The argument works with any positive lower bound for p_d, but to prove our claim the choice 1/2 is sufficient.) Assume that $q_0, \dots, q_{d-1}$ are so small that all roots of $Q(Y) = Y^d + q_{d-1}Y^{d-1} + \cdots + q_0$ lie in the disc with radius 4^{-d}. Then it is an easy exercise to show, that $|q_j| \le 2^{-d+j+1} \le p_d^{-d+j+1}$ $(j = 0, \dots, d-1)$. Hence $\psi^{-1}(\mathcal{E}_d^{(s)}(4^{-d})) \subseteq \mathcal{H}_d^+(s)$ and we get $F_d^{(s)}(p_d) \ge \lambda_d(\mathcal{E}_d^{(s)}(4^{-d})) > 0$ for all $p_d \ge 1/2$. Thus

$$\lambda_{d+1}(\mathcal{H}_d^+(s)) \ge \frac{1}{2}\left(\frac{1}{2}\right)^{d(d-1)/2} \lambda_d(\mathcal{E}_d^{(s)}(4^{-d})),$$

which is certainly a positive number.

Let $p_d X^d + p_{d-1}X^{d-1} + \cdots + p_0$ be a polynomial with indeterminate coefficients lying in a commutative ring. Then its discriminant $D = D(p_0, \dots, p_d)$ is a homogenous polynomial in $p_0, \dots, p_d$ of degree $d(d-1)$. Specializing the coefficient domain to $\mathbb{C}$ it is well-known that $D = 0$ if and only if either $p_d = 0$, or $p_d \ne 0$ and the polynomial has multiple roots. Using the latter fact Akiyama and Pethő [1] proved, see Theorem 7.1., that the inner boundary points of $\mathcal{E}_d^{(s)}(1)$ lie on the hypersurface S_D defined by the equation $D = 0$. Repeating that proof to $\mathcal{H}_d^+(s)$, one can see that its boundary is the union of finitely many pieces of S_D and the intersection of the hyperplane $p_d = 0$ with the hypercube $[-1, 1]^{d+1}$. $\square$

Corollary 3.1. *Let $B > 0$. Then $\mathcal{H}_d^+(s, B)$ and $\mathcal{H}_d^-(s, B)$ have positive Jordan measure and their boundaries are the unions of finitely many algebraic surfaces. Moreover*

$$\lambda_{d+1}(\mathcal{H}_d^+(s, B)) = \lambda_{d+1}(\mathcal{H}_d^-(s, B)) = \lambda_{d+1}(\mathcal{H}_d^+(s))B^{d+1}.$$

Proof. The assertion follows directly from Lemma 3.1 together with the fact that $(x_0,\dots,x_d)\in\mathcal{H}_d^+(s)$ if and only if $(Bx_0,\dots,Bx_d)\in\mathcal{H}_d^+(s,B)$. □

Now we are in the position to give the proof of our main result.

Proof of Theorem 2.1. For any $B>0$ we have

$$\mathcal{H}_d(s,B)=\mathcal{H}_d^+(s,B)\cup\mathcal{H}_d^-(s,B).$$

Thus, by Lemma 3.1 we get

$$\lambda(\mathcal{H}_d(s,B))=2\lambda(\mathcal{H}_d^+(s,B))=2\lambda(\mathcal{H}_d^+(s))B^{d+1}.$$

The boundary of $\mathcal{H}_d^+(s,B)$ is the union of finitely many algebraic surfaces. These are piecewise smooth up to a null set, i.e. their box counting dimension is at most d. This implies

$$\begin{aligned}|D_d^*(s,B)-2\lambda(\mathcal{H}_d^+(s,B))| &=\\ |D_d^*(s,B)-2B^{d+1}\lambda(\mathcal{H}_d(s))| &= O(B^d),\end{aligned}$$

and our theorem is proved. □

Remark 3.1. *In an earlier version we used a Theorem of Davenport* [3] *to estimate $D_d^*(s,B)$ by $\lambda(\mathcal{H}_d^+(s,B))$. The anonymous referee suggested the here presented proof. The application of Davenport's or similar estimates is only justified if one has more detailed description of the boundary.*

To prove our second theorem, we need the following result of Akiyama and Pethő [2].

