Error Correction for Discrete Tomography

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Dedicated to the memory of Carla Peri.

Abstract

Discrete tomography focuses on the reconstruction of functions from their line sums in a finite number d of directions. In this paper we consider functions $f : A \to R$ where A is a finite subset of $\mathbb{Z}^2$ and R an integral domain. Several reconstruction methods have been introduced in the literature. Recently Ceko, Pagani and Tijdeman developed a fast method to reconstruct a function with the same line sums as f. Up to here we assumed that the line sums are exact. Some authors have developed methods to recover the function f under suitable conditions by using the redundancy of data. In this paper we investigate the case where a small number of line sums are incorrect as may happen when discrete tomography is applied for data storage or transmission. We show how less than $d/2$ errors can be corrected and that this bound is the best possible. Moreover, we prove that if it is known that the line sums in k given directions are correct, then the line sums in every other direction can be corrected provided that the number of wrong line sums in that direction is less than $k/2$.

Keywords: discrete tomography, error correction, line sums, polynomial-time algorithm, Vandermonde determinant.

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1 Introduction

We consider functions $f : A \to R$ where $A = \{(i, j) \in \mathbb{Z}^2 : 0 \le i < m, 0 \le j < n\}$ for given positive integers m, n and integral domain R, e.g. $\mathbb{Z}, \mathbb{R}$ or a finite field. We assume that f is unknown, but that its line sums in a positive number d of directions are given. The line sums are often referred to as X-rays to highlight the link between discrete tomography and computed tomography scans. This type of discrete tomography problem has been widely studied, see e.g. [1–10]. Discrete tomography originated in the study of crystals which may be damaged when many X-rays are used [11, 12]. In such applications the possible values of the line sums are linear combinations of a finite set of positive real numbers. Later applications of discrete tomography were developed where the co-domain of f can be chosen, such as distributed storage [13, 14], watermarking [15, 16], image compression [17] and erasure coding [18–20].

It makes an essential difference whether the line sums are exact or not. If they are exact then there is at least one function satisfying the line sums, but there may be infinitely many. In 1978 Katz [21] gave a necessary and sufficient condition for the uniqueness of the solution. The structure of possible solutions of a discrete tomography problem has been studied by numerous authors. It turns out that proving the existence of a solution, and in case of existence a subsequent reconstruction, can be very hard if the range of the function on A is restricted to a fixed finite set. In 1991 Fishburn, Lagarias, Reeds and Shepp [11] gave necessary and sufficient conditions for uniqueness of reconstruction of functions $f : A \to \{1, 2, \ldots, N\}$ for some positive integer $N > 1$. See also the thesis of Wiegelmann [22]. In 1999 and 2000 Gardner, Gritzmann and Prangenberg [23, 24] showed under very general conditions that proving the existence or uniqueness of a function $f : A \to \mathbb{N}$ from its line sums in d directions is NP-complete. The crux of the NP-results is that the co-domain is not closed under subtraction.

If the co-domain R is an integral domain, then linear algebra techniques such as Gauss elimination provide polynomial time algorithms. This is useful for the practical reconstruction of f, see e.g. Batenburg and Sijbers [2]. In the present paper we investigate the theoretical structure of solutions and leave such computational techniques aside. If the solution is not unique, then any two solutions differ by a so-called ghost, a nontrivial function $g : A \to \mathbb{R}$ for which all the line sums in the d directions vanish. In 2001 Hajdu and Tijdeman [7] gave an explicit algebraic expression for the ghost of minimal size and showed that every ghost is a linear combination of shifts of it. It implies that arbitrary function values can be given to a certain set of points of A and that thereafter the function values of the other points of A are uniquely determined by the line sums, see Dulio and Pagani [25].

Suppose A is an m by n grid and the line sums of a function $f : A \to R$ in d directions are known. Recently a method was developed to construct a function $g : A \to R$ which has the same line sums as f has in time linear in dmn as to the number of operations such as addition and multiplication. This development started with four papers of Dulio, Frosini and Pagani [4, 26–28] with fast reconstruction results for corner regions of A in case $d = 2$ or 3. Subsequently Pagani and Tijdeman [29] did so for general d. In particular, their approach enables one to reconstruct f, if f is the only function which satisfies the line sums in the d directions. Finally Ceko, Pagani and Tijdeman [30] developed an algorithm to construct a function $g : A \to R$ in time linear in dmn such that g has the same line sums as f. This yields a parameter representation of all the functions $g : A \to R$ which have the same line sums as f. We think it is unlikely that there exists a general reconstruction method which requires essentially less than $\mathcal{O}(dmn)$ operations, if the solution is unique.

A remaining problem is to construct the most likely consistent set of line sums if the measured line sums contain errors. The most common cause of inconsistency of line sums is noise. This happens, for example, if the line sums are approximations of line integrals. Here the assumption is that many line sums may not be exact, but that for each line sum the difference between measured and actual sum is small. Many algorithms have been developed to deal with this situation in case $R = \mathbb{R}$, often approximation methods which work well in practice but do not guarantee optimality. See for example Parts 2 of the books edited by Herman and Kuba [9, 31] and the paper by Batenburg and Sijbers [2].

The consistent set of line sums nearest to the measured line sums in case of $R = \mathbb{R}$ can be constructed as follows. Consider the incidence matrix B of the lines in the d given directions and intersecting A. This is an L by mn matrix where L is the total number of such lines. Then the range of B forms a subspace in $\mathbb{R}^L$. By constructing the orthogonal projection of the vector of the measured line sums onto this subspace, we obtain the consistent line sum vector b_0 nearest to the measured one. Theorem 5.5.1 of [32] provides the standard tool from linear algebra to compute the vector b_0. For more details on this procedure see Theorem 4.1 of [33].

In the present paper we deal with another type of errors, viz. errors which may be arbitrary large, but are small in number. For literature in this direction, see e.g. [13, 14, 18, 19]. Again the properties of the co-domain of f make an essential difference. Alpers and Gritzmann [34] showed that for functions $f : A \to \{0, 1\}$ the Hamming distance between any two solutions with equal cardinality of the lattice sets is $2(d-1)$. They remarked that the problem of determining how the individual measurements should be corrected in order to provide consistency of the data is NP-complete whenever $d \geq 3$. The situation is again totally different and much more favourable if the co-domain of f is an integral domain.

A common approach for error correction in linear systems involves solving an L_1 minimization problem [35, 36]. Candes, Romberg and Tao showed that an object may be recovered exactly from incomplete frequency samples via convex optimization [37]. Chandra, Svalbe, Guédon, Kingston and Normand [18] used redundant image regions to reconstruct the original function f in linear time. They chose m, n and the directions appropriately and assumed that all the line sums in some directions were wrong. This result is comparable with our Theorem 2 which, however, is valid for any m, n and any finite set of directions. In Theorem 1 we show that if $f : A \to R$ and $g : A \to R$ do not have the same line sums, then at least d line sums are different and in case of $< d/2$ wrong line sums reconstruction of f can be done in polynomial time. (Even $< d$ wrong line sums if it is known which line sums are wrong). A simple example shows that here the bounds d and $d/2$, respectively, are the best possible (see Section 3). After having corrected the wrong line sums, we have a consistent set of line sums and the method from [29, 30] can be applied to find the optimal solution in the above described sense.

Theorems 1 and 2 are stated in the next section. Moreover, in Sections 9-11 we give a pseudocode algorithm and an example, and prove that the complexity is $\mathcal{O}(d^4mn)$ operations. Of course, it may happen that the number of errors is much smaller than $d/2$. We introduce F for the total number of errors and G for the maximum of the number of wrong line sums in some direction as parameters so that the amount of computation can be reduced if it is expected that there are far less than $d/2$ errors. Here F and G can be freely chosen such that $G \leq F < d/2$.

