# TERMS OF RECURRENCE SEQUENCES IN THE SOLUTION SETS OF GENERALIZED PELL EQUATIONS

L. HAJDU AND P. SEBESTYÉN

Abstract. In this paper we completely describe those recurrence sequences which have infinitely many terms in the solution sets of generalized Pell equations. Further, we give an upper bound for the number of such terms when there are only finitely many of them.

## 1. Introduction

Let $d, t$ be non-zero integers with $d > 1$ square-free, and consider the equation

$$x^2 - dy^2 = t \tag{1}$$

in integers $x, y$. If $t = \pm 1, \pm 4$, then (1) is called a Pell equation, while in case of general $t$ it is a generalized Pell equation.

There are several papers in the literature concerning recurrence sequences with terms occurring in the solution sets of (generalized) Pell equations. We mention a few such recent results; the interested reader may consult their references. In the papers [1, 2, 3, 4, 5, 6, 8, 11, 15, 16, 19] the authors provide various finiteness results concerning the values (or sums or products of values) of certain concrete recurrence sequences (such as Fibonacci, Tribonacci, generalized Fibonacci, Lucas, Padovan, Pell, repdigits) in the $x$ coordinate of equation (1), for the cases $t = \pm 1, \pm 4$. Concerning the $y$-coordinate, we are aware only of two related results. Faye and Luca [12] proved that for $t = 1$ and $d > d_0$ with some $d_0$, any fixed binary recurrence sequence has at most two terms among the $y$-coordinates of the solutions of (1), and the same authors [13] showed that again with $t = 1$, there are at most

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two solutions of (1) with $y$-coordinates of the form $2^n - 1$. For related result, for example concerning sums or linear combinations of integers with fixed prime factors in the solution sets of Pell equations, see e.g. the papers [7, 9, 18] and the references there.

In view of the above results, it seems to be interesting to consider the question under more general circumstances. In the present paper we completely describe those recurrence sequences which have infinitely many terms in either of the $x, y$ coordinates of the solution sets of generalized Pell equations, i.e. of (1) with arbitrary $t$. Furthermore, we establish an upper bound for the number of solutions in the case where there are only finitely many solutions. We note that equation (1) can be considered as a norm form equation of degree two. Recently, Fuchs and Heintze [14] obtained similar results concerning values of recurrence sequences in the coordinates of solutions of general norm form equations. However, their results concern only non-degenerate recurrences. Thus our results can also be considered as an extension of those in [14] in the case of norm form equations of degree two.

## 2. New results

To formulate our main result, we need to introduce some notation.

Write $X$ and $Y$ for the sets of solutions of equation (1) in $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$, respectively.

Let $r$ be a positive integer, $a_1, \dots, a_r \in \mathbb{Z}$ such that $a_r \neq 0$ and $U_0, \dots, U_{r-1} \in \mathbb{Z}$ not all zero. If

$$U_n = a_1 U_{n-1} + \cdots + a_r U_{n-r} \quad (n \geq r) \tag{2}$$

and $r$ is minimal such that $(U_n)$ satisfies a relation above, then we say that $U = (U_n) = (U_n)_{n \geq 0}$ is a linear recurrence sequence (of integers) of order $r$. Throughout the paper we always assume that a recurrence sequence is given by its minimal length relation (2). We shall also use the notation

$$U = U(a_1, \dots, a_r, U_0, \dots, U_{r-1}).$$

The characteristic polynomial of $(U_n)$ is defined by

$$f(x) := x^r - a_1 x^{r-1} - \cdots - a_r = \prod_{i=1}^{q} (x - \alpha_i)^{m_i} \tag{3}$$

where $\alpha_1, \dots, \alpha_q$ are distinct algebraic numbers and $m_1, \dots, m_t$ are positive integers. Then as it is well-known (see e.g. Theorem C1 in part C of [22]) we have a representation of the form

$$U_n = \sum_{i=1}^{q} g_i(n)\alpha_i^n \quad \text{for all } n \geq 0. \tag{4}$$

Here $g_i(x)$ is a polynomial of degree $m_i - 1$ $(i = 1, \ldots, q)$ with coefficients in the number field $\mathbb{Q}(\alpha_1, \ldots, \alpha_q)$. The sequence $(U_n)$ is *degenerate* if there are integers $i, j$ with $1 \leq i < j \leq q$ such that $\alpha_i/\alpha_j$ is a root of unity; otherwise it is *non-degenerate.*

Now we can give our main result about terms of recurrence sequences in the solution sets of generalized Pell equations.

