POLYNOMIAL VALUES OF SUMS OF HYPERBOLIC BINOMIAL COEFFICIENTS

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Abstract. Diophantine problems related to binomial coefficients have a vast literature. In this paper we investigate power and polynomial values of row sums of hyperbolic Pascal triangles, which are recently introduced generalizations of Pascal's classical triangle. We prove various effective and ineffective finiteness results.

1. Introduction

There is an extensive literature on Diophantine problems concerning Pascal's classical triangle and binomial coefficients. These problems often lead to Diophantine equations of the form

$$\binom{x}{n} = g(y), \tag{1}$$

i.e., to the study of the polynomial values of binomial coefficients. In 1951 Erdős [16] proved that the equation

$$\binom{x}{n} = y^\ell$$

has no solutions in integers x, n, y, ℓ with $y > 1$, $\ell > 1$, $n \geq 4$ and $x \geq 2n$. Győry [17] extended this result by showing that apart from the case $n = \ell = 2$ the above equation has the only integer solution $y = 140$, $\ell = 2$, $x = 50$, $n = 2$. Yuan [38] showed that apart from an obvious exception, the equation

$$a\binom{x}{n} = by^r + c$$

has only finitely many solutions and all these can be effectively determined. Stoll and Tichy [35] proved effective and ineffective finiteness

Date: July 31, 2019.

2010 *Mathematics Subject Classification.* 11D41, 05A10, 11B37.

Key words and phrases. Pascal triangle, hyperbolic Pascal triangle, Chebyshev polynomials, recurrence sequences, polynomial values, power values.

results on the Diophantine equation

$$a\binom{x}{m} + b\binom{y}{n} = c.$$

Kulkarni and Sury [21] used the ineffective finiteness criterion of Bilu and Tichy [9] to prove that, for irreducible $g \in \mathbb{Q}[x]$ with $\deg g \geq 3$, equation (1) has only finitely many integer solutions. (In fact, they formulated their result for polynomial values of products of consecutive integers, however, the two problems are equivalent.) For further results related to equation (1) we refer to [6, 7, 31, 8, 18] and the references therein. Diophantine properties of sums of binomial coefficients (or, as a strongly related problem, sums of products of blocks of consecutive integers) have also been intensively studied; see e.g. the works [23, 30, 20, 15, 19, 2, 36] and the references given there.

Let p and q be positive integers with $p \geq 3, q \geq 3$. It is well-known that, in the hyperbolic plane there are an infinite number of types of regular mosaics (see e.g. [12]), and they are assigned by Schläfli's symbol $\{p, q\}$ such that $(p-2)(q-2) > 4$. Pascal's classical triangle has several generalizations in the literature (see, e.g., [4, 10]). Following and generalizing the connection between Pascal's classical triangle and the Euclidean regular square mosaic $\{4, 4\}$ to the case of a regular mosaic $\{p, q\}$ on the hyperbolic plane, Belbachir, Németh and Szalay [3] introduced *hyperbolic Pascal triangles.* See [3] for details, figures and explanations. As a hyperbolic analogue of the binomial coefficients they used the notation $)^{n}_{k}($ for the k-th element in the n-th row of such a hyperbolic Pascal triangle. Let us fix $\{4, q\}$ with $q \geq 5$. (Note that for $q = 4$ we get back the original Pascal triangle.) In contrast to the classical Pascal triangle, in the hyperbolic case there are three types of elements $)^{n}_{k}($. Elements having two ascendants and $q-2$ descendants (called *elements of type A*), elements having one ascendant and $q-1$ descendant (called *elements of type B*), and the *winger elements.* Thus, $)^{n}_{k}($ is either the sum of the values of its two ascendants or the value of its unique ascendant.

The authors of [3] showed that the numbers a_n, b_n of elements of type A and B, respectively, the total number $s_n = a_n + b_n + 2$ of elements, the sum of elements of type A and B, respectively, as well as the sum of all elements in a row of the hyperbolic Pascal triangle linked to $\{4, q\}$ can be described by ternary homogeneous recurrence relations and explicit expressions. They also proved that in the special case $q = 5$, given any pair u, v of positive integers there exist $i, j \in \mathbb{N}$ such that $u =)^{i}_{j}($ and $v =)^{i}_{j+1}($. For further related results we refer to [25, 26, 29] and the references given there.

In this paper we are interested in the hyperbolic analogues of the Diophantine problems above. As we mentioned, by results from [3], $\left)\genfrac{}{}{0pt}{}{i}{j}\right($ can take any value infinitely often - so investigating say power values of these hyperbolic binomial coefficients seems to be pointless. (At least, no analogue of the mentioned results of Erdős and Győry exists.) So we shall focus on the *sums* of hyperbolic binomial coefficients in the rows of the hyperbolic Pascal triangles. As we noted above, related problems for sums of binomial coefficients have also been investigated earlier. In the hyperbolic case one can get various finiteness results as well, which can be considered to be the analogues of the corresponding theorems obtained in the classical case. For this purpose, write $\hat{s}_n(q)$ for the sum of all elements in the n-th row of the hyperbolic Pascal triangle linked to $\{4, q\}$ (with $q \geq 5$), that is

$$\hat{s}_n(q) = \sum_{k=0}^{s_n-1} \left)\genfrac{}{}{0pt}{}{n}{k}\right(.$$

Note that in [3] the notation $\hat{s}_n^q$ was used. However, since $\hat{s}_n(q)$ is a polynomial in q, this notation will be more appropriate for our purposes. Recalling the well-known identity

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n, \tag{2}$$

it is natural to study those row sums in the hyperbolic Pascal triangle, i.e., those values of $\hat{s}_n(q)$ which are powers of 2. More generally, our purpose is to study the power values and the polynomial values of $\hat{s}_n(q)$.