In the proofs of the theorems we use the fact that there is redundancy in the information given by the line sums. Hajdu and Tijdeman [38] pursued an analysis of the redundacy by Van Dalen [39]. Their line sum relation lemma (Lemma 9) is the basis of the present paper. Besides, some properties of Vandermonde determinants are derived and used. By the redundacy of data the values of the wrong line sums do not matter. The lines with correct line sums are detected and the right values of the wrong line sums are derived from the correct line sums.

2 The main results

Let d, m, n be positive integers and $A = \{(i, j) \in \mathbb{Z}^2 : 0 \leq i < m, 0 \leq j < n\}$. Let $D = \{(a_h, b_h) : h = 1, 2, \ldots, d\}$ be a set of pairs of coprime integers with $a_h \geq 0$ and $b_h = 1$ if $a_h = 0$. We call the elements of D directions. For $f : A \to \mathbb{R}$ we define the line sum in the direction of (a_h, b_h) by

$$\ell_{h,t} = \sum_{(i,j)\in A,\ b_h i - a_h j = t} f(i, j) \tag{1}$$

for $h = 1, 2, \ldots, d$ and $t \in \mathbb{Z}$. Denote for all h and t by $\ell^*_{h,t}$ the corresponding measured line sum. We call line sums with $\ell^*_{h,t} = \ell_{h,t}$ correct line sums and others wrong line sums. In this paper we suppose that all the line sums in the directions of D are measured and that there are less than $d/2$ wrong line sums and show how to correct them.

Theorem 1. *Let d, m, n be positive integers and let A and D be as defined above. Let $f : A \to \mathbb{R}$ be an unknown function such that for $h = 1, 2, \ldots, d$ the line sums $\ell^*_{h,t}$ in the direction of (a_h, b_h) are measured with in total less than $d/2$ wrong line sums. Then the correct line sums can be determined.*

It is remarkable that the bound depends only on d and is independent of m, n and the directions themselves. The restriction on the entries of the directions serves to choose one of the two directions (a, b) and $(-a, -b)$ which provide the same line sums.

If k directions with only correct line sums are known and there are not too many wrong line sums in some other direction, then these wrong line sums can be corrected:

Theorem 2. *Suppose, in the notation of Theorem 1, all the line sums in directions (a_h, b_h) are known to be correct for $h = 1, 2, \ldots, k$. Then the line sums can be corrected in each direction with less than $k/2$ wrong line sums, and, moreover, in each direction with at most k wrong line sums where it is known which line sums are wrong.*

In Section 3 we show that the bound $d/2$ cannot be improved. Sections 4 and 5 contain results related to Vandermonde determinants. The line sum relation lemma is proved in Section 6. Theorems 2 and 1 are derived in Sections 7 and 8, respectively. In Section 9 a pseudocode is provided, which details the steps of the algorithm to find the correct line sums. An example in Section 10 illustrates the algorithm. Section 11 provides an analysis of the complexity of the algorithm. In the final section we state some conclusions.

3 An example which shows that Theorem 1 cannot be improved

Change the value of f at one element of A. Then exactly d line sums change. Thus d of the pairs of corresponding line sums are different. It follows that the bound $d/2$ in Theorem 1 is the best possible.

Example 1. Consider the function $f : A \to \mathbb{Z}$ with one unknown value indicated by ?.

2	6	5	4
3	?	2	0
5	1	4	2
6	3	1	4

Suppose the measured horizontal line sum through ? is 7, the measured vertical line sum through ? is 12 and both measured diagonal line sums through ? are 13. Then $d = 4$, the horizontal and vertical line sums suggest that the value of ? is 2 whereas the diagonal line sums indicate that it should be 3. Both values result in $2 = d/2$ wrong line sums. □

Obviously this can be generalized for arbitrarily large m, n and d.

4 Vandermonde equations with variable coefficients

Let r be a positive integer. Let $c_0, c_1, \ldots, c_{2r-1}$ and $t_1, t_2, \ldots, t_r$ be given real numbers with $t_1, t_2, \ldots, t_r$ distinct. It is well known and very useful that a system of linear equations

$$\sum_{i=1}^{r} t_i^j x_i = c_j \tag{2}$$

for $j = 0, 1, \ldots, r-1$ in unknowns $x_1, x_2, \ldots, x_r$ has a unique solution which can be found by using a Vandermonde matrix. In this section we show how to solve the system of equations (2) for $j = 0, 1, \ldots, 2r-1$, if both $x_1, x_2, \ldots, x_r$ and $t_1, t_2, \ldots, t_r$ are unknowns.

The method is based on the following lemmas.

Lemma 3. *Let M be the r by r matrix with entries $M_{i,j} = \sum_{h=1}^{r} t_h^{i+j}$ for $i, j = 0, 1, \ldots, r-1$. Then*

$$\det(M) = \prod_{1 \le i < j \le r} (t_j - t_i)^2.$$

Proof. Observe that $M = V^T \cdot V$ where V is the Vandermonde matrix with $V_{i,j} = t_i^j$ for $i = 1, 2, \ldots, r$ and $j = 0, 1, \ldots, r-1$. Therefore

$$\det(M) = (\det(V))^2 = \left(\prod_{i<j} (t_j - t_i) \right)^2.$$

□

Lemma 4. *Let c_0, c_1, ..., c_{2r-1} be given real numbers. If t_1, t_2, ..., t_r and x_1, x_2, ..., x_r satisfy* (2) *for $j = 0, 1, \dots 2r-1$, then*

$$c_{j+r} - c_{j+r-1}B_1 + \cdots + (-1)^r c_j B_r = 0 \tag{3}$$

for $j = 0$, 1, ..., $r-1$ where B_1, B_2, ..., B_r are defined by

$$(z - t_1)(z - t_2)\dots(z - t_r) = z^r - B_1 z^{r-1} + \cdots + (-1)^r B_r. \tag{4}$$

Proof. We have

$$\begin{aligned} c_{j+r} = \sum_{i=1}^{r} t_i^{j+r} x_i &= \sum_{i=1}^{r} x_i \sum_{h=1}^{r} (-1)^{h-1} B_h t_i^{j+r-h} \\ &= \sum_{h=1}^{r} (-1)^{h-1} B_h \sum_{i=1}^{r} t_i^{j+r-h} x_i \\ &= \sum_{h=1}^{r} (-1)^{h-1} B_h c_{j+r-h}. \end{aligned}$$

□

We are now ready to show how system (2) can be solved if both x_1, x_2, ..., x_r and t_1, t_2, ..., t_r are unknowns.

Lemma 5. *Let r be a positive integer. Let c_0, c_1, ..., c_{2r-1} be given real numbers. If nonzero real numbers x_1, x_2, ..., x_r and distinct real numbers t_1, t_2, ..., t_r satisfy* (2) *for $j = 0, 1, \dots, 2r-1$, then B_1, B_2, ..., B_r defined by* (4) *can be determined by solving the linear system* (3) *for $j = 0$, 1, ..., $r-1$. Subsequently t_1, t_2, ..., t_r can be found by computing the zeros of the polynomial*

$$z^r - B_1 z^{r-1} + B_2 z^{r-2} + \cdots + (-1)^r B_r. \tag{5}$$

If t_1, t_2, ..., t_r are chosen, the values of x_1, x_2, ..., x_r can be found by solving system (2) *for $j = 0$, 1, ..., $r-1$.*

Proof. First we apply Lemma 4, where, by (2), we have to solve a system of r linear equations in r unknowns B_1, B_2, ..., B_r with coefficient matrix M^* with $M^*_{i,j} = (-1)^{r-i} \sum_{h=1}^{r} t_h^{i+j} x_h$ for i, $j = 0$, 1, ..., $r-1$. Note that

$$\det(M^*) = \pm x_1 x_2 \cdots x_r \cdot \det(M)$$

where M is the matrix from Lemma 3. Since $\det(M^*)$ is nonzero by Lemma 3, we can solve the system of r linear equations and so determine the numbers B_1, B_2, ..., B_r. By computing the zeros of (5) the numbers t_i are found. Note that the numbers t_1, t_2, ..., t_r cannot be distinguished and we may assume $t_1 < t_2 < \cdots < t_r$. The expression for x_i follows from solving system (2) for $j = 0$, 1, ..., $r-1$ using Cramer's rule. □

Remark. Later on we apply Lemma 5 in such a way that the number t_i corresponds to the line $b_h x - a_h y = t_i$ for which the line sum is wrong and the corresponding number x_i is equal to the difference between the measured line sum and the correct line sum. We refer to Section 9, in particular formula (17), for the way the x_i's are computed in practice.

