ON THE EQUATION $x(x+d)\ldots(x+(k-1)d)=by^2$

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ABSTRACT. In this paper we give a new bound for the solutions x of the title equation, provided that $k \geq 8$. This bound is polynomial in d. Moreover, under the same condition, a similar bound for the number of solutions in (x, k, y, l) is given.

1. INTRODUCTION

It is an old problem to describe those finite arithmetic progressions, for which the product of the terms yields a perfect power, or an 'almost' perfect power. The first result in this direction is due to Erdős [E1] and Rigge [R]. They independently proved that the equation

$$\text{(I)} \qquad x(x+1)\ldots(x+k-1) = y^l$$

for $l = 2$ has no solutions with $k \geq 2$ and $x \geq 1$, that is the product of two or more consecutive positive integers is never a perfect square. Later, Erdős and Selfridge in 1975 (cf. [ES]) obtained a deep generalization of this, namely that equation (I) for $l \geq 2$ has no solutions with $k \geq 2$ and $x \geq 1$, or in other words, the product of two or more consecutive positive integers is never a perfect power.

Another, closely related problem is to determine those binomial coefficients, which are perfect powers. This problem was studied by Erdős (see [E2]). He showed that the equation

$$\text{(II)} \qquad \binom{x+k-1}{k} = y^l$$

in positive integers x, k, y, l with $x \geq k+1$, $l \geq 2$ has no solutions if $k \geq 4$. Győry [Gy1] proved that the only solution of equation (II) with $k \geq 2$, $(l,k) \neq (2,2)$ is $(x,k,y,l) = (48,3,140,2)$. He settled the case $k = 3$ and pointed out that the case $k = 2$ is a consequence of a recent result of Darmon and Merel [DM]. $k = l = 2$ must be excluded because in this case equation (II) clearly has infinitely many solutions.

For a positive integer n denote by $P(n)$ the greatest prime divisor of n ($P(1) = 1$). A common generalization of equations (I) and (II) is

(III) $$x(x+1)\ldots(x+k-1) = by^l \quad \text{with} \quad l, k \geq 2, \ P(b) \leq k.$$

Using a result of Sylvester [Sy], it turns out that if $P(y) \leq k$ for a given k, then (III) has only 'small' solutions, and these solutions can be easily determined (cf. [Gy2]). On the other hand, if $P(y) > k$, then apart from the case $k = l = 2$, equation (III) has only the solution $(x, k, b, y, l) = (48, 3, 6, 140, 2)$. This result is due to Győry [Gy2] (case $k \leq 3$) and Saradha [Sa1] (case $k \geq 4$).

For further related results on the previous equations and on the more general equation

(IV) $$x(x+d)\ldots(x+(k-1)d) = by^l \text{ with } l \geq 2, \ k \geq 3, \ (x,d) = 1, \ P(b) \leq k$$

in positive integers x, d, k, b, y, l, we refer to the works of Beukers, Shorey and Tijdeman [BST], Győry [Gy3], Marszalek [M], Saradha [Sa1] and [Sa2], Shorey [Sh1] and [Sh2], Shorey and Nesterenko [SN1] and [SN2], Shorey and Tijdeman [ST1], [ST2] and [ST3], and Tijdeman [T1] and [T2], and the references given there.

In the present paper we deal with the equation

(1) $$x(x+d)\ldots(x+(k-1)d) = by^2 \text{ with } k \geq 3, \ (x,d) = 1, \ P(b) \leq k,$$

where x, d, k, b, y are unknown. If $d = 1$, then suppose that $P(y) > k$ in (1). Among several other inequalities, Shorey and Tijdeman [ST1] proved that if $k \geq C_1$ where C_1 is an effectively computable absolute constant then (1) implies that

$$x + (k-1)d \leq 17d^2k(\log k)^4,$$

see [ST1], Theorem 3. Combining this result with

$$2^{\omega(d)} > C_2 \frac{k}{\log k}$$

(cf. [ST1] Theorem 1.(a)) where C_2 is an effective computable absolute positive constant and $\omega(d)$ is the number of distinct prime divisors of d, one can see that x is bounded by $c_\varepsilon d^{2+\varepsilon}$, where c_ε is a constant depending only on ε, provided that k is large enough.