Lemma 3.2. *For d and s as above, write $\mathcal{C}_d^{(s)}(b)$ for the set of polynomials with integer coefficients of degree d and signature s, having all roots outside the unit circle, and having constant term $b>0$. Then we have*

$$|\mathcal{C}_d^{(s)}(b)|\gg b^{d-1},$$

where the implied constant depends only on d.

Proof. In view of $s\le d$, the statement is an immediate consequence of Theorem 5.1 of [2]. □

Now we can give the proof of our second theorem.

Proof of Theorem 2.2. The upper estimate trivially follows from the inequality $D_d(s,B)\le(2B+1)^d$. To prove the lower inequality, we shall apply Lemma 3.2. For this, observe that if a polynomial $P(x)=x^d+p_{d-1}x^{d-1}+\dots+p_1x+b\in\mathbb{Z}[x]$ belongs to $\mathcal{C}_d^{(s)}(b)$, then since all the roots of $P(x)$ are larger than one in absolute value, we have $|p_i|\le 2^db$ $(i=1,\dots,d-1)$. This immediately yields that $\mathcal{C}_d^{(s)}(b)\subseteq\mathcal{D}_d(s,B)$ for any integer $B>0$ and

$0 < b < \lfloor B/2^d \rfloor$. Hence, observing that the sets $\mathcal{C}_d^{(s)}(b)$ $(b = 1, \dots, \lfloor B/2^d \rfloor)$ are pairwise distinct, using Lemma 3.2 we obtain

$$D_d(s, B) \geq \sum_{b=1}^{\lfloor B/2^d \rfloor} |\mathcal{C}_d^{(s)}(b)| \gg \sum_{b=1}^{\lfloor B/2^d \rfloor} b^{d-1} \gg B^d,$$

and the theorem follows. □

Finally, we give the proof of our third theorem.

Proof of Theorem 2.3. The first statement concerning the formula given for $\lambda(\mathcal{H}_d^*(s, B))$ follows by a simple calculation from Theorem 2.1. of [1].

To prove the formula for $\lambda(\mathcal{H}_d^+(s, B))$ we start from

$$\lambda(\mathcal{H}_d^+(s, B)) = \int\limits_{\mathcal{H}_d^+(s,B)} 1 \; dp_0 \dots dp_d.$$

We apply the substitution

$$p_d = Bq_d, \quad p_i = Bq_d q_i \quad (i = 0, \dots, d-1).$$

Observe that the determinant of its Jacobian is $B^{d+1} q_d^d$. Thus we have

$$\lambda(\mathcal{H}_d^+(s, B)) = B^{d+1} \int\limits_{A} q_d^d \; dq_0 \dots dq_d,$$

where

$$A = \{(q_0, \dots, q_{d-1}, q_d) \in \mathbb{R}^{d+1} \; : \; X^d + q_{d-1}X^{d-1} + \dots + q_1 X + q_0$$
$$\text{has signature } s \text{ and } 0 < q_d \leq 1, \; -\frac{1}{q_d} \leq q_i \leq \frac{1}{q_d} \; (i = 0, \dots, d-1)\}.$$

Here we used the trivial fact that the signatures of the polynomials

$$X^d + q_{d-1}X^{d-1} + \dots + q_1 X + q_0$$

and

$$Bq_d X^d + Bq_d q_{d-1} X^{d-1} + \dots + Bq_d q_1 X + Bq_d q_0$$

are the same. Putting everything together, we have

$$\lambda_{d+1}(\mathcal{H}_d^+(s, B)) = B^{d+1} \int\limits_0^1 q_d^d \lambda_d \left(\mathcal{H}_d^* \left(s, \frac{1}{q_d} \right) \right) \; dq_d$$

which proves the theorem. □

4. Numerical results

In this section we give some numerical data regarding $\lambda(\mathcal{H}_d^*(s))$ and $\lambda(\mathcal{H}_d^+(s))$. We can calculate the precise values of $\lambda(\mathcal{H}_d^*(s))$ only for $d = 2, 3$, and of $\lambda(\mathcal{H}_d^+(s))$ only for $d = 2$. Evaluating the integrals appearing in Theorem 2.3 we obtain