**Theorem 2.1.** *Let $(U_n)$ be a non-degenerate linear recurrence sequence of integers of order $r$, such that its characteristic polynomial is not of the form $x^2 + ax \pm 1$ with $(a^2 \mp 4)/d$ being a square in $\mathbb{Q}$. Then the inclusion*

$$U_n \in X \cup Y \tag{5}$$

*holds only for finitely many indices $n$. Further, the number of such values $n$ is bounded by $c_1 = c_1(r, d, t)$, where $c_1$ is an effectively computable constant depending only on $r, d, t$.*

**Remark.** We note that the exclusion of the specific binary recurrence sequences from Theorem 2.1 is necessary. This is demonstrated by the following example. Take $d = 2$ and $t = -1$, that is consider the classical Pell equation

$$x^2 - 2y^2 = -1.$$

As it is well-known, its positive solutions are given by

$$x + \sqrt{2}y = (1 + \sqrt{2})^m \quad (m \geq 0).$$

Let $U_0 = 1$, $U_1 = 3$ and

$$U_{n+2} = 6U_{n+1} - U_n \quad (n \geq 0). \tag{6}$$

One can easily check that $(U_n)$ is contained in $X$ - this sequence just comes from the even values of $m$ in (6). On the other hand, here $a = 6$, and $(a^2 - 4)/2 = 16$ is a full square, yielding that the roots of the characteristic polynomials $x^2 - 6x + 1$ are units of the ring of integers of $\mathbb{Q}(\sqrt{2})$.

## 3. The proof of Theorem 2.1

In the proof of the theorem we shall use several lemmas. The first one describes the solutions of equation (1) in the particular, but very important case $t = 1$.

**Lemma 3.1.** *Let $u_0$ and $v_0$ be the smallest positive solutions (in $x$ and $y$, respectively) of the equation*

$$x^2 - dy^2 = 1. \tag{7}$$

*Then all positive integer solutions $u, v$ of (7) are given by*

$$u + \sqrt{d}v = \left(u_0 + \sqrt{d}v_0\right)^m \quad (m \geq 1).$$

*Proof.* The statement is Theorem 7.26 of [20] on p. 354. □

Our second lemma shows that the sets of the coordinates of the solutions of equation (1) are unions of finitely many non-degenerate binary linear recurrence sequences.

**Lemma 3.2.** *Let $u_0$ be as in Lemma 3.1. If equation* (1) *has a solution, then all its solutions are given by*

$$(x, y) = \left(G_n^{(i)}, H_n^{(i)}\right) \quad (i = 1, \ldots, I)$$

*with some binary recurrence sequences*

$$G^{(i)} = G^{(i)}(2u_0, -1, G_0^{(i)}, G_1^{(i)}),\ H^{(i)} = H^{(i)}(2u_0, -1, H_0^{(i)}, H_1^{(i)}).$$

*Here $I$ and $G_0^{(i)}, G_1^{(i)}, H_0^{(i)}, H_1^{(i)}$ $(i = 1, \ldots, I)$ are some positive integers with $I < c_2$ and*

$$|G_j^{(i)}|, |H_j^{(i)}| < c_3 \ (0 \leq j \leq 1,\ 1 \leq i \leq I), \tag{8}$$

*where $c_2$ is an effectively computable constant depending only on $t$, while $c_3$ is an effectively computable constant depending only on $d$ and $t$.*

*Proof.* The assertions of the lemma are long and well-known qualitatively. The present formulation is an immediate consequence of Lemma 3.2 in [18]. □

Our last lemma is a deep result of Schlickewei and Schmidt [21] concerning the finiteness of the solutions of so-called polynomial-exponential equations. For its formulation, we need to introduce some further notation.

Consider the equation

$$\sum_{\ell=1}^{k} P_\ell(\mathbf{x})\boldsymbol{\alpha}_\ell^{\mathbf{x}} = 0 \tag{9}$$

in variables $\mathbf{x} = (x_1, \ldots, x_s) \in \mathbb{Z}^s$, where the $P_\ell$ are polynomials with coefficients in an algebraic number field $K$, and

$$\boldsymbol{\alpha}_\ell^{\mathbf{x}} = \alpha_{\ell 1}^{x_1} \cdots \alpha_{\ell s}^{x_s}$$

with given non-zero $\alpha_{\ell 1}, \dots, \alpha_{\ell s} \in K$ $(\ell = 1, \dots, k)$.