We conclude this section with a simple application of the Vandermonde determinant.

Lemma 6. *Let t_1, t_2, ..., t_r be distinct integers. If $\sum_{i=1}^{r} t_i^j x_i = 0$ for $j = 0$, 1, ..., $k-1$, then $k < r$ or $x_1 = x_2 = \cdots = x_r = 0$.*

Proof. It follows from $\sum_{i=1}^{r} t_i^j x_i = 0$ for $j = 0$, 1, ..., $r-1$ that the t_i's are not distinct or all x_i's are 0. □

5 A Vandermonde-related determinant

We prove the following result.

Lemma 7. *Let $a_1, a_2, \ldots, a_{2k}, b_1, b_2, \ldots, b_{2k}$ be variables. Set $W_{u,v} = a_u b_v - a_v b_u$ for $u, v = 1, 2, \ldots, 2k$. Let M be a $k \times k$ matrix with entries $M_{h,H} = \left(\prod_{i=1}^{k} W_{i,k+H}\right) / W_{h,k+H}$ for $h, H = 1, \ldots, k$. Then*

$$\det(M) = (-1)^{k(k-1)/2} \left(\prod_{1 \le h_1 < h_2 \le k} W_{h_1,h_2} \right) \left(\prod_{1 \le H_1 < H_2 \le k} W_{k+H_1,k+H_2} \right).$$

Proof. Clearly, we can consider $\det(M)$ to be a polynomial in the unique factorization domain $\mathbb{R}[a_1, a_2, \ldots, a_{2k}, b_1, b_2, \ldots, b_{2k}]$. The degree of $\det(M)$ equals $2k^2 - 2k$. For each h_1, h_2 with $1 \le h_1 < h_2 \le k$ the columns numbered h_1 and h_2 are proportional if $a_{h_1} b_{h_2} = a_{h_2} b_{h_1}$ which implies $\det(M)$=0. Therefore $\det(M)$ is divisible by W_{h_1,h_2}. Similarly, for each H_1, H_2 with $1 \le H_1 < H_2 \le k$ the rows numbered H_1 and H_2 are proportional if $a_{k+H_1} b_{k+H_2} = a_{k+H_2} b_{k+H_1}$ which implies that $\det(M)$ is divisible by $W_{k+H_1,k+H_2}$. The product of these distinct and coprime factors,

$$\left(\prod_{1 \le h_1 < h_2 \le k} W_{h_1,h_2} \right) \left(\prod_{1 \le H_1 < H_2 \le k} W_{k+H_1,k+H_2} \right),$$

has degree $2k^2 - 2k$ too. Therefore there is a real number c such that

$$\det(M) = c \left(\prod_{1 \le h_1 < h_2 \le k} W_{h_1,h_2} \right) \left(\prod_{1 \le H_1 < H_2 \le k} W_{k+H_1,k+H_2} \right).$$

Since $a_u b_v$ is lexicographically smaller than $a_v b_u$ for $u < v$, we infer that

$$c \cdot a_1^{k-1} a_2^{k-2} \cdots a_{k-1} \cdot a_{k+1}^{k-1} a_{k+2}^{k-2} \cdots a_{2k-1} \cdot b_2 b_3^2 \cdots b_k^{k-1} \cdot b_{k+2} b_{k+3}^2 \cdots b_{2k}^{k-1}$$

is the smallest lexicographic element in the expansion of $\det(M)$. We claim that on comparing the exponents it turns out that this term can only be obtained by developing the main diagonal of M. The first column is the only one containing a_{k+1}'s. Since no b_1 should be chosen, the only possibility is to choose $-a_{k+1}b_2, -a_{k+1}b_3, \ldots, -a_{k+1}b_k$ from the leftmost element of the first column. The second column is the only one containing a_{k+2}'s. Therefore it has to be chosen $k-2$ times and the other has to involve a_1. Since b_2 should not be chosen anymore, we choose $a_1 b_{k+2}$ and $-a_{k+2}b_3, -a_{k+2}b_4, \ldots, -a_{k+2}b_k$ from the element at the second column of the main diagonal. Continuing in this way it turns out that the only possible choice of the factors is in the expansion of entry $M_{h,h}$ the term with factors

$$a_1 b_{k+h}, a_2 b_{k+h}, \ldots, a_{h-1} b_{k+h}, -a_{k+h} b_{h+1}, -a_{k+h} b_{h+2}, \ldots, -a_{k+h} b_k,$$

for $h = 1, 2, \ldots, k$. Since the coefficient of the resulting product is $(-1)^{k(k-1)/2}$, we conclude that $c = (-1)^{k(k-1)/2}$. □

Example 2. For $k = 3$ the matrix M is as follows, where the chosen elements to obtain the smallest lexicographic element are boldface.

$$\begin{pmatrix} (a_2b_4 - \mathbf{a_4b_2})(a_3b_4 - \mathbf{a_4b_3}) & (a_2b_5 - a_5b_2)(a_3b_5 - a_5b_3) & (a_2b_6 - a_6b_2)(a_3b_6 - a_6b_3) \\ (a_1b_4 - a_4b_1)(a_3b_4 - a_4b_3) & (\mathbf{a_1b_5} - a_5b_1)(a_3b_5 - \mathbf{a_5b_3}) & (a_1b_6 - a_6b_1)(a_3b_6 - a_6b_3) \\ (a_1b_4 - a_4b_1)(a_2b_4 - a_4b_2) & (a_1b_5 - a_5b_1)(a_2b_5 - a_5b_2) & (\mathbf{a_1b_6} - a_6b_1)(\mathbf{a_2b_6} - a_6b_2) \end{pmatrix}$$

An alternative version of Lemma 7 reads as follows.

Corollary 8. *Let $a_1, a_2, \ldots, a_{2k}, b_1, b_2, \ldots, b_{2k}$ be reals. Set $W_{u,v} = a_u b_v - a_v b_u$ for $u, v = 1, 2, \ldots, 2k$. Let $M^* = \{M^*_{h,H} : h = 1, 2, \ldots, k; H = 1, 2, \ldots, k\}$ be the matrix with entries $M^*_{h,H} = 1/W_{h,k+H}$. Then*

$$\det(M^*) = (-1)^{k(k-1)/2} \left(\prod_{1 \le h_1 < h_2 \le k} W_{h_1,h_2} \right) \left(\prod_{1 \le H_1 < H_2 \le k} W_{k+H_1,k+H_2} \right) \left(\prod_{h=1}^{k} \prod_{H=1}^{k} W_{h,k+H} \right)^{-1}.$$

Proof. Note that $M_{h,H} = (\prod_{i=1}^{k} W_{i,k+H}) \cdot M^*_{h,H}$ and that the factor within brackets is independent of h. Hence,

$$\det(M) = \left(\prod_{H=1}^{k} \prod_{i=1}^{k} W_{i,k+H} \right) \det(M^*).$$

□

6 The line sum relation lemma

The following result is of fundamental importance in our present study. It follows from Lemma 4.1 of [38]. For the convenience of the reader we give a direct proof here.

