ON SPECIAL EXTREMA OF POLYNOMIALS WITH APPLICATIONS TO DIOPHANTINE PROBLEMS

LAJOS HAJDU

Abstract. We prove that apart from explicitly given cases, described in terms of Dickson polynomials, a polynomial $f \in \mathbb{Q}[x]$ can have at most one shift $f(x) - \lambda$ ($\lambda \in \mathbb{C}$) of the form $u(g(x))^q (h(x))^k$ with $u \in \mathbb{C}$, $g, h \in \mathbb{C}[x]$ and either $\deg(g) = 2$, k is even, $q = k/2$ or $\deg(g) \leq 1$, $k \geq 2$, $q \geq 1$. This is shown by handling the case of two possible shifts, which was an open issue. As an application, we give a precise statement yielding a description of polynomials f having infinitely many shifted power (S-integral) values, and a complete description of superelliptic equations having infinitely many S-integral solutions when the polynomial involved is composite. In the case where there are finitely many solutions, our results yield effective bounds for them. Finally, as further applications, we give effective results for polynomial values in the solutions of Pell equations and in non-degenerate binary recurrence sequences.

1. Introduction

Let $f \in \mathbb{C}[x]$. A $\lambda \in \mathbb{C}$ is called an extremum of f if $f(x) - \lambda$ has a multiple root in $\mathbb{C}$. Extrema of polynomials are of wide interest. It is (at least partly) due to the fact that several Diophantine problems can be reduced to the investigation and description of them. Classical results of Siegel [42] and Faltings [16] show that irreducibility of curves is strongly related to the number of integral and rational points on them. Of particular interest are curves defined by a separable polynomial equations of the form $F(x) = G(y)$; see e.g. results of Davenport, Lewis and Schinzel [14], Bilu [8], Avanzi and Zannier [3, 4] and An and Diep [1] and the references there. In particular, Bilu and Tichy [11] gave a complete description of equations of the form $F(x) = G(y)$ allowing infinitely many solutions $x, y \in \mathbb{Z}$. In these results the extrema

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of polynomials played an important role. Further, we can mention papers of Beukers, Shorey and Tijdeman [7], Bilu, Brindza, Kirschenhofer, Pintér and Tichy [9], Rakaczki [33], Kulkarni and Sury [23], Bilu, Kulkarni and Sury [10], Stoll and Tichy [45], Hajdu, Laishram and Tengely [21], or Bazsó, Bérczes, Hajdu and Luca [5] (and in fact many more), see also the survey paper of Győry, Kovács, Péter and Pintér [20] and the references there, where certain special types of separable polynomial equations were considered involving some important families of combinatorial polynomials. Also in them, the extrema of the occurring polynomials were of particular importance.

In many cases, the main problem is to find or characterize those extrema λ of a given polynomial $f \in \mathbb{C}[x]$, where the shifted polynomial $f(x) - \lambda$ has 'many' multiple roots. In case of superelliptic equations (i.e. equations of the type $F(x) = y^k$ with say $F \in \mathbb{Q}[x]$ and fixed $k \geq 2$, in integer unknowns x, y) it is because of results of LeVeque [25] and Brindza [12], guaranteeing that the equation has only finitely many integral solutions, unless the root structure of $F(x)$ is 'degenerate' in some sense. Indeed, for example in the papers Pintér and Rakaczki [32] and Rakaczki [34, 35] the main emphasize is on guaranteeing that only a 'few' shifts of the actual (Bernoulli, Euler, Hermite) polynomials being investigated are 'degenerate' (i.e. have 'many' multiple roots), and then the finiteness of the solutions of the underlying Diophantine equations follow in an effective form. On the other hand, also in the general case $F(x) = G(y)$, extrema yielding 'many' multiple roots are of great importance as well; see e.g. Section 5 of Bilu and Tichy [11].

In this paper we completely characterize the polynomials $f \in \mathbb{Q}[x]$ which for any fixed t have precisely t shifts $f(x) - \lambda_i$ ($\lambda_i \in \mathbb{C}, i = 1, \dots, t$) of the form $u(g(x))^q(h(x))^k$ with $u \in \mathbb{C}$, $g, h \in \mathbb{C}[x]$ and either $\deg(g) = 2$, k is even, $q = k/2$ or $\deg(g) \leq 1$, $k \geq 2$, $q \geq 1$. By the above mentioned results of LeVeque [25] and Brindza [12] this is the decisive condition for many applications, e.g. concerning superelliptic equations. In this characterization the main novelty is the description of the case $t = 2$, the other cases are well-known and/or simple. Our main tool will be a deep result of Avanzi and Zannier [3] concerning a certain Pell-type equation for polynomials.

We apply our results to several Diophantine problems, which are interesting in themselves and are widely studied. First, we give a precise statement concerning the infinitude of shifted power values (in S-integers of an algebraic number field) of a polynomial $f \in \mathbb{Q}[x]$. Our result provides a complete solution to the problem and also an effective bound for the heights of the solutions when there are finitely many of them. This theorem can be considered to be an effective algebraic version of the main result of Bilu and Tichy [11] concerning equations of the form $f(x) = g(y)$, when $g(y)$ is of the shape $g(y) = \beta y^k + \alpha$. In fact, in our result $k \geq 2$ is also a variable for which we derive an upper bound, too. So our theorem concerns a family of polynomials $g_k(y)$ (of the given shape) rather than only an arbitrary, but fixed one.