In Section 2 we formulate three new theorems. Theorem 2.1 states that $\hat{s}_n(q)$ takes only finitely many perfect power values if the base of the power and q are fixed and all these values can be effectively determined. Theorem 2.2 provides an effectively computable upper bound on the solutions of the equation $\hat{s}_n(q) = ax^\ell + b$ for $n \geq 5$ fixed. Our third result (Theorem 2.3) provides finiteness for the polynomial values of $\hat{s}_n(q)$ with $n \geq 5$ fixed, which is effective in the case when the polynomial has degree 2. To prove our theorems, we need to combine powerful methods (such as Baker's method or the Bilu-Tichy theorem) with new arguments. For example, on our way we shall prove a strong connection between $\hat{s}_n(q)$ and the Chebyshev polynomials of the second kind, and based upon this we derive several important properties of the quantities $\hat{s}_n(q)$.

We mention that proving finiteness results for the polynomial values of $\hat{s}_n(q)$ where both n and q are free seems to be very hard - we leave this problem as an open issue.

2. New results

First, we formulate the result of Belbachir, Németh and Szalay [3] showing that the row sum $\hat{s}_n(q)$ satisfies a certain recurrence relation. For our present purposes, we may use this assertion as the defining property of $\hat{s}_n(q)$.

Theorem A. *Let $\hat{s}_n(q)$ denote the sum of all elements in the n-th row of the hyperbolic Pascal triangle linked to $\{4, q\}$, $q \geq 5$. Then the sequence $\{\hat{s}_n(q)\}_{n\in\mathbb{N}}$ satisfies the ternary homogeneous recurrence relation*

$$\hat{s}_n(q) = q\hat{s}_{n-1}(q) - (q+1)\hat{s}_{n-2}(q) + 2\hat{s}_{n-3}(q) \quad (n \geq 4)$$

with initial values

$$\hat{s}_1(q) = 2,\ \hat{s}_2(q) = 4,\ \hat{s}_3(q) = 2q.$$

Moreover,

$$\hat{s}_n(q) = \frac{q-1-\sqrt{D}}{2\sqrt{D}}\hat{\alpha}_q^n - \frac{q-1+\sqrt{D}}{2\sqrt{D}}\hat{\beta}_q^n + 2 \tag{3}$$

holds, where

$$D = q^2 - 2q - 7, \quad \hat{\alpha}_q = \frac{q-1+\sqrt{D}}{2}, \quad \hat{\beta}_q = \frac{q-1-\sqrt{D}}{2}. \tag{4}$$

Proof. The statement is a reformulation of a part of Theorem 2 of [3]. □

In this section we present some finiteness results concerning the power values and polynomial values of $\hat{s}_n(q)$, when one of n, q is fixed. Our starting point, motivated by the identity (2), is the Diophantine equation

$$\hat{s}_n(q) = 2^\ell \tag{5}$$

in positive integers n, q, ℓ.

First let $q \geq 5$ be a fixed integer. Then, by Theorem A, $\hat{s}_n(q)$ is a ternary homogeneous recurrence sequence. Our first result deals with a generalization of (5), namely with the equation

$$\hat{s}_n(q) = w^\ell \tag{6}$$

in positive integers n and ℓ where w is a fixed integer.

Theorem 2.1. *For any $q \geq 5$ and $w \in \mathbb{Z}$, equation* (6) *has only finitely many solutions in positive integers n, ℓ. Further, we have* $\max\{n, \ell\} < C_1(q, w)$*, where $C_1(q, w)$ is an effectively computable constant depending only on q and w.*

We note that our Theorem 2.1 implies that for any $q \geq 5$, there are only finitely many rows of the hyperbolic Pascal triangle corresponding to $\{4, q\}$ for which the sum of all elements is a power of 2, or more generally, a power of any fixed integer w.

Now let q be a variable, and let $n \geq 3$ be a fixed positive integer. In this case $\hat{s}_n(q)$ is a polynomial in q with integer coefficients of degree $n-2$. In this case we can get more general results, concerning the polynomial values of $\hat{s}_n(q)$. Consider first the Diophantine equation

$$\hat{s}_n(q) = ax^\ell + b \tag{7}$$

in integers q, x, ℓ with $q \geq 5$ and $\ell \geq 2$, where a, b are fixed rationals.

Concerning the above equation, we prove the following effective statement.

Theorem 2.2. *Let $n \geq 5$, $n \neq 6$, $a, b \in \mathbb{Q}$, $a \neq 0$. Assuming that $\ell \leq 3$ if $x \in \{-1, 0, 1\}$, for the solutions $q, x, \ell \in \mathbb{Z}$ of (7) with $q \geq 5$ and $\ell \geq 2$ we have*

$$\max(q, |x|, \ell) < C_2(n, a, b).$$

Here $C_2(n, a, b)$ is an effectively computable constant depending only on n, a and b.

Remark. For $n \leq 3$ equation (7) is trivial. In case of $n = 4$ one could easily find $a, b \in \mathbb{Q}$ such that (7) has infinitely many integer solutions q, x with $\ell = 2$. If $n = 6$, then one can find appropriate a, b again such that equation (7) has infinitely many integer solutions q, x with $\ell = 2$. Namely, as

$$\hat{s}_6(q) + 8 = 2(q^2 - 2q - 2)^2,$$

e.g. the equation

$$\hat{s}_6(q) = 2x^2 - 8$$

has infinitely many integer solutions, given by $q \geq 5$ arbitrary and $x = q^2 - 2q - 2$.

Still with fixed n, let now $g \in \mathbb{Q}[x]$ be an arbitrary polynomial, and consider the equation

$$\hat{s}_n(q) = g(x) \tag{8}$$

in $q, x \in \mathbb{Z}$ with $q \geq 5$. Observe that for fixed ℓ, this equation is a generalization of (7). For n even, put $T_n(x) = \hat{s}_n(\sqrt{x} + 1)$. (Note that in Section 4 we shall show that for n even $\hat{s}_n(x+1)$ is an even polynomial (see Corollary 4.3), thus $T_n \in \mathbb{Q}[x]$.)

Concerning equation (8), we prove the following result.