Lemma 9. *Let $A, D, f, \ell_{h,t}$ be as in Section 2. Let K be a subset of $\{1, 2, \ldots, d\}$. For $h = 1, 2, \ldots, k := |K| \ge 2$ define $E_{h,K}$ by*

$$E_{h,K} = (-1)^{h-1} \prod_{i,j \in K,\ i<j,\ i,j \ne h} (a_i b_j - a_j b_i). \tag{6}$$

Then

$$\sum_{h \in K} E_{h,K} \sum_{t \in \mathbb{Z}} t^{k-2} \ell_{h,t} = 0.$$

Proof. Without loss of generality we may assume that $K = \{1, 2, \ldots, k\}$ with $D_h = (a_h, b_h)$ for $h = 1, 2, \ldots, k$. Put $\mathbf{a}^s = (a_1^s, a_2^s, \ldots, a_k^s)$ and $\mathbf{b}^s = (b_1^s, b_2^s, \ldots, b_k^s)$ for $s = 0, 1, 2, \ldots$. We denote the determinant of the $m \times m$ matrix with i-th column vector $\mathbf{x}_i = (x_{1,i}, \ldots, x_{m,i})$ by $\det(\mathbf{x}_1, \ldots, \mathbf{x}_m)$. Furthermore, we denote the determinant of the matrix which we obtain by omitting its first column vector and its h-th row vector by $\det(\mathbf{x}_2, \ldots, \mathbf{x}_m)_h$.

Obviously, for $s = 0, 1, \ldots, k-2$ we have

$$\det(\mathbf{a}^s \mathbf{b}^{k-2-s}, \mathbf{a}^{k-2}, \mathbf{a}^{k-3}\mathbf{b}, \mathbf{a}^{k-4}\mathbf{b}^2, \ldots, \mathbf{b}^{k-2}) = 0.$$

(Here the product of vectors is defined termwise.) By developing by the first column we obtain, for $s = 0, 1, \ldots, k-2$,

$$\sum_{h=1}^{k} (-1)^{h-1} a_h^s b_h^{k-2-s} \det(\mathbf{a}^{k-2}, \mathbf{a}^{k-3}\mathbf{b}, \mathbf{a}^{k-4}\mathbf{b}^2, \ldots, \mathbf{b}^{k-2})_h = 0.$$

Observe that $(-1)^{h-1}\det(\mathbf{a}^{k-2}, \mathbf{a}^{k-3}\mathbf{b}, \mathbf{a}^{k-4}\mathbf{b}^2, \ldots, \mathbf{b}^{k-2})_h$ is the Vandermonde determinant $E_{h,k}$. Hence, for arbitrary $(i, j) \in A$,

$$\sum_{h=1}^{k} (b_h i - a_h j)^{k-2} E_{h,K} = \sum_{s=0}^{k-2} \binom{k-2}{s} i^{k-s-2} j^s \sum_{h=1}^{k} a_h^s b_h^{k-2-s} E_{h,k} = 0.$$

Since for every direction any element of A is on exactly one line in that direction, we get

$$0 = \sum_{(i,j) \in A} f(i,j) \sum_{h=1}^{k} (b_h i - a_h j)^{k-2} E_{h,K} = \sum_{h=1}^{k} \sum_{t \in \mathbb{Z}} \sum_{(i,j) \in A, b_h i - a_h j = t} f(i,j)(b_h i - a_h j)^{k-2} E_{h,K}.$$

Thus

$$0 = \sum_{h=1}^{k} \sum_{t \in \mathbb{Z}} t^{k-2} E_{h,K} \sum_{(i,j) \in A, b_h i - a_h j = t} f(i,j) = \sum_{h=1}^{k} \sum_{t \in \mathbb{Z}} E_{h,K} t^{k-2} \ell_{h,t}.$$

□

7 Error correction of line sums in one direction

For the proof of Theorem 2 we combine the preceding lemmas.

Proof of Theorem 2. Number the directions such that $D_1, D_2, \ldots, D_k$ are directions with only correct line sums and direction D_H for some fixed H with $k < H \leq d$ may have wrong line sums. For $j \leq k$ we apply Lemma 9 to the set $K_{j,H} := \{1, 2, \ldots, j, H\}$,

$$\sum_{h=1}^{j} E_{h,K_{j,H}} \sum_{t \in \mathbb{Z}} t^{j-1} \ell_{h,t} + E_{H,K_{j,H}} \sum_{t \in \mathbb{Z}} t^{j-1} \ell_{H,t} = 0. \tag{7}$$

Define and compute

$$c^*_{j,H} = \sum_{h=1}^{j} E_{h,K_{j,H}} \sum_{t \in \mathbb{Z}} t^{j-1} \ell^*_{h,t} + E_{H,K_{j,H}} \sum_{t \in \mathbb{Z}} t^{j-1} \ell^*_{H,t}. \tag{8}$$

From (7) and (8) we obtain,

$$\sum_{h=1}^{j} E_{h,K_{j,H}} \sum_{t \in \mathbb{Z}} t^{j-1} (\ell^*_{h,t} - \ell_{h,t}) + E_{H,K_{j,H}} \sum_{t \in \mathbb{Z}} t^{j-1} (\ell^*_{H,t} - \ell_{H,t}) = c^*_{j,H}. \tag{9}$$

The choice of D_1, D_2, …, D_k implies $\ell^*_{h,t} = \ell_{h,t}$ for $h = 1, 2, \ldots, k$ and all $t \in \mathbb{Z}$. Thus

$$E_{H,K_{j,H}} \sum_{t \in \mathbb{Z}} t^{j-1} (\ell^*_{H,t} - \ell_{H,t}) = c^*_{j,H} \tag{10}$$

for $j = 1, 2, \ldots, k$. Notice that by our assumption there are at most k non-zero terms $\ell^*_{H,t} - \ell_{H,t}$, for $t = t_1, t_2, \ldots, t_r$, say. Then we have a system of linear equations (2) with $x_i = \ell^*_{H,t_i} - \ell_{H,t_i}, c_j = c^*_{j,H}/E_{H,K_{j,H}}$. If $t_1, t_2, \ldots, t_r$ are known, then we can simply solve system (2) and find $\ell^*_{H,t_i} - \ell_{H,t_i}$ for $i = 1, 2, \ldots, r$ and determine the correct line sum ℓ_{H,t_i} for the line $b_H x - a_H y = t_i$ for $i - 1, 2, \ldots, r$. If it is unknown which lines $b_H x - a_H y = t_i$ have wrong line sums, then $r < k/2$. Let I be the largest integer less than $k/2$. Then we consider the system of linear equations

$$\sum_{i=1}^{I} t_i^{j+h} x_i = c_{j+h} \quad \text{for} \quad j = 0, 1, \ldots, I; h = 0, 1, \ldots, I.$$

The rank r of the matrix with element c_{j+h} for $j, h = 1, 2, \ldots, I$ equals the number r of line sums ℓ_{H,t_i} with wrong line sums. Lemma 5 enables us to compute successively $B_1, B_2, \ldots, B_r$ and $t_1, t_2, \ldots, t_r$, indicating the lines $b_H x - a_H y = t_i$ where the wrong line sums are, and $x_1, x_2, \ldots, x_r$, which represent the errors $\ell^*_{H,t_i} - \ell_{H,t_i}$ for $i = 1, 2, \ldots, r$. Thus we can compute the correct line sums ℓ_{H,t_i} for $i = 1, 2, \ldots, r$.

□

8 Detection of directions with wrong line sums

In this section we prove Theorem 1.

Proof of Theorem 1. We introduce two parameters which may be used to reduce the amount of computation time: we assume that we want to find the correct line sums if in total there are at most F wrong line sums and these wrong line sums are in at most G directions. Thus $G \le F < d/2$. We prove the following hypothesis by induction on k.

Hypothesis for k. *The number of directions with wrong line sums that we have already detected equals r_{k-1}. The remaining directions form a set R_k such that if there is a wrong line sum in direction $D_h \in R_k$, then there are at least k wrong line sums in direction D_h and for each direction $D_h \in R_k$,*

$$\sum_{t\in\mathbb{Z}} t^j \ell_{h,t} = \sum_{t\in\mathbb{Z}} t^j \ell^*_{h,t}$$

for $j = 0, 1, \ldots, k-2$.