At this point we also mention that superelliptic equations, that is equations of the form

$$f(x) = \beta y^k \tag{1}$$

where $k \geq 2$ and say $f \in \mathbb{K}[x]$, $\beta \in \mathbb{K}$ in unknown $x, y \in O_{\mathbb{K}}$ are of wide interest, with an extremely rich literature. (Here $\mathbb{K}$ is a number field with ring of integers $O_{\mathbb{K}}$.) The same is true for the even more general situation, when in (1) the exponent k is also a variable. The interested reader may check for example chapters 6 and 8 of [41] and [6], and the references there. We shall explicitly give some related results later (see Lemmas 4.1 from Shorey and Tijdeman [41] and 4.2 from Brindza [12]), which give explicit conditions under which (1) has only finitely many solutions in x, y and x, y, k, respectively (bounded in terms of the parameters involved). Our Theorem 3.2 can be considered as an extension of these results, too, to the case where on the right hand side of (1), instead of (almost) powers βy^k we have shifted powers $\beta y^k + \alpha$. Theorem 3.2 gives a characterization when this more general equation has infinitely many solutions, and provides effective bounds for the solutions x, y, k when there are finitely many of them. (In fact, we shall work in a slightly more general framework, with S-integers.)

We also give a description of composite polynomials assuming infinitely many S-integral power values. In the case where there are only finitely many solutions, the result is effective (i.e., the height of the solutions can be effectively bounded). After that, we give an effective result for polynomial values in the solutions of Pell equations. Finally, we effectively bound the values of polynomials in non-degenerate binary recurrence sequences. The latter problem attracted a lot of attention already, and our Theorem 3.5 generalizes and/or extends several results from the literature, e.g. those of Pethő [31] and Shorey and Stewart [39] about perfect powers in non-degenerate binary recurrence sequences, Nemes and Pethő [29] concerning values of general polynomials in certain binary recurrence sequences and of Kovács [24] on values of certain combinatorial polynomials in concrete, important binary recurrence sequences. We give more details before the formulation of Theorem 3.5.

The structure of the paper is the following. In the next section we introduce some objects needed later, and give some of their most important properties. In Section 3 we formulate our results. Their proofs (together with some lemmas) are given in the last section.

2. Preliminaries

In this section we introduce the objects we need later and collect some important facts about them.

2.1. **Extrema of polynomials.** In this subsection we partly follow the treatment and notation of Section 2.3 of [11]. All polynomials are assumed to be of degree at least one, unless stated otherwise. The root type of a polynomial $f(x) \in \mathbb{C}[x]$ having r distinct roots is the unordered list $[\mu_1, \ldots, \mu_r]$ of the multiplicities of its roots. Clearly, $\mu_1 + \cdots + \mu_r = \deg(f)$. The f-type of a complex number λ is the root type of the polynomial $f(x) - \lambda$. If $f(x) - \lambda$ has at least one multiple root (i.e. the f-type of λ is not $[1, \ldots, 1]$), then λ is called an extremum of $f(x)$. For λ of f-type $[\mu_1, \ldots, \mu_r]$, write

$$\delta_f(\lambda) = (\mu_1 - 1) + \cdots + (\mu_r - 1) = \deg(f) - r. \tag{2}$$

Observe that $\delta_f(\lambda) > 0$ if and only if λ is an extremum of $f(x)$. We have

$$\sum_{\lambda \in \mathbb{C}} \delta_f(\lambda) = \deg(f) - 1. \tag{3}$$

This follows from the fact that $\delta_f(\lambda) = \deg(\gcd(f(x) - \lambda, f'(x)))$, so the sum in (3) equals to $\deg(f') = \deg(f) - 1$.

Let $\lambda \in \mathbb{C}$ with f-type $[\mu_1, \ldots, \mu_r]$. We shall say that λ is *special* with respect to f, if there exists a $k \geq 2$ such that the tuple

$$\left[\frac{k}{\gcd(k, \mu_1)}, \ldots, \frac{k}{\gcd(k, \mu_r)}\right] \tag{4}$$

is one of

$$[2, 2, 1, \ldots, 1], \quad [q, 1, \ldots, 1] \ (q \geq 1).$$

If the choice of f is obvious, we simply say that λ is special. Note that as we shall see later, this condition is very important (in fact, decisive) concerning the infinitude of the solutions of superelliptic equations of the form $f(x) - \lambda = \beta y^k$.

Remark 1. Observe that if $\deg(f) \geq 3$ and λ is special with respect to f then λ is also an extremum of f. We shall use this fact throughout the paper without any further mentioning.

Here we point out that special shifts of polynomials (i.e. shifts with special extrema) are of special form.

Lemma 2.1. *Let $f(x) \in \mathbb{Q}[x]$ and $\lambda \in \mathbb{C}$. Then $f(x) - \lambda$ is of the form*

$$f(x) - \lambda = u(g(x))^q (h(x))^k \tag{5}$$

with $u \in \mathbb{C}$, $g, h \in \mathbb{C}[x]$ and either $\deg(g) = 2$, k is even, $q = k/2$ or $\deg(g) \leq 1$, $k \geq 2$, $q \geq 1$, if and only if λ is special with respect to f.

Proof. Assume first that λ is special with respect to f. If with some k, the tuple (4) is $[2, 2, 1, \dots, 1]$ then k is necessarily even and we have

$$f(x) - \lambda = u(g(x))^{k/2} (h(x))^k$$

with $u \in \mathbb{C}$, $g, h \in \mathbb{C}[x]$ and $\deg(g) = 2$. On the other hand, if the tuple (4) is $[q, 1, \dots, 1]$ with some $q \geq 1$, then

$$f(x) - \lambda = u(g(x))^q (h(x))^k$$

holds with $u \in \mathbb{C}$, $g, h \in \mathbb{C}[x]$ and $\deg(g) \leq 1$.