Theorem 2.3. *Let $g(x) \in \mathbb{Q}[x]$.*

(i) *If* $\deg g = 2$ *and* $n \geq 5$, $n \neq 6$, *then there exists an effectively computable constant* $C_3(n, g)$ *depending only on* n *and* g *such that* $\max(q, |x|) < C_3(n, g)$ *for each integer solutions* q, x *with* $q \geq 5$ *of equation* (8).

(ii) *If* $\deg g \geq 3$ *and* $n \geq 7$, *then equation* (8) *has only finitely many integer solutions* q, x *with* $q \geq 5$, *apart from the following cases:*
 (a) $g(x) = \hat{s}_n(h(x))$, *where* $h \in \mathbb{Q}[x]$ *with* $\deg h \geq 1$,
 (b) n *is even and* $g(x) = T_n(\widetilde{g}(x))$, *where* $\widetilde{g}$ *is a polynomial over* $\mathbb{Q}$ *whose square-free part has at most two zeroes.*

Moreover, in both cases, there are infinitely many choices of n *and* g *such that equation* (8) *has infinitely many solutions in integers* q, x *with* $q \geq 5$.

Note that we need to exclude the cases $n \leq 4$ and $n = 6$ in part (i) of the statement, see the remark after Theorem 2.2. Also, the assumption $n \geq 7$ in part (ii) is necessary. (See the Remark after the proof of this statement.)

In the proofs of Theorems 2.1 and 2.2, as well as of part (i) of Theorem 2.3 we shall use effective tools, so these results are effective. On the other hand, in the proof of part (ii) of Theorem 2.3 we apply the ineffective finiteness criterion of Bilu and Tichy [9], that is why this statement is ineffective. We mention that as a key ingredient of our proofs, we shall establish a strong connection between the polynomials $\hat{s}_n(q) \in \mathbb{Q}[q]$ (for $n \in \mathbb{N}$ fixed) and the Chebyshev polynomials of the second kind (cf. Theorem 4.2 and Corollary 4.3).

3. Proof of Theorem 2.1

First we recall a deep result of Shorey and Stewart [33] on non-degenerate recurrence sequences. Let $r_1, r_2, \ldots, r_k$ and $u_0, u_1, \ldots, u_{k-1}$ be integers with $r_k \neq 0$. Put

$$u_n = r_1 u_{n-1} + \ldots + r_k u_{n-k}, \quad \text{for } n = k,\ k+1, \ldots \ .$$

Let $\alpha_1, \ldots, \alpha_t$ be the distinct roots of the characteristic polynomial $F(x) = x^k - r_1 x^{k-1} - \ldots - r_k$ of the recurrence sequence u_n. If α_1 has multiplicity one, then for $n \geq 0$ we have

$$u_n = a_1 \alpha_1^n + P_2(n) \alpha_2^n + \ldots + P_t(n) \alpha_t^n, \tag{9}$$

where $P_i(n)$ is a polynomial with degree less than the multiplicity of α_i in $F(x)$ $(i = 1, \ldots, t)$, and where a_1 and the coefficients of $P_2(n), \ldots, P_t(n)$ are elements of the field $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$.

Theorem B. *Let d be a non-zero integer and let u_n satisfy* (9)*. If $|\alpha_1| > |\alpha_j|$ for $j = 2, \ldots, t$, a_1 and $u_n - a_1\alpha_1^n$ are non-zero and*

$$u_n = dx^\ell$$

for some integers x, ℓ with $x, \ell > 1$, then $\ell < C$, where C is an effectively computable number depending only on d, $\alpha_1, \ldots, \alpha_k$, a_1 and the coefficients and degrees of $P_2, \ldots, P_t$.

Proof. This is Theorem 3 in [33]. □

Proof of Theorem 2.1. A simple argument (by induction) shows that for every $n \geq 1$ we have $\hat{s}_n(q) \geq 2^n$. Hence without loss of generality we may assume that $w > 1$.

Let $q \geq 5$ be a fixed integer. Then, by Theorem A, the characteristic polynomial of $\hat{s}_n(q)$ is

$$F(x) = x^3 - qx^2 + (q+1)x - 2,$$

which has the roots

$$1, \quad \hat{\alpha}_q = \frac{q-1+\sqrt{q^2-2q-7}}{2}, \quad \hat{\beta}_q = \frac{q-1-\sqrt{q^2-2q-7}}{2}$$

from (4). Clearly, these roots are simple, and we have $\hat{\alpha}_q > 1 > \hat{\beta}_q > 0$. Note that now relation (9) is given by (3) (writing 2 as $2 \cdot 1^n$). An easy calculation gives that, for $q \geq 5$, neither of

$$a_1 = \frac{q-1-\sqrt{q^2-2q-7}}{2\sqrt{q^2-2q-7}}$$

and $\hat{s}_n(q) - a_1\hat{\alpha}_q^n$ is equal to zero. Thus Theorem B implies that $\ell < C(q)$ in (6), with an effectively computable constant $C(q)$ depending only on q. In view of equation (6), we then have

$$\hat{s}_n(q) < w^{C(q)},$$

whence our statement immediately follows. □

4. Proofs of Theorem 2.2 and Theorem 2.3 (*i*)

As we will see, the proofs of our effective theorems are based on knowledge about the root structure of the polynomials $\hat{s}_n(q)$ and on the observation that these polynomials are "almost" Chebyshev polynomials of the second kind.

We recall that the *Chebyshev polynomial $U_n(x)$ of the second kind* are defined by

$$U_n(x) = \sum_{m=0}^{[n/2]} (-1)^m \frac{(n-m)!}{m!(n-2m)!}(2x)^{n-2m} \quad (n \geq 0). \tag{10}$$

In the next lemma we summarize some well-known properties of Chebyshev polynomials of the second kind.

Lemma 4.1. *Let n be a nonnegative integer.*

(i) The polynomials $U_n(x)$ satisfy the binary recurrence relation

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, \dots \tag{11}$$

with initial terms $U_0(x) = 1$ and $U_1(x) = 2x$.