First we treat the case $k = 2$. We consider the sums ℓ^*_h of the line sums $\ell^*_{h,t}$ in each direction D_h. Since there are at most $G < d/2$ directions with wrong line sums, the majority of directions has the same correct value $\ell := \sum_{t\in\mathbb{Z}} \ell_{h,t}$ which is the sum of all f-values and therefore independent of h. We set the, r_1 say, directions which have a different sum of line sums apart. We continue with the set R_2 of the other $d - r_1$ directions. Observe that the directions in R_2 may have wrong line sums too, but that then in such a direction there are at least two errors, because the sum of the line sums is correct. Thus the Hypothesis holds for $k = 2$.

Suppose the hypothesis is true for k with $2 \le k < G$. It follows that the number of directions with wrong line sums in R_k is at most

$$(G - r_{k-1})/k < (d - 2r_{k-1})/(2k). \tag{11}$$

Hence all directions in R_k have correct line sums if $d - 2r_{k-1} \le 2k$ and if this inequality holds, the induction hypothesis is true for $k + 1$. In the sequel we assume

$$d - 2r_{k-1} \ge 2k + 1. \tag{12}$$

It follows that $|R_k| = d - r_{k-1} \ge 2k + 1$. By renumbering the directions we may assume $D_1, D_2, \ldots, D_k \in R_k$. For $h \in \{1, 2, \ldots, k\}$, $H > k$ and $K_{k,H} := \{1, 2, \ldots, k, H\}$ we define and compute

$$c^*_{k,H} = \sum_{h=1}^{k} E_{h,K_{k,H}} \sum_{t\in\mathbb{Z}} t^{k-1}\ell^*_{h,t} + E_{H,K_{k,H}} \sum_{t\in\mathbb{Z}} t^{k-1}\ell^*_{H,t}. \tag{13}$$

From (7) with $j = k$ and (13) we obtain, for all $D_H \in R_k, H > k$, similarly to (9),

$$\sum_{h=1}^{k} E_{h,K_{k,H}} \sum_{t\in\mathbb{Z}} t^{k-1}(\ell^*_{h,t} - \ell_{h,t}) + E_{H,K_{k,H}} \sum_{t\in\mathbb{Z}} t^{k-1}(\ell^*_{H,t} - \ell_{H,t}) = c^*_{k,H}. \tag{14}$$

We distinguish between the following two cases:

A) More than $(G - r_{k-1})/k$ directions $D_H \in R_k, H > k$ satisfy $c^*_{k,H} \ne 0$.

B) At most $(G - r_{k-1})/k$ directions $D_H \in R_k, H > k$ satisfy $c^*_{k,H} \ne 0$.

Case A) Because of the induction hypothesis the number of $H > k$ for which the direction D_H contains a wrong line sum does not exceed $(G - r_{k-1})/k$. Therefore there are at most $(G - r_{k-1})/k$ indices $H > k$ with $D_H \in R_k$ and $E_{H,K_{k,H}} \sum_{t\in\mathbb{Z}} t^{k-1}(\ell^*_{H,t} - \ell_{H,t}) \ne 0$. It follows that there is at least one direction D_H with $\sum_{h=1}^{k} E_{h,K_{k,H}} \sum_{t\in\mathbb{Z}} t^{k-1}(\ell^*_{h,t} - \ell_{h,t}) \neq 0$. This implies that there is an $h \in \{1,2,\ldots,k\}$ such that $\sum_{t\in\mathbb{Z}} t^{k-1}(\ell^*_{h,t} - \ell_{h,t}) \neq 0$. Thus there is a direction with a wrong line sum among $D_1, D_2, \ldots, D_k$.

Case B) At least $d-k-r_{k-1}-(G-r_{k-1})/k$ directions $D_H \in R_k$ with $H > k$ have no wrong line sum, hence satisfy $\ell^*_{H,t} = \ell_{H,t}$ for all t. We have, by (12),

$$d-k-r_{k-1}-\frac{d/2-r_{k-1}}{k} = \frac{d}{2}+\left(\frac{d}{2}-r_{k-1}\right)\left(1-\frac{1}{k}\right)-k \geq \frac{d}{2}+\frac{1}{2}-1-\frac{1}{2k} \geq \frac{d-1}{2}-1 \geq k-1.$$

Since $k < G < d/2$, at least k directions $d_H \in R_k$ with $H > k$ have no wrong line sums, hence satisfy $\ell^*_{H,t} = \ell_{H,t}$ for all t. Let $D_{k+1}, D_{k+2}, \ldots, D_{2k}$ be directions in R_k with $\sum_{t\in\mathbb{Z}} t^{k-1}(\ell^*_{H,t} - \ell_{H,t}) = 0$ for all t. Then we have, by (9), for $H \in \{k+1, k+2, \ldots, 2k\}$,

$$\sum_{h=1}^{k} E_{h,K_{k,H}} \sum_{t\in\mathbb{Z}} t^{k-1}(\ell^*_{h,t} - \ell_{h,t}) = c^*_H = 0. \tag{15}$$

Here we consider $E_{h,K_{k,H}}$ as coefficients and $\sum_{t\in\mathbb{Z}} t^{k-1}(\ell^*_{h,t} - \ell_{h,t})$ as unknowns. The coefficient matrix has as typical element

$$E_{h,K_{k,H}} = (-1)^{h-1} \prod_{i,j\in\{1,2,\ldots,k,H\},\ i<j,\ i,j\neq h} (a_i b_j - a_j b_i).$$

We claim that the corresponding determinant is nonzero. Observe that the h-th column has a nonzero factor

$$(-1)^{h-1} \prod_{i,j\in\{1,2,\ldots,k\},\ i<j,\ i,j\neq h} (a_i b_j - a_j b_i)$$

in common. By dividing it out for $h = 1,\ 2,\ \ldots,\ k$ the coefficient $E_{h,K_{k,H}}$ reduces to

$$E^*_{h,K_{k,H}} := \prod_{i\in\{1,2,\ldots,k\},\ i\neq h} (a_i b_H - a_H b_i).$$

It follows from Lemma 7 that the determinant of the matrix with typical entry $E^*_{h,D_{k,H}}$ equals

$$(-1)^{k(k-1)/2}\left(\prod_{1\leq h_1<h_2\leq k} (a_{h_1}b_{h_2} - a_{h_2}b_{h_1})\right)\left(\prod_{k+1\leq H_1<H_2\leq 2k} (a_{H_1}b_{H_2} - a_{H_2}b_{H_1})\right).$$

Since this expression is nonzero, the system (15) has the unique solution $\sum_{t\in\mathbb{Z}} t^{k-1}(\ell^*_{h,t} - \ell_{h,t}) = 0$ for $h = 1, 2, \ldots, k$.

By comparing the cases A and B we see that $C^*_{k,H} \neq 0$ for at most $(G-r_{k-1})/k$ directions $D_H \in R_k$ with $H > k$ if and only if $\sum_{t\in\mathbb{Z}} t^{k-1}(\ell^*_{h,t} - \ell_{h,t}) = 0$ for $h = 1, 2, \ldots, k$. Split the $d-r_{k-1}$ directions in R_k into subsets of k elements and a remainder subset of $< k$ elements. Then we have more than $(d-r_{k-1})/k-1$ k-subsets. Among them at most $(G-r_{k-1})/k < (d/2-r_{k-1})/k$ have a direction with a wrong line sum. Since, by (12),

$$\frac{d-r_{k-1}}{k} - 1 - \frac{d-2r_{k-1}}{2k} = \frac{d}{2k} - 1 > 0,$$

we see that there is at least one k-subset without wrong line sums. Renumber the directions such that this k-subset is $\{D_1, D_2, \ldots, D_k\}$. Then it follows as in (10) that

$$E_{H,D_H} \sum_{t\in\mathbb{Z}} t^{k-1}(\ell^*_{H,t} - \ell_{H,t}) = c^*_{k,H}$$

for $H > k$ with $D_H \in R_k$. We define the set R_{k+1} as the set of directions $d_1, d_2, \ldots, d_k$ together with the directions $d_H, H > k$ for which $c^*_{k,H} = 0$ and define r_k as $d - |R_k|$. For all $d_h \in R_{k+1}$ we have $\sum_{t\in\mathbb{Z}} t^{k-1}(\ell^*_{h,t} - \ell_{h,t}) = 0$. By the induction hypothesis this is also true for the lower powers of t. Therefore we have the system of equations $\sum_{t\in\mathbb{Z}} t^j(\ell^*_{h,t} - \ell_{h,t}) = 0$ for $h = 1, 2, \ldots, k$; $j = 0, 1, \ldots, k-1$. It follows from Lemma 6 that $\ell^*_{h,t} - \ell_{h,t} = 0$ for $h = 1, 2, \ldots, k$, if the number of nonzero terms is at most k. Thus we may assume that if there is a direction in R_{k+1} with a wrong line sum, then it has at least $k+1$ wrong line sums. This completes the induction step.