2. Results

Theorem 1. *If (x, d, k, b, y) is a solution of* (1) *with $k \geq 8$ and $d > 1$, then* $x < 4d^4(\log d)^4$.

Remarks. 1. We may assume that $d \geq 23$, as (1) in case $1 < d \leq 22$ was solved by Saradha [Sa2], and the only solutions are $(x, d, k, b, y) = (2, 7, 3, 2, 12)$, $(18, 7, 3, 1, 120)$ and $(64, 17, 3, 2, 504)$.

2. Erdős conjectured that relation (IV) implies that k is bounded by an absolute constant. In case of $l = 2$, under the further assumption $x \geq 4d^4(\log d)^4$, our Theorem gives an answer to this problem, with a good bound for k. For other

results and references concerning this and related conjectures of Erdős see [Sh1], [Sh2] and [Gy3].

3. Our argument makes it possible to assume only that $P(b) \le ck$ where c is "around" 1.1. The details are not worked out here.

4. Our theorem provides a method for determining all solutions of equation (1), with d fixed. The 'small' solutions can be found e.g. by a simple search, and 'large' solutions may occur only when $k \le 7$. However, in this case equation (1) can be reduced either to simultaneous Pell's equations, or to simple elliptic equations of the form $x'^3 + a'x' + b' = y'^2$, and these equations can be resolved easily. Using this approach, Filakovszky and Hajdu [FH] resolved equation (1) for several values of d.

A combination of Theorem 1 with some recent results of Bennett [B] and Saradha [Sa2] yields

Theorem 2. *Suppose that $d > 1$. Then equation (1) has at most cd^6 solutions in (x, k, b, y), where c is an effectively computable absolute constant.*

3. Auxiliary results

Every factor of the left hand side of (1) can be uniquely written in the form

$$(2) \qquad x + id = a_i x_i^2, \ a_i \text{ is square-free}, \ i = 0, \ldots, k-1.$$

Lemma 1. *For every solution (x, d, k, b, y) of (1) for which the a_i-s in (2) are all different, we have $k \le 7$.*

Proof. First we prove that under the assumptions of the Lemma, $k < 210$ holds.

Let p be a prime with $p \nmid d$, and write

$$A = \prod_{i=0}^{k-1} a_i \text{ and } B = \prod_{p \le k} p^{\lceil k/p \rceil}.$$

If $p > k$, then for every $i = 0, \ldots, k-1$, $p \nmid a_i$. On the other hand, if $p \le k$, then only every p-th number $x + id$ can be divisible by p, thus p has at most $\lceil k/p \rceil$ multiples among them, hence $A | B$. For every $i = 0, \ldots, k-1$ let a_i' be the odd part a_i, and put

$$A' = \prod_{i=0}^{k-1} a_i' \text{ and } B' = \prod_{3 \le p \le k} p^{\lceil k/p \rceil}.$$

Then we get $A' | B'$. Let $1 = h_1 < h_2 < \ldots$ be the sequence of the odd square-free numbers. Since every h_j may occur as a_i' at most twice, we obtain

$$A' \ge h_1 h_2 \ldots h_m h_1 h_2 \ldots h_{m'},$$

where $m = [k/2]$ and $m' = k - m = \lceil k/2 \rceil$.