(ii) The zeros of the polynomial $U_n(x)$ occur at

$$x = \cos \frac{(n-k+1)\pi}{n+1} \quad (k = 1, 2, \dots, n).$$

(iii) The extreme values of the polynomial $U_n(x)$ have magnitudes which increase monotonically as $|x|$ increases away from 0.

(iv) $U_n(-x) = (-1)^n U_n(x)$, i.e., $U_n(x)$ is an odd function if n is odd, and even if n is even.

Proof. See, e.g., Section 1.2.2. of [24] and Chapter 22. in [1]. □

Observe that by Theorem A, the sequence $\{\hat{s}_n(q) - 2\}_{n\in\mathbb{N}}$ is a binary homogeneous recurrence sequence with characteristic polynomial

$$F_q(x) = x^2 + (1-q)x + 2$$

(since $\hat{\alpha}_q + \hat{\beta}_q = q - 1$ and $\hat{\alpha}_q\hat{\beta}_q = 2$).

Let $t := q - 1$ and put $S_n(t) := \dfrac{\hat{s}_{n+1}(t+1) - 2}{2}$. Then clearly $\{S_n(t)\}_{n\in\mathbb{Z}_{\geq 0}}$ has characteristic polynomial $F_t(x) = x^2 - tx + 2$, and we have

$$S_{n+2}(t) = tS_{n+1}(t) - 2S_n(t), \tag{12}$$

and

$$S_n(t) = \frac{1}{\sqrt{t^2-8}}\hat{\alpha}_t^n - \frac{1}{\sqrt{t^2-8}}\hat{\beta}_t^n,$$

where

$$\hat{\alpha}_t = \frac{t + \sqrt{t^2-8}}{2} \quad \text{and} \quad \hat{\beta}_t = \frac{t - \sqrt{t^2-8}}{2}.$$

The first few elements of the sequence $\{S_n(t)\}_{n\in\mathbb{Z}_{\geq 0}}$ are

$$0;\ \ 1;\ \ t;\ \ t^2-2;\ \ t^3-4t;\ \ t^4-6t^2+4;\ \ t^5-8t^3+12t;$$

$$t^6-10t^4+24t^2-8;\ t^7-12t^5+40t^3-32t;\ t^8-14t^6+60t^4-80t^2+16;\dots.$$

In Theorem 4.2 below we show that there is a correspondence between the polynomials $S_n(t)$ and the Chebyshev polynomials of the second kind.

Theorem 4.2. *For every non-negative integer n, we have*

$$S_{n+1}(t) = \sqrt{2}^n \cdot U_n\left(\frac{t}{2\sqrt{2}}\right), \tag{13}$$

where U_n is the n-th Chebyshev polynomial of the second kind.

Proof. We prove by induction on n. Obviously, (13) holds for $n = 0, 1$ since

$$S_1(t) = 1 = \sqrt{2}^0 \cdot U_0\left(\frac{t}{2\sqrt{2}}\right)$$

and

$$S_2(t) = t = \sqrt{2} \cdot U_1\left(\frac{t}{2\sqrt{2}}\right),$$

respectively. We assume that (13) holds for some $n = k$ with $k \geq 1$. Then, by the recurrence relations (12) and (11), for $n = k+1$ we have

$$\begin{aligned} S_{k+2}(t) &= tS_{k+1}(t) - 2S_k(t) = \\ &= t\left(\sqrt{2}^k \cdot U_k\left(\frac{t}{2\sqrt{2}}\right)\right) - 2\left(\sqrt{2}^{k-1} \cdot U_{k-1}\left(\frac{t}{2\sqrt{2}}\right)\right) = \\ &= \sqrt{2}^{k+1}\left(2\frac{t}{2\sqrt{2}}U_k\left(\frac{t}{2\sqrt{2}}\right) - U_{k-1}\left(\frac{t}{2\sqrt{2}}\right)\right) = \\ &= \sqrt{2}^{k+1} \cdot U_{k+1}\left(\frac{t}{2\sqrt{2}}\right), \end{aligned}$$

which completes the proof. □

As an immediate consequence of the above theorem, we obtain

Corollary 4.3. *For $n \geq 2$ we have*

$$\hat{s}_n(q) = \sqrt{2}^n U_{n-2}\left(\frac{q-1}{2\sqrt{2}}\right) + 2, \tag{14}$$

where U_{n-2} is the $(n-2)$-th Chebyshev polynomial of the second kind. In particular, for n even, $\hat{s}_n(q+1)$ is an even polynomial.

In the following lemma we describe the root structure of the derivative $\hat{s}'_n(q)$ of the polynomial $\hat{s}_n(q)$. Note that for $n \leq 3$ the polynomials $\hat{s}'_n(q)$ have no roots.

Lemma 4.4. *For $n \geq 4$, all the roots of the polynomial $\hat{s}'_n(q)$ are real and simple.*

Proof. Let $n \geq 4$. Then by Corollary 4.3 we have

$$\hat{s}_n(q) - 2 = \sqrt{2}^n U_{n-2}\left(\frac{q-1}{2\sqrt{2}}\right). \tag{15}$$

Clearly, the number of roots as well as the multiplicities of the roots of a polynomial remain unchanged if we replace its variable by a linear polynomial of that. Thus we infer from part (ii) of Lemma 4.1 and (15) that all the roots of the polynomial $\hat{s}_n(q) - 2$ are real and simple. Hence, our statement follows from Rolle's theorem. □

The next lemma will be crucial in the proof of part (i) of Theorem 2.3.