We stop detecting directions with wrong line sums at level k if $r_{k-1} = G$ or $k > F - \rho_g$ where ρ_g denotes the number of already detected wrong line sums. If $r_{k-1} = G$, then a wrong line sum in a direction $D_h \in R_k$ would lead to a total of $G+1$ directions with wrong line sums which contradicts the definition of G. If $k > F - \rho_g$, then a new direction with wrong line sums would give a total of $\rho_g + k > F$ wrong line sums, contradicting the definition of F. If a direction in R_k is detected with wrong line sums, then ρ_g is augmented by k. When we stop, we have found all directions with wrong line sums or the assumptions on F and G are not satisfied. In the latter case one might try a higher value of F or G.

It remains to show how the errors can be found and corrected for every direction which contains wrong line sums. For this we proceed as in the proof of Theorem 2. $\square$

9 An Error Correction Algorithm

In this section, we explicitly describe an algorithm for finding directions which contain wrong line sums, and correcting the wrong line sums. Since there may be relatively few errors in practice, we allow the user to specify the maximum number of errors F which have been made in at most G directions, where $G \le F < d/2$. If F, G are not chosen, set $F = G = \lfloor (d-1)/2 \rfloor$. We use $\leftrightarrow$ to denote swapped elements. When the line sums of two directions are swapped, $\ell^*_{i,t} \leftrightarrow \ell^*_{j,t}$, it is implicitly meant that this occurs for all t.

The algorithm finds the directions that contain wrong line sums, then the wrong line sums themselves, and therafter uses the correct line sums to repair them. We use the variable g to count the number of detected directions containing an erroneous line sum, and order the directions $D = \{D_1, \ldots, D_g, D_{g+1}, \ldots, D_d\}$, where D_i contains wrong line sums for $i \le g$. We denote the total number of already detected wrong line sums by ρ_g and the contribution of D_g to it by ρ^*_g. Steps 1-7 of the algorithm find all directions for which the sum of line sums does not match the majority and therefore have a wrong line sum. Steps 8-27 detect directions with at least $k \ge 2$ errors which were not detected yet. Steps 28-40 determine the wrong line sums themselves and correct them.

In Step 29 we introduce parameter $S = F - \rho_g + \rho^*_H$ which is an upper bound for the number of wrong line sums in direction D_H, since direction D_H already contributed ρ^*_H to ρ_g. In Step 31 we use direction D_{g+2S}. Since

$$g + 2S \le g + 2F - 2(r_g - r_H) \le g + 2F - 2(g-1) = 2F - g + 2 \le 2F + 1 \le d, \qquad (16)$$

this value of S is permitted. In Step 33 the exact number s of wrong line sums in direction D_H is determined.

The computation in Step 35 may not be exact. This is no problem, since the roots $t_1, t_2, \ldots, t_s$ are integers and can be found by rounding.

To apply Lemma 5 in Steps 36-39, we use Cramer's rule in the form of the matrix determinant lemma. Let $V = (t_j^{i-1})_{i,j=1}^s$ be the Vandermonde matrix and fix i. Let $u^T = (u_j)_{j=1}^r$, $v^T = (v_j)_{j=1}^r$ be column vectors where $u_j = c_j - t_j^{i-1}$ and v_j is equal to 1 for element i, and zero elsewhere. Then we can write x_i as

$$x_i = \frac{\det(V + uv^T)}{\det(V)} = 1 + v^T V^{-1} u = 1 + \sum_{j=1}^{r} V_{i,j}^{-1} u_j. \tag{17}$$

Therefore, we do not need to compute determinants for each i. Instead, a Vandermonde inverse matrix is computed once.

The algorithm may also work well for values of F and G greater than $d/2$. This depends on the way the errors in the line sums are distributed. If, after all, a function $f^* : A \to R$ has been computed, then an easy check reveals whether the line sums of f^* agree with the measured line sums.

Algorithm 1 Line sum error correction

Input: A finite set of (primitive) directions $D = \{(a_h, b_h) : h = 1, 2, \ldots, d\}$ and (measured) line sums $\ell^*_{h,t}$ in the directions of D of a function $f : A \to \mathbb{R}$ such that $\ell^*_{h,t}$ contains at most F errors in at most G directions where $G \le F < d/2$ (F, G may optionally be specified).

Output: Corrected line sums $\ell_{h,t}$.

for $h \leftarrow 1$ **to** d **do** // Find directions with a wrong line sum
    $\ell^*_h \leftarrow \sum_{t\in\mathbb{Z}} \ell^*_{h,t}$
$g \leftarrow 0$, $\rho_g \leftarrow 0$
for $h \leftarrow 1$ **to** d **do**
    if $\ell^*_h \neq \text{median}(\{\ell^*_1, \ell^*_2, \ldots, \ell^*_d\})$ **then**
        $g \leftarrow g + 1;\ \rho_g \leftarrow \rho_g + 1, \rho^*_g \leftarrow 1$
        $D_g \leftrightarrow D_h;\ \ell^*_{g,t} \leftrightarrow \ell^*_{h,t}$
$k \leftarrow 2$ // Find other directions with $k \ge 2$ wrong line sums
while $k \le F - \rho_g$ **and** $g \le G$ **do**
    $\mathit{maxDirections} \leftarrow \min(G - g, \lfloor (F - \rho_g)/k \rfloor)$
    $i \leftarrow g - k + 1$
    repeat
        $i \leftarrow i + k$
        $\mathit{count} \leftarrow 0$
        for $H \leftarrow g + 1, \ldots, i - 1, i + k, \ldots, d$ **do**
            $K_0 \leftarrow \{i, \ldots, i + k - 1, H\}$
            $c_H \leftarrow \sum_{h=1}^{k} E_{g+h,K_0} \sum_{t\in\mathbb{Z}} t^{k-1} \ell^*_{h+i-1,t} + E_{H,K_0} \sum_{t\in\mathbb{Z}} t^{k-1} \ell^*_{H,t}$ // cf.(8)
            if $c_H \neq 0$ **then**
                $\mathit{count} \leftarrow \mathit{count} + 1$
                if $\mathit{count} > \mathit{maxDirections}$ **then**
                    break
    until $\mathit{count} \le \mathit{maxDirections}$
    for $H \leftarrow g + 1, \ldots, i - 1, i + k, \ldots, d$ **do**
        if $c_H \neq 0$ **then**
            $g \leftarrow g + 1;\ \rho_g \leftarrow \rho_{g-1} + k;\ \rho^*_g \leftarrow k$
            $D_g \leftrightarrow D_H;\ \ell^*_{g,t} \leftrightarrow \ell^*_{H,t}$
    $k \leftarrow k + 1$
for $H \leftarrow 1$ **to** g **do** // Correct errors in direction D_H
    $S \leftarrow F - \rho_g + \rho^*_H$ // An upper bound for the number of wrong line sums in D_H
    for $j \leftarrow 1$ **to** $2S$ **do**
        $K_0 \leftarrow \{g + 1, \ldots, g + j, H\}$

32: $\qquad c_j \leftarrow \sum_{t\in\mathbb{Z}} t^{j-1}\ell^*_{H,t} + \sum_{h=1}^{j} \frac{E_{g+h,K_0}}{E_{H,K_0}} \sum_{t\in\mathbb{Z}} t^{j-1}\ell^*_{g+h,t}$ // cf. (8)