$$\lambda(\mathcal{H}_2^*(0)) = \frac{13}{6} = 2.1667, \ \lambda(\mathcal{H}_2^*(1)) = \frac{11}{6} = 1.8333,$$

$$\lambda(\mathcal{H}_3^*(0)) = \frac{766}{1215} + \frac{\log(3)}{6} = 0.8136, \ \lambda(\mathcal{H}_3^*(1)) = \frac{8954}{1215} - \frac{\log(3)}{6} = 7.1865,$$

and

$$\lambda(\mathcal{H}_2^+(1)) = \frac{31}{18} - \frac{1}{3}\log(2) = 1.4912, \ \lambda(\mathcal{H}_2^+(0)) = \frac{41}{18} + \frac{1}{3}\log(2) = 2.5088.$$

Here and later on, to perform our calculations we used the program package Mathematica, and the values are always given with four digit precision.

Observe that $\lambda(\mathcal{H}_2^*(s))$ $(s = 0, 1)$ are rational, but $\lambda(\mathcal{H}_2^+(s))$ and $\lambda(\mathcal{H}_3^*(s))$ $(s = 0, 1)$ are transcendental. We think that $\lambda(\mathcal{H}_d^+(s))$ and $\lambda(\mathcal{H}_d^*(s))$ $(s = 0, \dots, \lfloor d/2 \rfloor)$ are all transcendental for $d \geq 2$ and $d \geq 3$, respectively. In contrast, Akiyama and Pethő [1], Theorem 5.1., proved that the analogous values $v_d^{(s)}$ are rational for all d, s.

For larger values of d we were unable to evaluate the integrals appearing in Theorem 2.3. The reason is that when we split up the original domain into subdomains according to the signature, the boundary (coming from the discriminant surface) is so complicated that Mathematica is not able to handle the situation. So to get some numerical data, we needed another approach. We used the Monte Carlo method to get approximate results both for $\lambda(\mathcal{H}_d^*(s))$ and $\lambda(\mathcal{H}_d^+(s))$ for $2 \leq d \leq 15$. The main principle behind the method is that we choose a 'large' number of randomly generated polynomials inside the basic region, and check their signatures. Then, heuristically, the frequency of polynomials having a prescribed signature s gives an approximation of the volume. Certainly, this approach is not capable to give a theoretical bound for the error terms, but it still provides some information about the studied volumes. More precisely, we do the following.

1. For approximating $\lambda(\mathcal{H}_d^*(s))$, we randomly choose (using uniform distribution) a vector from $[-1, 1]^d$, say $(p_0, \dots, p_{d-1})$. For approximating $\lambda(\mathcal{H}_d^+(s, 1))$ we do the same, but now the vector is in $[0, 1] \times [-1, 1]^d$.
2. We construct the polynomial $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1 X + p_0$ or $P(X) = p_d X^d + p_{d-1}X^{d-1} + \cdots + p_1 X + p_0$, respectively.
3. We determine the signature of $P(X)$.
4. After a 'large' number of iterations (in our case we used $200,000$ loops for each d) we calculate the ratio of the number of polynomials with a given signature and the total number of polynomials, which approximate the value of $\lambda(\mathcal{H}_d^*(s))$ or $\lambda(\mathcal{H}_d^+(s))$, respectively.

In the following tables we give the results of the above method for $2 \leq d \leq 15$, that is, the approximate values of $\lambda(\mathcal{H}_d^*(s))$ and $\lambda(\mathcal{H}_d^+(s))$, respectively (for all possible values of s). We note that comparing the approximate values with the precise values given above for $d = 2, 3$ and $d = 2$, respectively, we see that in those cases the errors are around 1%. Thus we expect that the other approximate values are rather close to the actual data, as well.