Suppose now that $f(x) - \lambda$ is of the form (5). Then one can readily check that λ is special with respect to f. □

2.2. **Algebraic numbers and S-integers.** Let $f \in \mathbb{Z}[x]$. Then by the (naive) height of f we mean the maximum of the absolute values of its coefficients. If β is an algebraic number, then $H(\beta)$ denotes its height, which is the height of the (primitive) defining polynomial $f_\beta \in \mathbb{Z}[x]$ of β.

Let $\mathbb{K}$ be an algebraic number field, and let S be a finite set of prime ideals in $\mathbb{K}$. We say that $\beta \in \mathbb{K}$ is an S-integer, if writing the principal (fractional) ideal (β) as $(\beta) = A/B$ with some coprime ideals in $\mathbb{K}$, the ideal B has no prime ideal divisor outside S. The ring of S-integers will be denoted by $O_{S,\mathbb{K}}$. Note that if β is an algebraic integer of $\mathbb{K}$, then we clearly have $\beta \in O_{S,\mathbb{K}}$.

2.3. **Some families of special polynomials.** In this subsection we introduce some families of polynomials which will appear in our statements. In fact, they will yield the exceptional cases.

The first family we need is formed by shifts of 'almost power' rational polynomials:

$$\mathcal{P}_1 := \{u(g(x))^q (h(x))^k + v : u, v \in \mathbb{Q},\ u \neq 0,\ g, h \in \mathbb{Q}[x],\ \text{and}$$
$$\text{either } \deg(g) = 2,\ k \text{ is even},\ q = k/2 \text{ or } \deg(g) \leq 1,\ k \geq 2,\ q \geq 1\}.$$

The second family is related to Dickson polynomials. Let a be a nonzero rational number and d a positive integer. Denote by $D_d(x, a)$

the d-th Dickson polynomial with parameter a, given by

$$D_d(x,a) = \sum_{i=0}^{\lfloor d/2 \rfloor} \frac{d}{d-i} \binom{d-i}{i} (-a)^i x^{d-2i}.$$

Dickson polynomials can be characterized by having exactly two extrema. More precisely, we have the following assertion, which is a simple consequence of Proposition 5.1 and Theorem 5.2 of Bilu and Tichy [11].

Lemma 2.2. *If $d \geq 3$ then $D_d(x,a)$ has exactly two extrema. If d is odd then both are of type $[1,2,\dots,2]$. If d is even, then one of them is of type $[2,\dots,2]$ and the other is of the type $[1,1,2,\dots,2]$.*

On the other hand, let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree d having exactly two extrema in $\mathbb{C}$. Moreover, let all its extrema be of one of the following types:

$$[2,\dots,2], \quad [1,2,\dots,2], \quad [1,1,2,\dots,2].$$

Then $f(x) = uD_d(x+v,a) + w$, where $a,u,v,w \in \mathbb{Q}$ with $au \neq 0$.

We need some further objects. Let $G(x)$ be a square-free monic polynomial with rational coefficients, of degree four, and consider the equation

$$F(x)^2 - G(x)H(x)^2 = \gamma \tag{6}$$

in polynomials $F,H \in \mathbb{Q}[x]$ and $\gamma \in \mathbb{Q} \setminus \{0\}$. By the degree of a solution triple $(F(x),H(x),\gamma)$ of equation (6) we mean $\deg(F(x))$.

We shall need the description of the solutions of (6) which has been completely given by Avanzi and Zannier [3]. For this description, consider the elliptic curve

$$\mathcal{C}: \quad y^2 = G(x)$$

over $\mathbb{Q}$, by choosing one of the two points at infinity (which are rational as G is monic), as the identity element. Denote by π the other point at infinity.

The following deep result of Avanzi and Zannier [3] gives a precise description of the solutions of (6).

Theorem A. *Equation (6) is solvable in $F,H \in \mathbb{Q}[x]$ and $\gamma \in \mathbb{Q} \setminus \{0\}$ if and only if π is a torsion point of $\mathcal{C}$.*

Further, if π is a torsion point of order exactly N, then there exists a solution of degree dN for any positive integer d, and, for a fixed degree, the solution with F monic is unique (up to the sign of H). Let $(F_1(x),H_1(x),\gamma_1)$ be the solution with $F_1(x)$ monic of minimal degree

N. Then the solution $(F_d(x), H_d(x), \gamma_d)$ with $F_d(x)$ monic of degree dN is given by

$$F_d(x) = D_d(F_1(x), \gamma_1/4), \quad \gamma_d = 4^{1-d}\gamma_1^d,$$

where D_d is the d-th Dickson polynomial.

Remark 2. Here we mention two points.
1. Importantly, based upon the method of their proof, Avanzi and Zannier could parametrize and give all solutions of (6), see Section 5 of [3]. The parametrized solutions are explicitly listed in the PhD thesis of Avanzi [2].
2. Using the well-known connection between Dickson- and Chebyshev polynomials, the above statement could be formulated by the help of (normalized) Chebyshev polynomials. In fact, this is what is done in [2] (see Theorem 2 there). However, as [3] is easier to access, we follow the formulation there (with Dickson polynomials).