Lemma 4.5. *For $n \geq 1$, the polynomial $\hat{s}_n(q)$ has at most two equal extrema.*

Proof. The statement can be easily checked for $n \leq 6$. Let $n \geq 7$. Then, by Corollary 4.3 and part (iv) of Lemma 4.1, it follows that the graph of $|\hat{s}_n(q)|$ is symmetric about the straight line $q = 1$. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_{n-3}$ denote the roots of $\hat{s}'_n(q)$. Clearly, $\hat{s}_n(q)$ has a local extremum at each α_i. By the symmetry observed above, we have

$$|\hat{s}_n(\alpha_j)| = |\hat{s}_n(\alpha_{n-2-j})| \quad (j = 1, \dots, n-3). \tag{16}$$

Note that $\alpha_{(n-2)/2} = 1$ if n is even, while, if n is odd, we have

$$\alpha_1 < \cdots < \alpha_{(n-3)/2} < 1 < \alpha_{(n-3)/2+1} < \cdots < \alpha_{n-3}. \tag{17}$$

In both cases, there are $\lfloor (n-3)/2 \rfloor$ pairs of coinciding extreme values of $|\hat{s}_n(q)|$.

From Corollary 4.3 and part (iii) of Lemma 4.1 we deduce that

$$|\hat{s}_n(\alpha_{\lfloor (n-3)/2 \rfloor + 1})| < |\hat{s}_n(\alpha_{\lfloor (n-3)/2 \rfloor + 2})| < \cdots < |\hat{s}_n(\alpha_{n-3})|, \tag{18}$$

whence our statement follows. □

Let $f(x) \in \mathbb{Z}[x]$ be a nonzero polynomial of degree d. Write H for the height (i.e. the maximum of the absolute values of the coefficients) of f. Further, let a be a nonzero integer. Consider the Diophantine equation

$$f(x) = ay^\ell. \tag{19}$$

The next lemma is a special case of a result of Bérczes, Brindza and Hajdu [5]. For the first results of this type, we refer to [32] and [37].

Lemma 4.6. *If $f(x)$ has at least two distinct roots and $|y| > 1$, then, in* (19)*, we have $\ell < C_4(d, H, a)$, where $C_4(d, H, a)$ is an effectively computable constant depending only on d, H and a.*

For a finite set of rational primes S, let $\mathbb{Z}_S$ denote the set of rational numbers whose denominator (in reduced form) has no prime divisor outside S. By the *height* $h(a)$ of a rational number $a = u/v$ with $u, v \in \mathbb{Z}$, $\gcd(u, v) = 1$, we mean $h(a) = \max\{|u|, |v|\}$.

The following result is a special case of a theorem of Brindza [11].

Lemma 4.7. *Let S be a finite set of primes. If, in* (19)*, either $\ell = 2$ and $f(x)$ has at least three roots of odd multiplicities, or $\ell \geq 3$ and $f(x)$ has at least two roots of multiplicities coprime to n, then for each solutions $x, y \in \mathbb{Z}_S$ of* (19) *we have $\max(h(x), h(y)) < C_5(d, H, a, \ell)$, where $C_5(d, H, a, \ell)$ is an effectively computable constant depending only on d, H, a and ℓ.*

Now we are ready to prove our effective results.

Proof of Theorem 2.3 (i). Let $n \geq 5$ be an integer, and let $g(x) \in \mathbb{Q}[x]$ be a polynomial with $\deg g = 2$. Assume that equation (8) holds. Then there exist rational numbers a, b, c with $a \neq 0$ such that

$$\hat{s}_n(q) = ax^2 + bx + c. \tag{20}$$

Obviously, we can rewrite (20) as

$$\hat{s}_n(q) + v = a(x + u)^2, \tag{21}$$

where $u = \frac{b}{2a}$ and $v = \frac{b^2-4ac}{4a}$. Thus, in view of Lemma 4.7, it is sufficient to show that the polynomial $\hat{s}_n(q) + v$ has at least three roots of odd multiplicity. Assuming the contrary, we can write

$$\hat{s}_n(q) + v = (Aq^2 + Bq + C)(w(q))^2, \tag{22}$$

with some $A, B, C \in \mathbb{Q}$, $w(q) \in \mathbb{Q}[q]$. Taking derivatives in relation (22), we obtain

$$\hat{s}_n'(q) = w(q)\left((2Aq + B)w(q) + 2(Aq^2 + Bq + C)w'(q)\right). \tag{23}$$

Hence, every root of $w(q)$ is also a root of $\hat{s}_n'(q)$. Denote the roots of $w(q)$ by q_i. For each root q_i, by (22), we have

$$\hat{s}_n(q_i) = -v. \tag{24}$$

Moreover, the numbers q_i are stationary points of the polynomial $\hat{s}_n(q)$. Thus, by Lemma 4.4, we get that $\hat{s}_n(q)$ has $\deg w$ equal extrema, or equivalently, $\hat{s}_n(q) - 2$ has $\deg w$ equal extrema. Note that $\deg w$ depends on the choice of A, B, C and on the parity of n.

If n is odd (i.e., $\deg \hat{s}_n(q) = n - 2$ is also odd), then (22) implies that $A = 0$, $B \neq 0$ and $\deg w = \frac{n-3}{2}$. Thus, in view of Lemma 4.5, we get a contradiction if $n \geq 9$.

If n is even, then, again by (22), we have either $\deg w = \frac{n-4}{2}$ (when $A \neq 0$) or $\deg w = \frac{n-2}{2}$ (when $A = B = 0$). Again, we get a contradiction with Lemma 4.5 if $n \geq 10$.

For $n = 5, 7, 8$, an easy computation in e.g. Maple gives the discriminant of $\hat{s}_n(q) + v$, as a polynomial in v. (Recall that $n = 6$ is excluded.)

If $n = 5, 7$, this polynomial has no rational roots and thus in these cases $\hat{s}_n(q)+v$ cannot be of the form shown in (22). For $n = 8$, the discriminant of $\hat{s}_8(q)+v$ has one rational root, namely, $v = 14$. However, the polynomial

$$\hat{s}_8(q) + 14 = 2(q-3)(q+1)(q^2-2q-5)(q-1)^2 \qquad (25)$$

has four simple roots which, by Lemma 4.7, implies our statement in this case. This completes the proof. □

Proof of Theorem 2.2. To prove that ℓ is bounded, by Lemma 4.6 it is sufficient to show that $\hat{s}_n(q) - b$ has two distinct roots for every $b \in \mathbb{Q}$. Assume to the contrary that it is not the case. Then for some $b \in \mathbb{Q}$ we have

$$\hat{s}_n(q) - b = r(uq+v)^{n-2}$$

with some $r, u, v \in \mathbb{Q}$. However, as $n \geq 5$, this implies that $\hat{s}'_n(q)$ has a multiple root, which contradicts Lemma 4.4. Hence ℓ is bounded as required.