TABLE 1. The approximated values of $\lambda(\mathcal{H}_d^*(s))$ for $2 \leq d \leq 15$

d/s	0	1	2	3	4	5	6	7
2	2.1652	1.8348	–	–	–	–	–	–
3	0.8192	7.1808	–	–	–	–	–	–
4	0.0880	10.2833	5.6286	–	–	–	–	–
5	0.0021	6.3378	25.6602	–	–	–	–	–
6	0.0003	1.6330	43.9437	18.4230	–	–	–	–
7	0.0000	0.1542	34.128	93.7178	–	–	–	–
8	0.0000	0.0051	12.4442	179.8340	63.7171	–	–	–
9	0.0000	0.0000	2.0838	163.8780	346.0380	–	–	–
10	0.0000	0.0000	0.1434	72.8678	728.5040	222.4840	–	–
11	0.0000	0.0000	0.0102	16.0154	744.4378	1287.5366	–	–
12	0.0000	0.0000	0.0000	1.6589	382.8122	2909.0406	802.4883	–
13	0.0000	0.0000	0.0000	0.0410	98.0173	3227.6070	4866.3347	–
14	0.0000	0.0000	0.0000	0.0000	10.6496	1847.4598	11599.2986	2926.5920
15	0.0000	0.0000	0.0000	0.0000	0.8192	574.4230	13800.5709	18392.1869

TABLE 2. The approximated values of $\lambda(\mathcal{H}_d^+(s))$ for $2 \le d \le 15$

d/s	0	1	2	3	4	5	6	7
2	2.5054	1.4946	–	–	–	–	–	–
3	1.7540	6.2460	–	–	–	–	–	–
4	0.6301	11.3332	4.0361	–	–	–	–	–
5	0.1061	10.7558	21.1381	–	–	–	–	–
6	0.0128	5.5776	45.8112	12.5984	–	–	–	–
7	0.0013	1.6326	52.2074	74.1587	–	–	–	–
8	0.0000	0.2163	33.6922	180.6090	41.4822	–	–	–
9	0.0000	0.0154	12.4595	232.6550	266.8700	–	–	–
10	0.0000	0.0051	2.6317	171.8940	706.3810	143.0890	–	–
11	0.0000	0.0000	0.3174	74.3629	998.0621	975.2576	–	–
12	0.0000	0.0000	0.0000	18.2886	814.0595	2754.4576	509.1942	–
13	0.0000	0.0000	0.0000	2.6214	400.1792	4165.5501	3263.5493	–
14	0.0000	0.0000	0.0000	0.4096	123.2896	3719.1680	10721.5258	1819.6070
15	0.0000	0.0000	0.0000	0.0000	22.1184	1965.7523	17215.8157	13564.3136

The graphs of the functions $\lambda(\mathcal{H}_d^*(s))$, $\lambda(\mathcal{H}_d^+(s))$ and of $v_d^{(s)}$ from [1] seem to have similar fashion. For small s their values tend rapidly to zero. This was proved for $v_d^{(0)}$ in [1] Theorem 6.1. and for $v_d^{(1)}$ by Kirschenhofer and Weitzer [6].

In the following figures we illustrate our results in a more comprehensive way. On the top of the bars we indicate the values of s.