Set $H(x) = \#\{i : h_i \le x\}$, and let $F(x) = [x] - [x/2]$, that is $F(x)$ is the number of the positive odd integers not greater than x. By a sieve formula we have

$$H(x) = \sum_{2 \nmid t} \mu(t) F(x/t^2) \le \sum_{t | 15} \mu(t) F(x/t^2).$$

Indeed, the right hand side expression enumerates those odd integers not exceeding x, which are free of 3^2 and 5^2. The inequality $\mid F(x) - x/2 \mid\, \leq 1/2$, $(x \geq 0)$ yields

$$\left| \sum_{t|15} \mu(t)F(x/t^2) - (32/75)x \right| \leq 2,$$

whence
$$H(x) \leq (32/75)x + 2.$$

Especially,
$$H(h_j) = j \leq (32/75)h_j + 2,$$

and we get
$$h_j \geq (75/32)(j - 2).$$

Using this estimate for $j \geq 4$ together with $h_1 = 1$, $h_2 = 3$ and $h_3 = 5$, we obtain

$$h_1 \ldots h_m \geq (75/32)^m (m - 2)!.$$

A similar estimate is valid for $h_1 \ldots h_{m'}$. By multiplying these inequalities, we have

$$A' > (75/32)^{m+m'} (m - 2)!(m' - 2)!.$$

Using

$$(m - 2)!(m' - 2)! = (k - 4)! \binom{k - 4}{m - 2}^{-1} > (k - 4)!2^{-(k-4)} > (16/k^4)2^{-k}k!,$$

we conclude
$$A' > (16/k^4)(75/64)^k k!.$$

Now we estimate B'. By Legendre's formula for the prime factorization of $k!$

$$(3) \qquad\qquad \sum_{p \geq 3}[k/p]\log p = \log k! - S$$

where
$$S = \sum_{p^t \in Q}[k/p^t]\log p \quad \text{with} \quad Q = \{2\} \cup \{p^j : j \geq 2\}.$$

Let $Q_0 = Q \cap [1, 500]$. Then

$$(4) \qquad S \geq \sum_{p^t \in Q_0}[k/p^t]\log p \geq k\sum_{p^t \in Q_0}\log p/p^t - \sum_{p^t \in Q_0}\log p \geq \alpha k - \beta,$$

where $\alpha = 1.046874$ and $\beta = 27.8$. The identity

$$\lceil k/p \rceil = 1 + [(k - 1)/p]$$

implies
$$B' \leq \prod_{p \leq k} p \prod p^{[(k-1)/p]}.$$

For every k we have $\prod_{p \leq k} p < e^{\gamma k}$, with $\gamma = 1.001102$ (cf. [RS] Theorem 6.). Using this together with (3) and (4), we obtain

$$B' < e^{\beta - \alpha(k-1) + \gamma k}(k-1)!.$$

This inequality with $A' \leq B'$ leads to

$$(16/k^4)(75/64)^k k! < e^{(\gamma - \alpha)k + \alpha + \beta}(k-1)!,$$

and by a simple computation we obtain $k < 210$.

Now suppose that $9 \leq k \leq 209$, and let $p \nmid d$ be a prime. It is clear that among the numbers $x + id$, $i = 0, \ldots, k - 1$, there are at most

$$r(p) := \sum_{t=1}^{\left[\frac{\log k}{2 \log p}\right]+1} (\lceil k/p^{2t-1}\rceil - [k/p^{2t}]),$$

ones, whose factorizations contain p on an odd exponent. However, after fixing k and calculating the values of $r(p)$, it turns out that it is impossible to construct k different a_i-s from these primes. Indeed, let k be any integer satisfying $9 \leq k \leq 209$, and determine the value of $r(p)$ for every prime $p \nmid d$ not exceeding k. Among the primes involved, choose the smallest one p', for which $r(p') < 2^{\pi(p')-1}$ holds, where $\pi(x)$ denotes the number of positive primes not greater than x. Clearly, we can construct at most

$$2^{\pi(p')-1} + \sum_{p' \leq p \leq k} r(p)$$

different a_i-s from our primes. However, a computation yields that the number of these a_i-s is always less than k in this case, and we obtain $k \leq 8$.