33: $s \leftarrow \text{rank} \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_S \\ c_2 & c_3 & c_4 & \cdots & c_{S+1} \\ c_3 & c_4 & c_5 & \cdots & c_{S+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_S & c_{S+1} & c_{S+2} & \cdots & c_{2S-1} \end{bmatrix}$

34: $\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_s \end{bmatrix} \leftarrow \begin{bmatrix} c_s & -c_{s-1} & \cdots & (-1)^{s-1}c_1 \\ c_{s+1} & -c_s & \cdots & (-1)^{s-1}c_2 \\ \vdots & \vdots & \ddots & \vdots \\ c_{2s-1} & -c_{2s-2} & \cdots & (-1)^{s-1}c_s \end{bmatrix}^{-1} \begin{bmatrix} c_{s+1} \\ c_{s+2} \\ \vdots \\ c_{2s} \end{bmatrix}$ // cf. (3)

35: $t_1, \ldots, t_s \leftarrow \text{roots}(z^s - B_1 z^{s-1} + B_2 z^{s-2} - \cdots + (-1)^s B_s)$ // cf. (4)

36: $V \leftarrow (t_j^{i-1})_{i,j=1}^{s}$

37: $W \leftarrow V^{-1}$ // cf. (17)

38: **for** $i \leftarrow 1$ **to** s **do**

39: $\qquad \ell_{H,t_i} \leftarrow \ell^*_{H,t_i} - \sum_{j=1}^{s} W_{i,j} c_j$

40: $\rho_g \leftarrow \rho_g + s - \rho^*_H$ // Total of detected errors

10 An example

To illustrate the algorithm we give an example. To show the various aspects of the algorithm, we choose $d = 16, F = 7, G = 4$. Let the directions be given by $D = \{D_1, D_2, \ldots, D_{16}\}$ with $D_1 = (1,0)$, $D_2 = (0,1)$, $D_3 = (1,1)$, $D_4 = (1,-1)$, $D_5 = (2,1)$, $D_6 = (2,-1)$, $D_7 = (1,2)$, $D_8 = (1,-2)$ and eight other directions, e.g. $D_9 = (3,1)$, $D_{10} = (3,-1)$, $D_{11} = (1,3)$, $D_{12} = (1,-3)$, $D_{13} = (3,2)$, $D_{14} = (3,-2)$, $D_{15} = (2,3)$, $D_{16} = (2,-3)$. For reason of transparency we assume that all the correct line sums are 0. We suppose that there are seven wrong line sums in three directions, $\ell^*_{3,0} = -3$, $\ell^*_{3,4} = 3$, $\ell^*_{6,-6} = -2$, $\ell^*_{6,1} = 1$, $\ell^*_{8,3} = 2$, $\ell^*_{8,5} = -4$, $\ell^*_{8,7} = 2$. Thus we have two wrong line sums in direction $(1,1)$, two in direction $(2,-1)$ and three in direction $(1,-2)$. We indicate the effects of the steps of the algorithm. Comments are given within square brackets.

Steps 4-7. [Selection of directions with deviant sum of line sums. Exchange of D_1 and D_6.] $\text{median}(\{\ell^*_1, \ell^*_2, \ldots, \ell^*_d\}) = 0 = \ell^*_h$ for all ℓ^*_h's except for $\ell^*_6 = -1$, $g \leftarrow 1$, $\rho_1 \leftarrow 1$, $\rho^*_1 \leftarrow 1$, $D_1 \leftarrow (2,-1)$, $D_6 \leftarrow (1,0)$, $\ell^*_{1,-6} \leftarrow -2$, $\ell^*_{1,1} \leftarrow 1$, $\ell^*_{6,-6} \leftarrow \ell^*_{6,1} \leftarrow 0$.

Steps 8-21. [$k = 2$, first try. Case A. This will fail since the test directions, D_2 and D_3, are assumed to have correct line sums, but D_3 has wrong line sums. See (6) for E.]
$k \leftarrow 2$, $\mathit{maxDirections} \leftarrow 3$, $i \leftarrow 0$, $i \leftarrow 2$, $\mathit{count} \leftarrow 0$, $H \leftarrow 4$, $K_0 \leftarrow \{2,3,4\}$, $c_4 \leftarrow 12$, $\mathit{count} \leftarrow 1$, $H \leftarrow 5$, $K_0 \leftarrow \{2,3,5\}$, $c_5 \leftarrow 24$, $\mathit{count} \leftarrow 2$, $H \leftarrow 6$, $K_0 \leftarrow \{2,3,6\}$, $c_6 \leftarrow 12$, $\mathit{count} \leftarrow 3$, $H \leftarrow 7$, $K_0 \leftarrow \{2,3,7\}$, $c_7 \leftarrow 12$, $\mathit{count} \leftarrow 4$, break.

Steps 12-27. [$k = 2$, second try, Case B. This succeeds since the test directions, D_4 and D_5, have correct line sums. The double error in D_3 will be detected. Exchange of D_2 and D_3. The triple error in D_8 will not be detected, since $\sum_t \ell^*_{8,t} = \sum_t t\ell^*_{8,t} = 0$.]
$i \leftarrow 4$, $\mathit{count} \leftarrow 0$, $H \leftarrow 2$, $K_0 \leftarrow \{4,5,2\}$, $c_2 \leftarrow 0$, $H \leftarrow 3$, $K_0 \leftarrow \{4,5,3\}$, $c_3 \leftarrow 36$, $\mathit{count} \leftarrow 1$, $c_H \leftarrow 0$ for $H = 6,7,...,16$, $H \leftarrow 3$, $g \leftarrow 2$, $\rho_2 \leftarrow 3$, $\rho^*_2 \leftarrow 2$, $D_2 \leftarrow (1,1)$, $D_3 \leftarrow (0,1)$, $\ell^*_{2,0} \leftarrow -3$, $\ell^*_{2,4} \leftarrow 3$, $\ell^*_{3,0} \leftarrow \ell^*_{3,4} \leftarrow 0$, $k \leftarrow 3$.

Steps 9-27. [$k = 3$. Since the test directions D_3, D_4, D_5 have only correct line sums, this

is Case B and the triple error in D_8 will be detected. Exchange of D_8 and D_3.]
maxDirections $\leftarrow 1$, $i \leftarrow 0$, $i \leftarrow 3$, *count* $\leftarrow 0$, $K_0 \leftarrow \{3,4,5,H\}$, $c_6 \leftarrow c_7 \leftarrow 0$, $c_8 \leftarrow -96$, *count* $\leftarrow 1$, $c_i \leftarrow 0$ for $i = 9, 10, \dots, 16$, $g \leftarrow 3$, $\rho_3 \leftarrow 6$, $\rho_3^* \leftarrow 3$, $D_3 \leftarrow (1,-2)$, $D_8 \leftarrow (0,1)$, $\ell_{3,3}^* \leftarrow 2$, $\ell_{3,5}^* \leftarrow -4$, $\ell_{3,7}^* \leftarrow 2$, $\ell_{8,3}^* \leftarrow \ell_{8,5}^* \leftarrow \ell_{8,7}^* \leftarrow 0$, $k \leftarrow 4$.

[Condition 9 on F is no longer satisfied. The directions with wrong line sums have been detected: $(2,-1),(1,1),(1,-2)$, now D_1, D_2, D_3.]

Steps 28-40: [Correction of the line sums for direction $(2,-1)$. Note that since $\ell_{g+h,t}^* = 0$ for $g+h > 3$, as in (10), c_j reduces to $c_j = \sum_{t\in\mathbb{Z}} t^{j-1}\ell_{H,t}^*$.]
$H \leftarrow 1$, $S \leftarrow 2$, $j \leftarrow 1$, $K_0 \leftarrow \{4,1\}$, $c_1 \leftarrow -1$, $j \leftarrow 2$, $K_0 \leftarrow \{4,5,1\}$, $c_2 \leftarrow 11$,
$j \leftarrow 3$, $K_0 \leftarrow \{4,5,6,1\}$, $c_3 \leftarrow -71$, $K_0 \leftarrow \{4,5,6,7,1\}$, $c_4 \leftarrow 431$,
$s \leftarrow 2$, $B_1 \leftarrow -7$, $B_2 \leftarrow 6$, $t_1 \leftarrow -6$, $t_2 \leftarrow -1$, $V \leftarrow (1,1;-6,-1)$, $W \leftarrow \frac{1}{5}(-1,-1;6,1)$, $\ell_{1,-6} \leftarrow \ell_{1,1} \leftarrow 0$, $\rho_g \leftarrow 7$.