Now we can introduce the second family of polynomials we shall need. First, for given G as above, write $\mathcal{F}_{G,n}(x)$ for the unique monic solution $F(x)$ of (6) of degree n (if such solution exists), and let $\gamma(G, n)$ be the corresponding γ-value in (6). Put

$$\begin{aligned}\mathcal{P}_2 := \{uD_d(x+v,a)+w : d \in \mathbb{N},\ a,u,v,w \in \mathbb{Q},\ au \neq 0\} \cup \\ \{u\mathcal{F}_{G,n}(x)+v : G \text{ is as above},\ n \text{ is even},\ u,v \in \mathbb{Q},\ u \neq 0, \\ \mathcal{F}_{G,n}(x) \neq \left(\mathcal{F}_{G,n/2}(x)\right)^2 - \gamma(G,n/2)/2\}.\end{aligned}$$

Remark 3. By classical results of Mazur [26, 27] we know that in Theorem A we have $2 \leq N \leq 12$ and $N \neq 11$. Thus $\mathcal{P}_2$ contains only linear transforms of polynomials of the shape $D_d(F(x), a)$ with $\deg(F) \leq 12$, $\deg(F) \neq 11$. Observe further that the polynomials $\mathcal{F}_{G,n}(x)$ excluded from $\mathcal{P}_2$ are constant multiples of shifts of full squares in $\mathbb{Q}[x]$.

2.4. **Pell equations.** Let a, b, c be integers with $abc \neq 0$. We shall be interested in polynomial values in the solution sets of Pell equations of the shape

$$ax^2 - by^2 = c. \tag{7}$$

If $ab > 0$ then (7) is called a Pell equation, and as it is well-known, it can have infinitely many solutions in integers x, y. In case of $ab < 0$, equation (7) trivially admits only finitely many solutions in integers x, y.

2.5. **Binary recurrence sequences.** A sequence of integers $(U_n)_{n=0}^{\infty}$ defined by

$$U_{n+2} = AU_{n+1} + BU_n \quad (n \geq 0)$$

with some $A, B, U_0, U_1 \in \mathbb{Z}$ is called a binary recurrence sequence. We shall always assume that $U_0^2 + U_1^2 > 0$ and that $(U_n)_{n=0}^{\infty}$ is non-degenerate, that is $AB \neq 0$ and the ratio of the roots α, β of the characteristic polynomial $x^2 - Ax - B$ of the sequence is not a root of unity. It is well-known that we have

$$U_n = \frac{a\alpha^n - b\beta^n}{\alpha - \beta} \quad (n \geq 0),$$

where $a = U_0 - U_1\beta$, $b = U_1 - U_0\alpha$. Putting $V_n = a\alpha^n + b\beta^n$, one can easily check that $V_n \in \mathbb{Z}$ and

$$V_n^2 - (A^2 + 4B)U_n^2 = 4(U_1^2 - AU_1U_0 - BU_0^2)(-B)^n \quad (n \geq 0) \tag{8}$$

holds. It is important to note that if $(U_n)_{n=0}^{\infty}$ is non-degenerate and $B = \pm 1$, then we have

$$A^2 + 4B \neq 0 \quad \text{and} \quad U_1^2 - AU_1U_0 - BU_0^2 \neq 0$$

in (8).

3. New results

Our principal result is the following.

Theorem 3.1. *For $i \geq 1$, write $\mathcal{S}_i$ for the set of polynomials $f \in \mathbb{Q}[x]$ with $\deg(f) > 4$ for which precisely i complex numbers are special. Then $\mathcal{S}_i$ is empty for $i \geq 3$, while $\mathcal{S}_2 = \mathcal{P}_2$ and $\mathcal{S}_1 = \mathcal{P}_1 \setminus \mathcal{P}_2$.*

Remark 4. We give several comments on Theorem 3.1.
1. In view of the theorem, for a monic $f \in \mathbb{Q}[x]$ of degree at least 5 there exists no special $\lambda \in \mathbb{C}$ if and only if $f \notin \mathcal{P}_1 \cup \mathcal{P}_2$.
2. As we shall see, the fact that $\mathcal{S}_i$ is empty for $i \geq 3$ is rather simple. This (under slightly more special assumptions) was essentially proved by Rakaczki, see Lemma 3 in [34].
3. Clearly, the cases $\deg(f) \leq 4$ could be handled without any problem. Naturally, in these cases some special parametric families of polynomials (which could be easily described) arise. To avoid (trivial) technical complications, we do not give the details here.
4. In view of Remark 3, if $f(x) \in \mathbb{Q}[x]$ is explicitly given, or if $f(x)$ is a member of a parametric family of some special polynomials (e.g. with combinatorial background), then like in case of the applications of the Bilu-Tichy theorem from [11] (see e.g. [9, 33, 23, 45, 21, 5]), in principle one should be able to decide (relatively easily) whether $f(x) \in \mathcal{P}_2$ or

not. (In our case, one may further use the parametrization of $\mathcal{F}_{G,n}(x)$ given by Avanzi and Zannier [3] and Avanzi [2] if necessary.) This means that Theorem 3.1 and our other theorems are hopefully easily accessible for various applications.
5. The above theorem has an interesting connection to elliptic integrals. Namely, one can show that the polynomials $f(x) \in \mathcal{P}_2$ can be expressed by the help of so-called pseudo-elliptic integrals. (As a related paper, see e.g. [30].) We suppress the details.

Combining Theorem 3.1 with (generalizations of) classical results of Schinzel and Tijdeman [38] and Brindza [12] we get the following statement, concerning shifted power values of a polynomial.