FIGURE 1. Approximate values of $\lambda(\mathcal{H}_d^*(s))$ for even $d < 15$

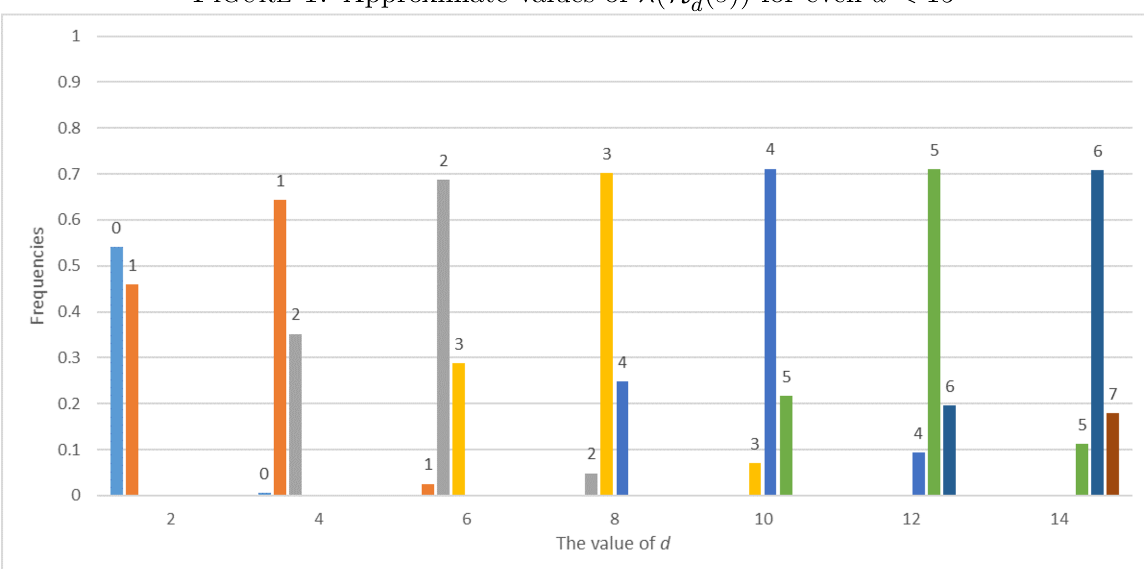


Figure 2. Approximate values of $\lambda(\mathcal{H}_d^*(s))$ for odd $d \leq 15$