Now we show that q and $|x|$ can also be bounded. For this, by what we have proved already, we may assume that ℓ is fixed. Further, in view of part (i) of Theorem 2.3, we may suppose that ℓ is odd. Clearly, without loss of generality we may assume that in fact ℓ is an odd prime. Rewrite (7) as

$$\hat{s}_n(q) - b = ax^{\ell}. \qquad (26)$$

By Lemma 4.7, it suffices to show that the polynomial on the left hand side of (26) has at least two roots of multiplicities coprime to ℓ. Suppose to the contrary that we have

$$\hat{s}_n(q) - b = (Aq+B)(w(q))^{\ell}, \qquad (27)$$

with some $A, B \in \mathbb{Q}$, $w(q) \in \mathbb{Q}[q]$. Taking derivatives in (27), we obtain

$$\hat{s}'_n(q) = w(q)^{\ell-1}\left(Aw(q) + \ell(Aq+B)w'(q)\right).$$

Thus, every root of $w(q)$ is a root of $\hat{s}'_n(q)$ of multiplicity at least $\ell - 1$, which contradicts Lemma 4.4. Hence q and $|x|$ are also bounded as required, and the theorem follows. □

5. Proof of Theorem 2.3 *(ii)*

In what follows we recall the finiteness criterion of Bilu and Tichy [9]. To do this, we need to define five kinds of so-called standard pairs of polynomials.

kind	standard pair	parameter restrictions
first	$(x^d, \alpha x^r v(x)^d)$	$0 \le r < d, (r,d) = 1,$ $r + \deg v > 0$
second	$(x^2, (\alpha x^2 + \beta) v(x)^2)$	-
third	$(D_\mu(x, \alpha^\nu), D_\nu(x, \alpha^\mu))$	$(\mu, \nu) = 1$
fourth	$(\alpha^{\frac{-\mu}{2}} D_\mu(x, \alpha), -\beta^{\frac{-\nu}{2}} D_\nu(x, \beta))$	$(\mu, \nu) = 2$
fifth	$((\alpha x^2 - 1)^3, 3x^4 - 4x^3)$	-

TABLE 1. Standard pairs

Let α, β, δ be nonzero rational numbers, $\mu, \nu, d > 0$ and $r \ge 0$ be integers, and let $v(x) \in \mathbb{Q}[x]$ be a nonzero polynomial (which may be constant). Denote by $D_\mu(x, \delta)$ the μ-th Dickson polynomial, given by

$$(28)\quad D_\mu(x,\delta) = \sum_{i=0}^{\lfloor \mu/2 \rfloor} d_{\mu,i} x^{\mu-2i} \qquad \text{with} \qquad d_{\mu,i} = \frac{\mu}{\mu - i}\binom{\mu - i}{i}(-\delta)^i.$$

For properties of Dickson polynomials, we refer to [22].

Two polynomials $F(x)$ and $G(x)$ are said to form a *standard pair over* $\mathbb{Q}$ if one of the ordered pairs $(F(x), G(x))$ or $(G(x), F(x))$ belongs to the list in Table 1.

Now we state the main result of [9], which will be a key ingredient in the proof of part (ii) of Theorem 2.3.

Theorem C. *Let $f(x), g(x) \in \mathbb{Q}[x]$ be nonconstant polynomials. Then the following two assertions are equivalent.*

(i) *The equation $f(x) = g(y)$ has infinitely many rational solutions x, y with a bounded denominator.*

(ii) *We have $f = \varphi \circ F \circ \lambda$ and $g = \varphi \circ G \circ \kappa$, where $\lambda(x), \kappa(x) \in \mathbb{Q}[x]$ are linear polynomials, $\varphi(x) \in \mathbb{Q}[x]$, and $F(x), G(x)$ form a standard pair over $\mathbb{Q}$ such that the equation $F(x) = G(y)$ has infinitely many rational solutions x, y with a bounded denominator.*

In [34], Stoll introduced Dickson-type polynomials in the following way. Polynomials $f_n \in \mathbb{R}[x]$ (resp. $\mathbb{Q}[x]$) with

$$(29)\quad f_0(x) = B, \quad f_1(x) = x, \quad f_{n+1}(x) = x f_n(x) - A f_{n-1}(x), \; n \ge 1,$$

where $A, B \in \mathbb{R}$ (resp. $\mathbb{Q}$) are called *Dickson-type recursive polynomials over* $\mathbb{R}$ (resp. $\mathbb{Q}$). By means of the ineffective finiteness criterion of Bilu and Tichy [9], he proved for Dickson-type recursive polynomials f_n over $\mathbb{Q}$ with $A \neq 0, B \neq 2, n \ge 3$ and for $g \in \mathbb{Q}[x]$ with $\deg g \ge 3$, that apart from eight special cases the Diophantine equation $f_n(x) = g(y)$ has only finitely many solutions x, y in rational numbers with a bounded denominator.

One can easily see from (11) that the polynomials $U_n(x/2)$ satisfy (29) with $A = 1$ and $B = 1$. Although by Corollary 4.3 the polynomial family $\{\hat{s}_n(q) - 2\}$ is very similar to $\{U_n(q/2)\}$, it is easy to check that it does not satisfy (29) over $\mathbb{Q}$. Hence we cannot apply the above mentioned finiteness result of Stoll [34] in the proof of part (ii) of our Theorem 2.3.