Let $\mathcal{P}$ be a partition of the set $\Lambda = \{1, \dots, k\}$. Then the system of equations

$$\sum_{\ell \in \lambda} P_\ell(\mathbf{x}) \boldsymbol{\alpha}_\ell^{\mathbf{x}} = 0 \quad (\lambda \in \mathcal{P}) \tag{10}$$

yields a refinement of (9). Let $\mathcal{S}(\mathcal{P})$ be the set of solutions of (10) which are not solutions of (10) with any proper refinement of $\mathcal{P}$. Set $\ell \overset{\mathcal{P}}{\sim} m$ if $\ell$ and $m$ lie in the same subset $\lambda$ of $\mathcal{P}$. Let $G(\mathcal{P})$ be the subgroup of $\mathbb{Z}^s$ consisting of $\mathbf{z}$ with

$$\boldsymbol{\alpha}_\ell^{\mathbf{z}} = \boldsymbol{\alpha}_m^{\mathbf{z}} \quad \text{for any } \ell, m \text{ with } \ell \overset{\mathcal{P}}{\sim} m.$$

**Lemma 3.3.** *Using the above notation, if $G(\mathcal{P}) = \{\mathbf{0}\}$ then we have*

$$|\mathcal{S}(\mathcal{P})| < 2^{35A^3} D^{6A^2}$$

*with $D = \deg(K)$ and*

$$A = \max\left(s, \sum_{\ell \in \Lambda} \binom{s + \delta_\ell}{s}\right),$$

*where $\delta_\ell$ is the total degree of the polynomial $P_\ell$.*

*Proof.* The statement is Theorem 1 in [21]. □

Now we can give the proof of Theorem 2.1.

*Proof of Theorem 2.1.* Let $(U_n)$ be a linear recurrence sequence of order $r$, satisfying the assumptions of the statement. In view of Lemma 3.2, $X \cup Y$ is the union of at most $2c_2$ binary recurrence sequences $(V_m)$ satisfying

$$V_{m+2} = 2u_0 V_{m+1} - V_m \quad (m \geq 0)$$

with some $V_0, V_1$ obeying

$$|V_0|, |V_1| \leq c_3.$$

Here $c_2$ and $c_3$ are the constants appearing in Lemma 3.2. Since $u_0 > 1$, by (4) we get that

$$V_m = B\beta^m + C\gamma^m \quad (m \geq 0),$$

where $\beta, \gamma$ are the (real) roots of the polynomial $x^2 - 2u_0 x + 1$ and $B, C$ are non-zero conjugated elements of $\mathbb{Q}(\sqrt{d})$. In particular, $\beta, \gamma$ are units and conjugates in $\mathbb{Q}(\sqrt{d})$.

Thus, writing $U_n$ in the form (4), the inclusion $U_n \in X \cup Y$ is equivalent with

$$B\beta^m + C\gamma^m = \sum_{i=1}^{q} g_i(n)\alpha_i^n \tag{11}$$

for some $m \geq 0$. Note that by our convention on the minimality on $r$ in (2), here none of $g_i$ and $\alpha_i$ is zero; further, the degrees of the $g_i$ are bounded by $r$. We shall show that (11) has only finitely many solutions in $(n, m)$, whose number is effectively bounded in terms of $r, d, t$. This, in view of that $B, C, \beta, \gamma$ are coming from a finite set of at most $2c_2$ elements, implies our theorem. Observe that as $\deg(g_i) \leq r$, the number of those values of $n$ which are roots of one of these polynomials, is bounded by $r^2$. So in what follows we shall assume that $n$ is not a root of $g_i$ $(1 \leq i \leq q)$.

We shall handle equation (11) by Lemma 3.3. For this, introduce the following notation. Suppose that the positive integers $n, m$ are solutions of (11). Write $\mathbf{N} = (n, m)$,

$$h_i(\mathbf{N}) = \begin{cases} g_i(n), & \text{if } i = 1, \dots, q, \\ -B, & \text{if } i = q+1, \\ -C, & \text{if } i = q+2, \end{cases}$$

and

$$\boldsymbol{\delta}_i = \begin{cases} (\alpha_i, 1), & \text{if } i = 1, \dots, q, \\ (1, \beta), & \text{if } i = q+1, \\ (1, \gamma), & \text{if } i = q+2. \end{cases}$$

Let $\mathcal{P}$ be a partition of the set $\{1, \dots, q, q+1, q+2\}$ such that we have

$$\sum_{i \in \lambda} h_i(\mathbf{N})\boldsymbol{\delta}_i^{\mathbf{N}} = 0 \quad (\lambda \in \mathcal{P}), \tag{12}$$

but (12) does not hold for any proper refinement of $\mathcal{P}$. Observe that all solutions $(n, m)$ of (11) are solutions of (12) with some $\mathcal{P}$. Now we distinguish subcases according to the structure of $\mathcal{P}$.