Steps 28-40. [Correction of the line sums for direction $(1,1)$.]
$H \leftarrow 2$, $S \leftarrow 2$, $c_1 \leftarrow 0$, $c_2 \leftarrow 12$, $c_3 \leftarrow 48$, $c_4 \leftarrow 192$, $s \leftarrow 2$, $B_1 \leftarrow 4, B_2 \leftarrow 0$, $t_1 \leftarrow 0$, $t_2 \leftarrow 4$, $V \leftarrow (1,1;0,4)$, $W \leftarrow \frac{1}{4}(4,-1;0,1)$, $\ell_{2,0} \leftarrow 0$, $\ell_{2,4} \leftarrow 0$, $\rho_g \leftarrow 7$.

Steps 28-40. [Correction of the line sums for direction $(1,-2)$.]
$H \leftarrow 3$, $S \leftarrow 3$, $c_1 \leftarrow 0$, $c_2 \leftarrow 0$, $c_3 \leftarrow 16$, $c_4 \leftarrow -240$, $c_5 \leftarrow 2464; c_6 \leftarrow -21600$, $s \leftarrow 3$, $B_1 \leftarrow -15, B_2 \leftarrow 71, B_3 \leftarrow -105$, $t_1 \leftarrow -7, t_2 \leftarrow -5, t_3 \leftarrow -3$, $V \leftarrow (1,1,1;-7,-5,-3;49,25,9)$, $W \leftarrow \frac{1}{8}(15,8,1;-42,-20,-2;35,12,1)$, $\ell_{3,-7} \leftarrow \ell_{3,-5} \leftarrow \ell_{3,-3} \leftarrow 0$, $\rho_g \leftarrow 7$.

[All the wrong line sums have been detected and corrected. After all, it can be checked whether a correct solution has been found indeed by computing the new line sums. If not, the number of wrong line sums exceeded F or the number of directions with wrong line sums exceeded G.]

11 Complexity

In order to compute the complexity of the above algorithm we make some preliminary observations. If there is an h such that $a_h \geq m$ or $|b_h| \geq n$, then each line sum in direction (a_h, b_h) is the f^*-value of exactly one point. Without loss of generality we may then assume that $a_h = m$ or $|b_h| = n$, respectively. Hence, for each h the value of $|t|$ in (1) is at most $2mn$ and the number of directions d does not exceed mn. We further use that $g \leq G \leq F < d/2, h, H \leq d$ and $k \leq G$.

In our complexity computation we count an addition, subtraction, multiplication, division and a comparison of two values as one operation. When computing the complexity we do not take into account the size of the terms. (This can be quite high because of the factors t^j.) An operation may therefore mean a multi-precision operation. We assume that the numbers t^j for $0 \leq j < 2d-1, |t| \leq 2mn$ are computed once. This involves $\mathcal{O}(dmn)$ operations.

The numbers $t_1, t_2, \dots, t_s$ which are computed in Step 35 of the algorithm are the numbers t indicating the lines of the wrong line sums in direction H. By checking for each integer $\leq 2mn$ with Horner's method whether it is a zero of the polynomial, Step 35 takes $\mathcal{O}(dmn)$ operations.

In our analysis, we follow the steps of the pseudocode given in Section 9.

- *Steps 1-2* (by the above remark on the number of line sums) require $\mathcal{O}(dmn)$ operations (additions).
- *Steps 3-7* altogether need $\mathcal{O}(dmn)$ operations. (By the Floyd-Rivest algorithm the median can be calculated in linear time, see [40].)
- *Steps 9-11, 27* require $\mathcal{O}(F)$ operations,
- *Steps 12-14, 22* mean $\mathcal{O}(d/k)$ operations,

- *Steps 15-17* involve $\mathcal{O}(dk^2mn)$ operations. (According to (9) the computation of the E's takes $\mathcal{O}(k^2)$ operations; k does not exceed F.)
- *Steps 18-21* take $\mathcal{O}(G)$ operations,
- *Steps 23-26* involve $\mathcal{O}(dmn)$ operations.

By the structure of this block, the complexity of Steps 9-27 is

$$O_{9-11,27}O_{12-14,22}(O_{15-21} + O_{23-26}) = \mathcal{O}(d^2mnF^2)$$

where O_i denotes the number of operations in Steps i.

- *Step 28*, in view of $g \leq G$, implies $\mathcal{O}(G)$ repetitions,
- *Step 29* needs $\mathcal{O}(G)$ additions,
- *Step 30* implies $\mathcal{O}(F)$ repetitions,
- *Steps 31-32*, since $j \leq 2F$, require $\mathcal{O}(mnF^3)$ operations,
- *Step 33* needs $\mathcal{O}(F^2)$ operations,
- *Step 34*, by Algorithm 2.3.2 on p. 58 of [41], altogether takes $\mathcal{O}(F^3)$ operations,
- *Step 35*, by Algorithm 2.2.2 on p. 50 of [41], needs $\mathcal{O}(F^3)$ operations,
- *Step 36*, by an earlier remark, needs $\mathcal{O}(dmn)$ operations,
- *Steps 37-39* take $\mathcal{O}(F^2)$ operations,
- *Step 40*, by Algorithm 2.2.2 on p. 50 of [41], requires $\mathcal{O}(F^3)$ operations,
- *Steps 41-43* need $\mathcal{O}(F^2)$ operations.

By the structure of this block, the complexity of Steps 28-43 is given by

$$O_{28}(O_{29-32} + O_{33-36} + O_{37-40} + O_{41-43}) = \mathcal{O}(mnG(F^3 + d)) = \mathcal{O}(dmnF^2G)$$

where O_i denotes the number of operations implied by the corresponding Steps. Thus the algorithm can be completed in $\mathcal{O}(d^2mnF^2)$ operations.

12 Concluding remarks

After Ceko, Pagani and Tijdeman [30] had developed a fast method to reconstruct a consistent discrete tomography problem, the next logical step was to determine what is the most likely set of consistent line sums in case of inconsistency of line sums. If many line sums are almost correct, we refer to Section 4 of [33]. In the present paper we study the case that only a small number of line sums is wrong and show how to rectify the wrong line sums. We present an algorithm which performs the task in $\mathcal{O}(d^4mn)$ operations. However, the numbers involved in an operation may become quite large.

If the domain of f is finite, but not a rectangular grid, then the algorithm can be applied by choosing A as the smallest rectangular grid with sides parallel to the coordinate axes containing the domain and defining function value 0 for each point of A which does not belong to the domain of f. In this way the domain of f is extended to A. Hereafter the given algorithm can be used.

An obvious question is whether the reconstruction method for cases with only few wrong line sums can be extended to dimension three and higher. This seems to be hard for two reasons.

Firstly a higher dimensional version of Lemma 9 is wanted in order to be able to detect and correct the wrong line sums. Secondly a three-dimensional version of the algorithm of Ceko, Pagani and Tijdeman is only known under special conditions, see [42]. The general case might be quite complicated, because it is much more complex to describe the convex hull of the union of all ghosts in dimension > 2 than in dimension 2. Thus reconstruction is much more difficult.

Another question is whether it is possible to correct $d/2$ or more errors in the line sums. In a generic case more wrong line sums will be corrected by the algorithm. The example in Section 3 shows that it is not always possible to correct the line sums in $d/2$ or more directions, since the f-value of one point is uncertain. Theorem 2 shows that if the directions with wrong line sums can be detected, per direction quite a few wrong line sums can be corrected.

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