By a *decomposition* of a polynomial $F(x)$ over a field $\mathbb{K}$ we mean writing $F(x)$ as

$$F(x) = G_1(G_2(x)) \quad (G_1(x), G_2(x) \in \mathbb{K}[x]),$$

which is *nontrivial* if

$$\deg G_1 > 1 \quad \text{and} \quad \deg G_2 > 1.$$

Two decompositions $F(x) = G_1(G_2(x))$ and $F(x) = H_1(H_2(x))$ are said to be *equivalent* if there exists a linear polynomial $\ell(x) \in \mathbb{K}[x]$ such that $G_1(x) = H_1(\ell(x))$ and $H_2(x) = \ell(G_2(x))$. The polynomial $F(x)$ is called *decomposable* over $\mathbb{K}$ if it has at least one nontrivial decomposition over $\mathbb{K}$; otherwise it is said to be *indecomposable*.

The decomposability of polynomial families satisfying (29) and related Diophantine equations were considered by Dujella and Tichy [14] for $B = 1$ and $A \in \mathbb{Z}$. Dujella and Gusić [13] described the decomposability of Dickson-type recursive polynomials over $\mathbb{Q}$ in the general case. Stoll [34] proved the following refinement of their result.

Lemma 5.1. *The Dickson-type polynomials f_n over $\mathbb{R}$ defined in* (29) *with $A \neq 0,\ B \neq 2$ are decomposable over $\mathbb{C}$ if and only if $n = 2k$ with $k \geq 2$. In that case,*

$$f_n(x) = \mathfrak{h}_k(x^2) \quad \textit{with } \mathfrak{h}_k(x) = f_{2k}(\sqrt{x}) \in \mathbb{Q}[x], \tag{30}$$

and $\mathfrak{h}_k$ is decomposable over $\mathbb{C}$ if and only if $B = -2$, $n = 8$ when

$$f_8(x) = \mathfrak{h}_4(x^2) \quad \textit{with } \mathfrak{h}_4(x) = (x^2 - 2Ax)^2 - 4A^2(x^2 - 2Ax) - 2A^4. \tag{31}$$

Moreover, all nontrivial decompositions of f_n are equivalent to (30) *and* (31)*.*

Proof. This is Theorem 5 of [34]. □

Proof of Theorem 2.3 (ii). Let $n \geq 7$ and $g(x) \in \mathbb{Q}[x]$ be a polynomial with $\deg g \geq 3$. Suppose that equation (8) has infinitely many solutions in integers q, x. Then Theorem C implies that there exist $\lambda(x), \kappa(x), \varphi(x) \in \mathbb{Q}[x]$ with $\deg \lambda = \deg \kappa = 1$ such that

$$\hat{s}_n(x) = \varphi(F(\lambda(x))) \quad \text{and} \quad g(x) = \varphi(G(\kappa(x))), \tag{32}$$

where $F(x), G(x)$ form a standard pair over $\mathbb{Q}$.

As it was mentioned above, the transformed second kind Chebyshev polynomial $U_n(x/2)$ is a Dickson-type recursive polynomial, i.e., satisfies the recurrence (29). By Corollary 4.3, we infer that every nontrivial decomposition of $U_n(x/2)$ is equivalent to a nontrivial decomposition of $\hat{s}_n(q)$. Thus, Lemma 5.1 implies that, in (32), we have

$$\deg \varphi \in \left\{1, \frac{n-2}{2}, n-2\right\}.$$

First, suppose that $\deg \varphi = n-2$. Then, by (32), we observe that $\deg F = 1$. Thus $\hat{s}_n(x) = \varphi(t(x))$, where $t(x) = F(\lambda(x)) \in \mathbb{Q}[x]$ is a linear polynomial. Clearly, $t^{-1}(x) \in \mathbb{Q}[x]$ is also linear. By (32), we obtain $\hat{s}_n(t^{-1}(x)) = \varphi(t(t^{-1}(x))) = \varphi(x)$. Hence,

$$g(x) = \varphi(G(\kappa(x))) = \hat{s}_n(t^{-1}(G(\kappa(x)))) = \hat{s}_n(h(x)),$$

where $h(x) = t^{-1}(G(\kappa(x)))$. So, if in this case equation (8) has infinitely many solutions, then $g(x)$ is of the form $\hat{s}_n(h(x))$, where $h \in \mathbb{Q}[x]$ with $\deg h \geq 1$. It is obvious that by these choices, equation (8) has infinitely many solutions in integers q, x with $q \geq 5$ for any n.

Next, suppose that $\deg \varphi = \dfrac{n-2}{2}$. Then n is even, and we obtain from Lemma 5.1 and Corollary 4.3 that $\hat{s}_n(x) = T_n((x-1)^2)$, and up to equivalence, this is the only decomposition of $\hat{s}_n(x)$. Thus we get that $\varphi(x) = T_n(\sigma(x))$, where $\sigma(x)$ is a linear polynomial. Hence $g(x)$ is of the form $g(x) = T_n(\widetilde{g}(x))$ with some $\widetilde{g}(x) \in \mathbb{Q}[x]$. Further, theorem C implies that the equation $(x-1)^2 = \widetilde{g}(y)$ has infinitely many rational solutions with a bounded denominator. Then, by Lemma 4.7 the polynomial $\widetilde{g}(x)$ can have at most two roots of odd multiplicities. Thus we get case (b) of our statement. Moreover, it is clear that for any even n, there are infinitely many choices for g such that equation (8) has infinitely many solutions in integers q, x with $q \geq 5$.

Finally, suppose that $\deg \varphi = 1$. Then there exist $\varphi_1, \varphi_0 \in \mathbb{Q}$ with $\varphi_1 \neq 0$ such that $\varphi(x) = \varphi_1 x + \varphi_0$. We study now the five kinds of standard pairs. In view of our assumptions on n and $\deg g$, it follows that $F(x)$, $G(x)$ cannot form a standard pair of the second or fifth kind.

Suppose that $F(x)$, $G(x)$ form a standard pair of the first kind over $\mathbb{Q}$. Then we have either

(a) $\hat{s}_n(x) = \varphi_1 \lambda(x)^d + \varphi_0$, or
(b) $\hat{s}_n(x) = \varphi_1 \alpha \lambda(x)^r v(\lambda(x))^d + \varphi_0$, where $0 \leq r < d$, $(r, d) = 1$ and $r + \deg v(x) > 0$.