Assume first that there is a subset $\lambda$ of $\mathcal{P}$ such that $\lambda \subseteq \{1, \dots, q\}$. Here $|\lambda| = 1$, as $\alpha_i \neq 0$ $(1 \leq i \leq q)$, is not possible. Thus $|\lambda| \geq 2$. We shall use Lemma 3.3 to prove our claim. For this, observe that by the non-degeneracy of $(U_n)$ we have $G(\mathcal{P}) = \{(0, 0)\}$. Hence by Lemma 3.3 we get an upper bound for the number of these values of $n$ in terms of $r$.

Suppose next that there is a subset $\lambda$ of $\mathcal{P}$ such that $q+1, q+2 \in \lambda$. Since $\beta/\gamma$ is not a root of unity, we see that $G(\mathcal{P}) = \{(0, 0)\}$ again. So

Lemma 3.3 yields an upper bound for the number of such values of $n$ in terms of $r$ also in this case.

Thus we are left with the case where $\mathcal{P}$ consists of precisely two sets, say $\lambda_1$ and $\lambda_2$ with $q+1 \in \lambda_1$, $q+2 \in \lambda_2$. Obviously, $|\lambda_1|, |\lambda_2| \geq 2$. Assume that one of these sets, say $\lambda_1$, has more than two elements. Then there exists $i, j$ with $1 \leq i < j \leq q$ with $i, j \in \lambda_1$. Since $\alpha_i/\alpha_j$ is not a root of unity, we get that $G(\mathcal{P}) = \{(0,0)\}$. Thus Lemma 3.3 provides an upper bound in terms of $r$ for the number of these values of $n$ once again. That is, we may assume that $|\lambda_1| = |\lambda_2| = 2$ and that (12) (switching back to the notation used in (11)) reads as

$$\begin{cases} B\beta^m = g_1(n)\alpha_1^n, \\ C\gamma^m = g_2(n)\alpha_2^n. \end{cases} \tag{13}$$

Importantly, we also see that the characteristic polynomial of $(V_n)$ has precisely two distinct roots. In particular, $\alpha_1, \alpha_2$ are either rational, or conjugated quadratic algebraic numbers. If in (13) we have $G(\mathcal{P}) = \{(0,0)\}$, then we can bound the number of solutions in the usual way. So we may assume that $G(\mathcal{P}) \neq \{(0,0)\}$. Thus there exist $t_1, t_2 \in \mathbb{Z}$ for which

$$\beta^{t_1} = \alpha_1^{t_2}, \quad \gamma^{t_1} = \alpha_2^{t_2}.$$

If $\alpha_1, \alpha_2 \in \mathbb{Q}$, then taking conjugates in $K := \mathbb{Q}(\beta) = \mathbb{Q}(\gamma)$, we get that $\alpha_1 = \alpha_2$, a contradiction. So $\alpha_1, \alpha_2$ are conjugated quadratic integers. This easily implies that $\alpha_1, \alpha_2 \in K$. Since $\beta$ and $\gamma$ are units in $\mathbb{Z}[\sqrt{d}]$, $\alpha_1, \alpha_2$ are units in $O_K$. Multiplying the left and right hand sides of (13), using that $\beta\gamma = 1$ and that $\alpha_1, \alpha_2$ are conjugated units of $O_K$, we obtain

$$BC = (\pm 1)^n g_1(n) g_2(n).$$

Hence, if any of $g_1, g_2$ is not a constant polynomial, we get at most $2r^2$ solutions for $n$ in this case. Thus we may assume that $g_1$ and $g_2$ are constant, that is, the characteristic polynomial of $(V_n)$ is

$$T(x) := (x - \alpha_1)(x - \alpha_2) \in \mathbb{Z}[x].$$

However, as the roots of $T(x)$ are units of $O_K$, its constant term is $\pm 1$ and the square-free part of its discriminant equals $d$. So we are just in the exceptional case excluded from the theorem, and the statement follows.

□

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L. Hajdu
University of Debrecen, Institute of Mathematics
H-4002 Debrecen, P.O. Box 400.
and Alfréd Rényi Institute of Mathematics
H-1367 Budapest, P.O. Box 127.
Hungary
*Email address*: hajdul@science.unideb.hu, hajdu@renyi.hu

P. Sebestyén
University of Debrecen, Institute of Mathematics
H-4002 Debrecen, P.O. Box 400.
Hungary
*Email address*: sebestyen.peter@science.unideb.hu