Theorem 3.2. *Let $f \in \mathbb{Q}[x] \setminus \mathcal{P}_2$ with $\deg(f) > 4$ and $\mathbb{K}$ be an algebraic number field. Then for any $\alpha, \beta \in \mathbb{K}$ with $\alpha \notin \mathbb{Q}$ and any finite set S of prime ideals of $\mathbb{K}$, the equation*

$$f(x) = \beta y^k + \alpha \tag{9}$$

has only finitely many solutions in $x, y \in O_{S,\mathbb{K}}$ and $k \in \mathbb{Z}$ with $k \geq 2$, and we have $\max(H(x), H(y), k) \leq C_1$. Here $C_1 = C_1(f, \alpha, \beta, S, \mathbb{K})$ is an effectively computable constant depending only on $f, \alpha, \beta, S, \mathbb{K}$, and we apply the convention that $k \leq \mathrm{ord}(y) + 1$ if y is a root of unity in $\mathbb{K}$ and $k = 2$ if $y = 0$.

Further, if $f \notin \mathcal{P}_1 \cup \mathcal{P}_2$ then the same conclusions hold with $\alpha \in \mathbb{Q}$, as well.

Remark 5. Importantly, in view of Theorem 3.1 and Lemma 4.2 the condition $f \notin \mathcal{P}_2$ (or $f \notin \mathcal{P}_1 \cup \mathcal{P}_2$ if α is rational) is necessary. That is, Theorem 3.2 is best possible.

Now we give an application for power values of decomposable polynomials. For this, let $f(x) \in \mathbb{Q}[x]$, $\varphi(x)$ a polynomial having algebraic coefficients, $\mathbb{K}$ an algebraic number field, $\beta \in \mathbb{K}$ and S a finite set of prime ideals of $\mathbb{K}$. Consider the equation

$$\varphi(f(x)) = \beta y^k \tag{10}$$

in $k \in \mathbb{Z}$ with $k \geq 2$ and $x, y \in O_{S,\mathbb{K}}$.

Theorem 3.3. *Suppose that $f(x) \in \mathbb{Q}[x] \setminus \mathcal{P}_2$ with $\deg(f) > 4$ and that φ has two non-rational roots of coprime multiplicities. Then equation* (10) *has only finitely many solutions x, y, k with $x, y \in O_{S,\mathbb{K}}$ and $k \in \mathbb{Z}$ with $k \geq 2$, under the convention $k \leq \mathrm{ord}(y) + 1$ if y is a root of unity in $\mathbb{K}$ and $k = 2$ if $y = 0$. Further, we have $\max(H(x), H(y), k) \leq C_2$, where $C_2 = C_2(\varphi, f, \beta, S, \mathbb{K})$ is an effectively computable constant depending only on $\varphi, f, \beta, S, \mathbb{K}$.*

Remark 6. At this point we also mention several things.
1. If all the multiplicities of the roots of φ are multiples of some $k \geq 2$, then certainly, equation (10) may have infinitely many solutions, for any f. Hence the condition concerning the multiplicities of the roots of φ cannot be dropped.
2. If there is only one root α of f with multiplicity coprime to k, and that root is rational, then in case of $f \in \mathcal{P}_1$ the statement would fail, too. Hence the condition that there are two *non-rational* roots of coprime multiplicities, is necessary as well.
3. For fixed k, Theorem 3.3 ultimately reduces to the question whether $\varphi(f(x)) - \beta y^k$ has a genus zero factor or not. Hence in this case the statement practically follows from Ritt's Second Theorem [36]. For details, related results, remarks and explanations see e.g. [17], Section 5 of [37], [47], [18], [3]. However, Theorem 3.3 is much more general than that: as k is a variable, the statement concerns an infinite family of curves of the shape $\varphi(f(x)) - \beta y^k$.

Now we give effective bounds for polynomial values in solutions of Pell equations and in binary recurrence sequences.

Theorem 3.4. *Let a, b, c be integers with $abc \neq 0$. Let S_X and S_Y be the X and Y coordinates, respectively, of the solutions $(X, Y) \in \mathbb{Z}^2$ of the equation*
$$aX^2 - bY^2 = c.$$
Then for any $f \in \mathbb{Q}[x] \setminus \mathcal{P}_2$ with $\deg(f) > 4$ there exists only finitely many integers x for which $f(x) \in S_X \cup S_Y$. Further, there exists an effectively computable constant $C_3(a, b, c, f)$ depending only on a, b, c and f such that $|x| < C_3(a, b, c, f)$ for all such x.

As we shall see, the above theorem immediately implies the next statement, concerning polynomial values in certain binary recurrence sequences. As this problem has an extensive literature, first we survey the related results which are most important from our viewpoint. That is, we restrict our attention to binary recurrences: in the general case, it would be an enormous task to survey the related literature. For related works concerning general recurrences we only refer to the papers of Nemes and Pethő [28], Kiss [22], Corvaja and Zannier [13], Fuchs and Tichy [19] and the references there.

Independently, Pethő [31] and Shorey and Stewart [39] provided an effective bound for the perfect powers in non-degenerate binary recurrence sequences. Stewart [44] and Shorey and Stewart [40] gave effective finiteness results for shifted powers (i.e. polynomial values of the form $x^k + c$) in certain binary recurrences. Nemes and Pethő [29] gave a necessary condition (involving Chebyshev polynomials) for the shape of polynomials $f(x) \in \mathbb{Q}[x]$ having infinitely many values in certain non-degenerate binary recurrence sequences. They did not formulate an effective statement, but they described the structure of the solution set in case of infinitely many solutions. Using the special properties of the problem (in fact, assertion (8)), they could reduce the question to the solution of a polynomial Pell-type equation of the form (6) - however, with $\deg(G(x)) = 2$. Thus to prove their theorem, they could use the corresponding results of Dorey and Whaples [15]. Finally, we mention that there are many results in the literature where values of special polynomials in specific binary recurrence sequences are studied (see e.g. the paper Kovács [24] and the references given there).