Now suppose that $k = 8$. In this case one can construct eight different a_i-s, however, as it was pointed out by Professor Tijdeman, it is impossible to arrange them into a 'valid' order. This fact can be proved by a simple combinatorial argument. Hence the Lemma is proved.

$\square$

We remark that the bound $k < 210$ obtained in the first part of the proof of the Lemma could be made sharper, but in view of the second part of the proof, it was not necessary.

Lemma 2. *Equation* (1) *with* $d > 1$ *implies that* $k < 4d(\log d)^2$.

Proof. The case $d \geq 23$ is just Theorem 3 in [Sa2]. Furthermore, in [Sa2] all the solutions of (1) with $1 < d \leq 22$ are determined, and the Lemma follows. $\square$

Lemma 3. *Let* a *and* b *be positive nonsquare integers and let* u *and* v *be nonzero integers with* $av \neq bu$. *Then the system of simultaneous Pell-equations*

$$x^2 - az^2 = u, \quad y^2 - bz^2 = v$$

in positive integers (x, y, z) *has at most* $c2^{\min\{\omega(u),\omega(v)\}} \log(|u|+|v|)$ *solutions, where* c *is an effectively computable absolute constant and* $\omega(n)$ *denotes the number of the distinct prime factors of* n.

Proof of Lemma 3. This is the main result in [B]. $\square$

4. Proofs of the Theorems

Proof of Theorem 1. Let (x, d, k, b, y) be any solution of (1) with $k \geq 8$. As mentioned above, we may suppose that $d \geq 23$. Using our Lemma we obtain that there exist $0 \leq j < i \leq k - 1$ such that in (2) $a_i = a_j$. This yields

$$(k-1)d \geq (x + id) - (x + jd) = a_j(x_i^2 - x_j^2) \geq a_j(2x_j + 1),$$

therefore

$$(k-1)^2 d^2/4 > x.$$

The last estimate together with Lemma 2 implies

$$x < 4d^4 (\log d)^4,$$

and the Theorem is proved. $\square$

Proof of Theorem 2. First suppose that in (1) $k \geq 8$ holds. Then by Theorem 1 we have $x < 4d^4 (\log d)^4$. Furthermore, by Lemma 2 we have $k < 4d(\log d)^2$, and in this case Theorem 2 follows.

Now suppose that in (1) $k \leq 7$ holds. Using (2), for $i = 0, 1, 2$ we can write $x = a_0 x_0^2$, $x + d = a_1 x_1^2$ and $x + 2d = a_2 x_2^2$, and we obtain the simultaneous Pell-equations

$$(5) \qquad\qquad a_1 x_1^2 - a_0 x_0^2 = d, \quad a_2 x_2^2 - a_0 x_0^2 = 2d.$$

Clearly, for the coefficients a_i, $i = 0, 1, 2$ we have $a_i | 2 \cdot 3 \cdot 5 \cdot 7$. Hence we have to consider at most 2^{12} simultaneous Pell-equations of the form (5). Thus to prove Theorem 2, it is sufficent to give an appropriate upper bound for the number of solutions of (5). Multiplying the equations of (5) by a_1 and a_2, respectively, and putting $y_0 = x_0$, $y_1 = a_1 x_1$ and $y_2 = a_2 x_2$, we obtain the system of equations

$$(6) \qquad\qquad y_1^2 - a_0 a_1 y_0^2 = a_1 d, \quad y_2^2 - a_0 a_2 y_0^2 = 2a_2 d.$$

We may suppose that $a_0 \neq a_1$ and $a_0 \neq a_2$. Indeed, otherwise just as in the proof of Theorem 1, we obtain $x < 4d^4 (\log d)^4$, which yields a much better bound for the number of solutions of (1) than stated. Now one can check easily that the assumptions of Lemma 3 are fulfilled, and we obtain that the number of solutions in (y_1, y_2, y_3) to (6) is less than $c 2^{\omega(d)}$, where c is an effectively computable absolute constant, and the Theorem is proved. $\square$

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