In case (a), we have $q = n-2$. Taking derivatives we obtain $\hat{s}_n'(x) = d\varphi_1\lambda(x)^{d-1}$. Since, by Lemma 4.4, all the roots of $\hat{s}_n'(x)$ are simple, we get $d \leq 2$, i.e., $n \leq 4$ which contradicts our assumption $n \geq 7$.

In the second case (b), we have $g(x) = \varphi_1\kappa(x)^d + \varphi_0$. Since $\deg g \geq 3$, we have $d \geq 3$. Put $\lambda(x) = \lambda_1 x + \lambda_0$. Taking derivatives in relation (b), we obtain

$$\hat{s}_n'(x) =$$
$$\varphi_1\alpha\lambda_1(\lambda_1x+\lambda_0)^{r-1}v(\lambda_1x+\lambda_0)^{d-1}(rv(\lambda_1x+\lambda_0)+d(\lambda_1x+\lambda_0)v'(\lambda_1x+\lambda_0)),$$

which implies that the polynomial $\hat{s}_n'(x)$ has a root of multiplicity at least $d-1 \geq 2$. This is impossible by Lemma 4.4.

Finally, consider the case when, in (32), $F(x), G(x)$ form a standard pair of the third or fourth kind over $\mathbb{Q}$. Then, putting again $\lambda(x) = \lambda_1x+\lambda_0$ ($\lambda_1 \neq 0$), whence $\lambda^{-1}(x) = c_1x+c_0$ with $c_1 = \frac{1}{\lambda_1}$ and $c_0 = -\frac{\lambda_0}{\lambda_1}$, we have

$$\hat{s}_n(c_1x + c_0) = e_1 D_\nu(x, \delta) + e_0 \tag{33}$$

with some $e_0 \in \mathbb{Q}, e_1, \delta \in \mathbb{Q} \setminus \{0\}$, where $D_\nu(x, \delta)$ is the ν-th Dickson polynomial. Clearly, we have $\nu = n-2 \geq 5$.

Moreover, by Corollary 4.3 and relation (10) we get that the coefficients of x^{n-3} in $\hat{s}_n(c_1x + c_0)$ is given by

$$2(n-2)c_1^{n-3}(c_0-1),$$

while by (28) the coefficient of x^{n-3} in $e_1D_\nu(x, \delta) + e_0$ is zero. This gives

$$c_0 = 1. \tag{34}$$

Thus using Corollary 4.3 and relation (10) again, we obtain

$$\hat{s}_n(c_1x+c_0) = 2c_1^{n-2}x^{n-2}-4(n-3)c_1^{n-4}x^{n-4}+4(n-4)(n-5)c_1^{n-6}x^{n-6}+\dots . \tag{35}$$

Thus in view of (28) and (35), we infer, by comparing the leading coefficients on both sides of (33), that

$$2c_1^{n-2} = e_1. \tag{36}$$

Comparing now the coefficients of x^{n-4} in (33), we obtain, by (28) and (35), that

$$-4(n-3)c_1^{n-4} = -e_1\nu\delta, \tag{37}$$

which, by substituting (36) together with $\nu = n-2$ into (37), implies that

$$c_1^2 = \frac{2(n-3)}{(n-2)\delta}. \tag{38}$$

Recall that $n \geq 7$. Then, again by (28) and (35), the equality the coefficients of x^{n-6} on both sides of (33) gives

$$4(n-4)(n-5)c_1^{n-6} = \frac{e_1(\nu-3)\nu\delta^2}{2},$$

whence, in view of (36) and $\nu = n-2$, it follows that

$$c_1^4 = \frac{4(n-4)}{(n-2)\delta^2}. \tag{39}$$

After substituting (38) into (39), we obtain $(n-3)^2 = (n-2)(n-4)$, which is a contradiction. Hence our theorem follows. $\square$

Remark. We note that the assumption $n \geq 7$ in part (ii) of Theorem 2.3 is necessary. As $\deg(\hat{s}_n(q)) = n-2$, it is obvious for $n = 3$. (The cases $n = 1, 2$ are trivial.) When $n = 4$, we have

$$\hat{s}_4(q) + 2 = 2(q-1)^2.$$

Thus obviously, equation (8) has infinitely many solution for $n = 4$ with $g(y) = 2y^\ell - 2$, for any $\ell \geq 2$. In the case $n = 5$, using the notation of the above proof it is clear that in (32), $F(x)$ and $G(x)$ can only be a standard pair of the third or fourth kind. From (33) we infer that $\nu = 3$ thus the standard pair $F(x), G(x)$ cannot be of the fourth kind. We further infer from (33) that

$$\lambda_1 = \frac{1}{c_1},\ \lambda_0 = -\frac{1}{c_1},\ e_1 = \varphi_1 = 2c_1^3,\ e_0 = \varphi_0 = 2 \ \text{ and } \delta = \alpha^\mu = \frac{4}{3c_1^2}$$

with an arbitrary nonzero rational number c_1. Clearly, there are infinitely many choices for c_1 such that $g(x) = \varphi(G(\kappa(x)))$ where $G(x) = D_\mu(x, \alpha^3)$ with $(\mu, 3) = 1$ whence, by Theorem C, equation (8) has infinitely many rational solutions with a bounded denominator. Finally, when $n = 6$, as we already mentioned, we have

$$\hat{s}_6(q) + 8 = 2(q^2 - 2q - 2)^2.$$

Thus, as one can easily check, for $n = 6$ equation (8) has infinitely many solutions e.g. with $g(y) = 8y^4 - 8$.

Acknowledgements

The research of the authors was supported in part by the Hungarian Academy of Sciences, by the NKFIH grants K115479 and K128088 and by the projects EFOP-3.6.1-16-2016-00022 and EFOP-3.6.2-16-2017-00015, co-financed by the European Union and the European Social Fund.

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