Now we formulate our result concerning polynomial values in binary recurrence sequences.

Theorem 3.5. *Let $(U_n)_{n=0}^{\infty}$ be a non-degenerate binary recurrence sequence with $B = \pm 1$ and $f \in \mathbb{Q}[x] \setminus \mathcal{P}_2$ with $\deg(f) > 4$. Then there exists an effectively computable constant $C_4(A, U_0, U_1 f)$ depending only on A, U_0, U_1 and f such that $\max(n, |x|) < C_4(A, U_0, U_1, f)$ for all solutions $n \geq 0$ and $x \in \mathbb{Z}$ of the equation*

$$U_n = f(x).$$

Remark 7. As we see, Theorem 3.5 can be considered to be a generalization of the above mentioned results from [31], [39], [44], [40] and [24]. Further, it provides an effective version of the main result from [29].

4. Proofs

Proof of Theorem 3.1. As we already mentioned, it is easy to see that $\mathcal{S}_i$ is empty for $i \geq 3$. (See e.g. Lemma 3 of [34] where the problem is considered in a slightly less general situation.) However, for the sake of completeness, and also because we shall need some information for the case $i = 2$ obtained on our way, we give a complete argument covering the case $i \geq 3$, as well.

We start with investigating the values $\delta_f(\lambda)$ of special extrema of a polynomial $f(x) \in \mathbb{Q}[x]$ of degree $n > 4$. We show that up to rather restricted cases, we have

$$\delta_f(\lambda) \geq \frac{n}{2}. \tag{11}$$

Let $[\mu_1, \dots, \mu_r]$ be the root type of a special extremum $\lambda \in \mathbb{C}$ of f. Then with some $k \geq 2$ the tuple

$$\left[\frac{k}{\gcd(k,\mu_1)}, \dots, \frac{k}{\gcd(k,\mu_r)}\right]$$

is one of

$$[2,2,1,\dots,1], \quad [q,1,\dots,1]\ (q \geq 1).$$

Assume first that

$$\left[\frac{k}{\gcd(k,\mu_1)}, \dots, \frac{k}{\gcd(k,\mu_r)}\right] = [q,1,\dots,1]\ (q \geq 1).$$

Then $k \mid \mu_i$ $(i = 2, \dots, r)$. Write $\mu_i = ku_i$ $(i = 2, \dots, r)$. Assume that contrary to (11) we have

$$\delta_f(\lambda) \leq \frac{n-1}{2}.$$

Then by (2) and

$$\mu_1 + \mu_2 + \cdots + \mu_r = n \tag{12}$$

we have

$$\mu_1 + \cdots + \mu_r \leq \frac{\mu_1 + \cdots + \mu_r - 1}{2} + r \tag{13}$$

whence

$$\mu_1 + k(u_2 + \cdots + u_r) \leq 2r - 1.$$

This immediately gives $k = 2$ and

$$[\mu_1, \dots, \mu_r] = [1, 2, \dots, 2],$$

yielding

$$\delta_f(\lambda) = \frac{n-1}{2}.$$

Suppose next that

$$\left[\frac{k}{\gcd(k,\mu_1)}, \dots, \frac{k}{\gcd(k,\mu_r)}\right] = [2,2,1,\dots,1].$$

Then k is even, $k/2$ divides μ_1, μ_2 and $k \mid \mu_i$ $(i = 3, \dots, r)$. Write $\mu_i = (k/2)u_i$ for $i = 1, 2$ and $\mu_i = ku_i$ $(i = 3, \dots, r)$. Assume that contrary to (11) we have

$$\delta_f(\lambda) \leq \frac{n-1}{2}.$$

Then using (2) and (12) again, through (13) we obtain

$$k(u_1/2 + u_2/2 + u_3 + \cdots + u_r) \leq 2r - 1.$$

After a simple calculation, using that $n \geq 5$, this yields that $k = 2$ and

$$[\mu_1, \ldots, \mu_r] = [1, 2, \ldots, 2], \; [1, 1, 2, \ldots, 2], \tag{14}$$

with

$$\delta_f(\lambda) = \frac{n-1}{2}, \; \frac{n-2}{2}, \tag{15}$$

respectively.

Altogether, we conclude that for any special extrema of f (11) holds, unless $k = 2$ together with (14) and (15).

Hence using (3) we immediately obtain that for $f(x) \in \mathbb{Q}[x]$ with $n = \deg(f) > 4$, no three distinct special extrema exist.

Suppose that α_1, α_2 are distinct special extrema of f. Then by (3) again, one of α_1 and α_2 must be of one of the types (14). Suppose that say α_1 is of the type

$$[\mu_1, \ldots, \mu_r] = [1, 2, \ldots, 2].$$

Then n is odd, and α_2 must be of the same type. However, then by (3) we see that f has only two extrema, and by Lemma 2.2 f is a linear transform of a Dickson polynomial. Hence in this case, our theorem follows. Assume next that say α_1 is of the type

$$[\mu_1, \ldots, \mu_r] = [1, 1, 2, \ldots, 2]. \tag{16}$$

Now if $\delta_f(\alpha_2) \geq n/2$ then by (3) we get that f has only two extrema again, and by Lemma 2.2 we are done. So we may suppose that α_2 is also of the type (16).