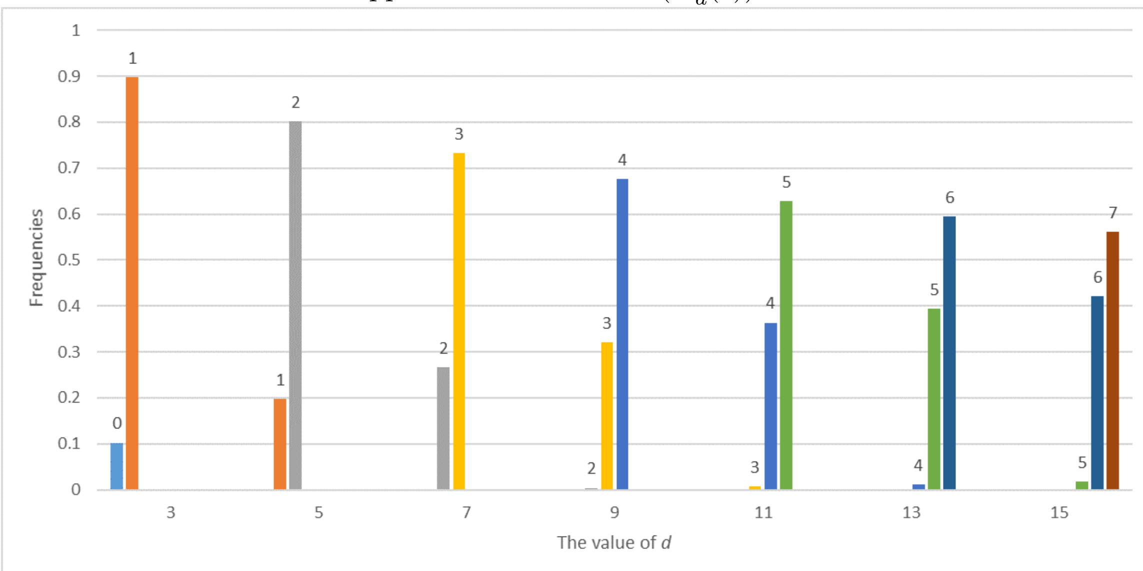


Figure 3. Approximate values of $\lambda(\mathcal{H}_d^+(s))$ for even $d < 15$

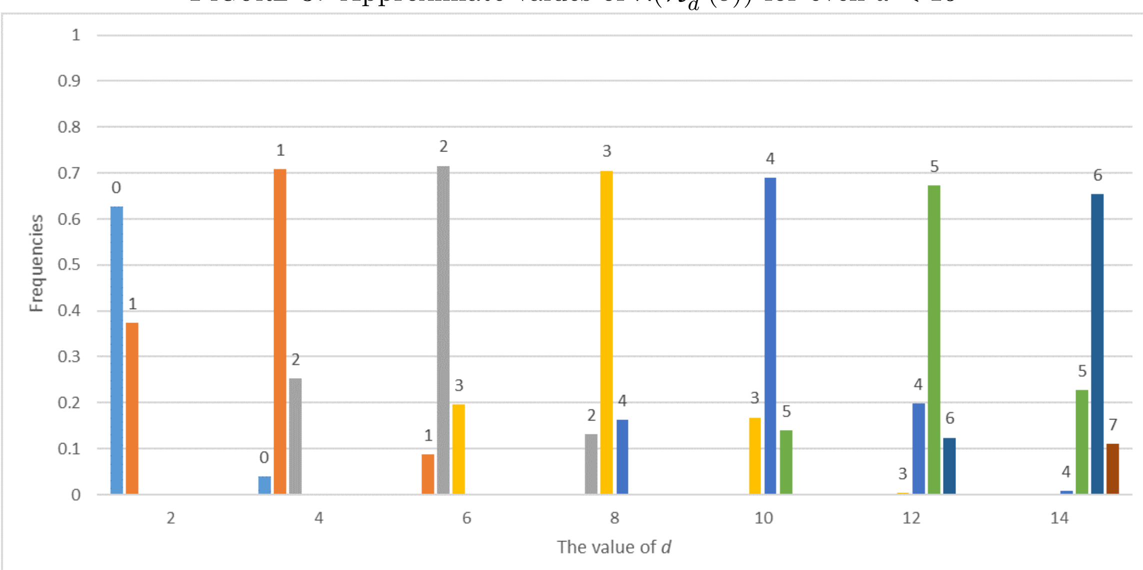
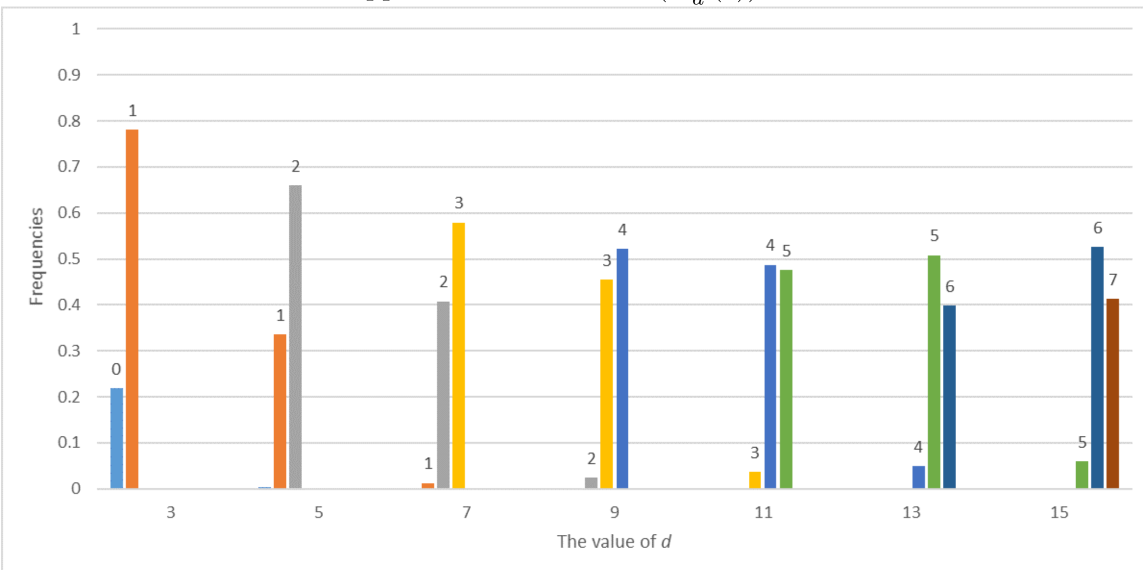


FIGURE 4. Approximate values of $\lambda(\mathcal{H}_d^+(s))$ for odd $d \le 15$



5. Open problems

Similarly as in case of $D_d^*(s, B)$ Dubickas and Sha [4], we propose the following questions concerning $D_d(s, B)$.

Does the limit

$$\lim_{B \to \infty} \frac{D_d(s, B)}{B^d}$$

exist for any d and s? If yes, what is its value?

Based upon our (rather restricted) calculations, we also ask the following questions. Is it true that $\lambda(\mathcal{H}_d^+(s))$ and $\lambda(\mathcal{H}_d^*(s))$ $(s = 0, \ldots, \lfloor d/2 \rfloor)$ are transcendental, for all $d \ge 2$ and $d \ge 3$, respectively? Are they linear combinations of 1 and $\log(d)$ with some rational coefficients?

6. Acknowledgement

We are very indebted for the anonymous referee for his trustworthiness. Using his suggestions our presentation became shorter and clearer.

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