Thus we can write

$$f(x) - \alpha_1 = g_1(x)h_1(x)^2, \quad f(x) - \alpha_2 = g_2(x)h_2(x)^2 \tag{17}$$

with some $g_1, g_2, h_1, h_2 \in \mathbb{C}[x]$, $\deg(g_1) = \deg(g_2) = 2$. Here we need some further investigation. If any of α_1, α_2, say α_1 is transcendental, then as it is well-known, $\mathbb{Q}(\alpha_1)$ is isomorphic to the field of rational fractions $\mathbb{Q}(z)$. We may assume that $g_1, h_1 \in \mathbb{Q}(\alpha_1)[x]$ in (17). So replacing α_1 by some transcendental $\alpha_3 \in \mathbb{C}$ different from α_1, α_2, and replacing f_1, g_1 by $f_3 = \sigma(f_1)$ and $g_3 = \sigma(g_1)$ where σ is the natural isomorphism from $\mathbb{Q}(\alpha_1)[x]$ to $\mathbb{Q}(\alpha_3)[x]$, we get

$$f(x) - \alpha_3 = g_3(x)(h_3(x))^2.$$

This is a contradiction, implying that both α_1 and α_2 are algebraic. A similar argument shows that either $\alpha_1, \alpha_2 \in \mathbb{Q}$, or α_1, α_2 are conjugate quadratic algebraic numbers. In both cases, $\alpha_1 + \alpha_2, \alpha_1\alpha_2 \in \mathbb{Q}$, so multiplying the left and right hand sides of the equalities in (17) we get

$$(f(x))^2 + 2uf(x) + v = g(x)(h(x))^2$$

with some $u, v \in \mathbb{Q}$, whence

$$(f(x)+u)^2 - u^2 + v = g(x)(h(x))^2. \tag{18}$$

A simple consideration shows that here we may assume that the polynomials

$$g(x) = g_1(x)g_2(x) \quad \text{and} \quad h(x) = h_1(x)h_2(x)$$

have rational coefficients. Observe that $\deg(g) = 4$ in (18). As

$$\gcd(f(x)-\alpha_1, f(x)-\alpha_2) = 1,$$

by (17) we have that $g(x)$ is square-free. Further, since $f(x)+u$ has two shifts (with $\alpha_1 - u$ and $\alpha_2 - u$) of root type (16), by (3) it cannot have a shift yielding a square of a polynomial. Thus the inclusion $\mathcal{S}_2 \subseteq \mathcal{P}_2$ follows from Theorem A.

To prove the 'only if' part, let $f(x) \in \mathcal{P}_2$. Clearly, without loss of generality we may assume that f is monic. If f is a shifted Dickson polynomial then by Lemma 2.2 we are immediately done. So let $f(x)$ be a rational shift $f(x) = F(x) + v$ of a (monic) solution $F(x)$ of equation (6), with certain $G, H \in \mathbb{Q}[x]$ and $\gamma \in \mathbb{Q}$ with $\gamma \neq 0$. Reordering (6), after factorization we get

$$(F(x)+\sqrt{\gamma})(F(x)-\sqrt{\gamma}) = G(x)H(x)^2.$$

Since

$$\deg(F(x)+\sqrt{\gamma}) = \deg(F(x)-\sqrt{\gamma})$$

and

$$\gcd(F(x)+\sqrt{\gamma}, F(x)-\sqrt{\gamma}) = 1$$

in $\mathbb{C}[x]$, we must have either

$$F(x)+\sqrt{\gamma} = G_1(x)H_1(x)^2 \quad \text{and} \quad F(x)-\sqrt{\gamma} = G_2(x)H_2(x)^2$$

with some $G_1, G_2, H_1, H_2 \in \mathbb{K}[x]$ with $\deg(G_1) = \deg(G_2) = 2$, where $\mathbb{K} = \mathbb{Q}(\sqrt{\gamma})$, or

$$F(x)+\sqrt{\gamma} = G_1(x)H_1(x)^2 \quad \text{and} \quad F(x)-\sqrt{\gamma} = H_2(x)^2$$

(or vice versa) with some $G_1, H_1, H_2 \in \mathbb{K}[x]$ with $\deg(G_1) = 4$. In the first case we are done, so we may assume that the second possibility holds. Then clearly $\sqrt{\gamma} \in \mathbb{Q}$, and we must have $G_1(x) = G(x)$. Hence we get

$$H_2(x)^2 + 2\sqrt{\gamma} = G(x)H_1(x)^2.$$

That is, by Theorem A we obtain that

$$H_2(x) = \mathcal{F}_{G,\frac{n}{2}}(x) \quad \text{and} \quad -2\sqrt{\gamma} = \gamma\left(G, \frac{n}{2}\right).$$

(Note that n is even in this case.) This shows that this possibility, by our condition

$$F(x) = \mathcal{F}_{G,n}(x) \neq (\mathcal{F}_{G,\frac{n}{2}}(x))^2 - \frac{\gamma(G,\frac{n}{2})}{2}$$

is excluded. Hence the statement concerning $\mathcal{S}_2$ follows.

Finally, assume that f has only one special extremum. Observe that by a similar argument as above, we then get that $\alpha \in \mathbb{Q}$ must hold. Then the statement (in view of Lemma 2.1 and the definition of $\mathcal{P}_1$) becomes obvious. □

To prove Theorem 3.2 we need two further lemmas.

Let $h(x)$ be a nonzero polynomial having algebraic coefficients. Moreover, let $\mathbb{L}$ be an algebraic number field, β a nonzero element of $\mathbb{L}$, and T is a finite set of prime ideals of $\mathbb{L}$. Consider the Diophantine equation

$$h(x) = \beta y^k \tag{19}$$

in unknown $k \in \mathbb{Z}$ with $k \geq 2$ and $x, y \in O_{T,\mathbb{L}}$.

The next lemma is a simple consequence of Theorem 10.5 of Shorey and Tijdeman [41]. For the first results of this type, we refer to [46] and [38].

Lemma 4.1. *If $h(x)$ has at least two distinct roots and y is not zero and is not a root of unity, then in* (19) *we have $k < C_2(h, \beta, T, \mathbb{L})$, where $C_2(h, \beta, T, \mathbb{L})$ is an effectively computable constant depending only on h, β, T, $\mathbb{L}$.*

The following result is a consequence of the main result of Brindza [12]. See also Theorem 8.3 in [41].

Lemma 4.2. *Let $k \geq 2$ also be fixed, and write $[\mu_1, \ldots, \mu_r]$ for the root type of $h(x)$ in* (19). *If the tuple*

$$\left[\frac{k}{\gcd(k,\mu_1)}, \ldots, \frac{k}{\gcd(k,\mu_r)}\right]$$

is none of

$$[2, 2, 1, \ldots, 1], \quad [q, 1, \ldots, 1] \ (q \geq 1),$$

then we have $\max(H(x), H(y)) < C_3(h, k, \beta, T, \mathbb{L})$ *for each solution $x, y \in O_{T,\mathbb{L}}$ of equation* (19), *where $C_3(h, k, \beta, T, \mathbb{L})$ is an effectively computable constant depending only on h, k, β, T, $\mathbb{L}$.*

Proof of Theorem 3.2. If $\alpha \notin \mathbb{Q}$, by our assumptions using Theorem 3.1 we have that α is not special with respect to f. Indeed, otherwise $\hat{\alpha}$ would also be special with respect to f, where $\hat{\alpha}$ is any algebraic conjugate of α. Thus Lemmas 4.1 and 4.2 imply the statement in this case.

The statement when $\alpha \in \mathbb{Q}$ but $f \notin \mathcal{P}_1 \cup \mathcal{P}_2$, immediately follows from Theorem 3.1 and Lemmas 4.1 and 4.2. $\square$

Proof of Theorem 3.3. Write

$$\varphi(x) = \delta(x-\alpha_1)^{t_1} \dots (x-\alpha_r)^{t_r}$$

where δ is a non-zero algebraic number and $\alpha_1, \dots, \alpha_r$ are the distinct (complex) roots of φ, of multiplicities $t_1, \dots, t_r$, respectively. Note that $\alpha_1, \dots, \alpha_r$ are algebraic numbers. As $r \geq 2$, by Lemma 4.1 we get that k is bounded in (10).

So we may assume that k is fixed. Clearly, without loss of generality we may assume that $\delta, \alpha_1, \dots, \alpha_r$ all belong to $\mathbb{K}$. Indeed, otherwise we can replace $\mathbb{K}$ by the number field $\mathbb{L}$ obtained by adjoining all these elements to $\mathbb{K}$, and S by the set T of prime ideals of $\mathbb{L}$ which divide the prime ideals of $\mathbb{K}$ in S (embedded into $\mathbb{L}$). Suppose that $x, y \in O_{S,\mathbb{K}}$ is a solution of (10). Then, as it is well-known (see e.g Lemma 8.1 in [41]), we can find non-zero $\nu_1, \dots, \nu_r, \tau_1, \dots, \tau_r \in O_{\mathbb{K}}$ with heights bounded in terms of $\varphi, \beta, f, \mathbb{K}$ such that

$$f(x) - \alpha_i = \frac{\nu_i}{\tau_i} z_i^{\ell_i} \quad (i = 1, \dots, r), \tag{20}$$

with some $z_i \in O_{\mathbb{K}}$, where $\ell_i = k/\gcd(k, t_i)$. Using our assumptions, without loss of generality we may assume that α_1 and α_2 are non-rational and $\gcd(t_1, t_2) = 1$. Then one of t_1 and t_2, say t_1 is not divisible by k. So $\ell_1 \geq 2$, and by Theorem 3.2 the statement immediately follows from (20) with $i = 1$. $\square$

Proof of Theorem 3.4. Suppose that $f(x) \in S_X$, the case $f(x) \in S_Y$ is similar. Then we can write

$$(af(x) + \sqrt{ac})(af(x) - \sqrt{ac}) = abY^2.$$

Similarly as in the proof of Theorem 3.3 from this we get that with some non-zero algebraic integers u_1, u_2, v_1, v_2 from $\mathbb{K} = \mathbb{Q}(\sqrt{ac})$ with heights bounded in terms of a, b, c, f we have

$$u_1(af(x) + \sqrt{ac}) = v_1 Y_1^2, \quad u_2(af(x) - \sqrt{ac}) = v_2 Y_2^2$$

with some algebraic integers Y_1, Y_2 from $\mathbb{K}$. As $f(x) \notin \mathcal{P}_2$, one of $\pm\sqrt{c/a}$ is not special with respect to f. Thus by Lemma 4.2, one of the above equations implies that x is bounded as required, and the statement follows. $\square$

Proof of Theorem 3.5. The statement concerning $|x|$ immediately follows from Theorem 3.4 using identity (8). Then using the well-know fact that $|U_n|$ tends to infinity as n tends to infinity (see e.g. Theorem

3.1 of [41], which is a reformulation of a result of Stewart [43]), the statement concerning n also follows. □

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Institute of Mathematics, University of Debrecen, H-4002 Debrecen, P.O. Box 400, Hungary

E-mail address: `hajdul@science.unideb.hu`