THE RESOLUTION OF THE DIOPHANTINE
EQUATION $x(x+d)\ldots(x+(k-1)d) = by^2$ FOR FIXED d

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ABSTRACT. In this paper we provide an algorithm for the resolution of the title equation, which works for any d. To illustrate the simplicity of the method, we extend a result of Saradha by giving all solutions with $23 \leq d \leq 30$, $k \geq 3$, $(x,d) = 1$ and $P(b) \leq k$, in positive integers x, d, k, b, y.

1. Introduction

A classical problem of Number Theory is to determine those finite arithmetical progressions, for which the product of the terms yields a perfect power, or an 'almost' perfect one. Erdős and Selfridge in 1975 (cf. [2]) proved that the product of two or more consecutive positive integers is never a perfect power, i.e. the equation

$$x(x+1)\ldots(x+k-1) = y^l$$

has no solutions with $k, l \geq 2$ and $x \geq 1$. There are many results in the literature concerning the various generalizations of the above equation, see e.g. the extensive survey papers [8], [9], [10], [11], or the very recent papers [1], [4], [5], [6], [7], and the references given there.

Let $P(b)$ denote the greatest prime factor of a positive integer $b > 1$, and put $P(1) = 1$. In this paper we investigate the following equation:
(1)
$$x(x+d)\ldots(x+(k-1)d) = by^2 \text{ with } d > 1, k \geq 3, \ (x,d) = 1, \ P(b) \leq k,$$

in positive integers x, d, k, b, y. In [7] Saradha proved that equation (1) has only the solutions

$$(x, d, k, b, y) = (2, 7, 3, 2, 12), (18, 7, 3, 1, 120), (64, 17, 3, 2, 504),$$

provided that $d \leq 22$ holds. In fact she gave an algorithm for the resolution of (1) for fixed values of d, and used her method to compute all solutions with $1 < d < 23$. The main steps of her method are the following. Put $C = (k-1)^2 d^2/4$, and suppose first that for a solution (x, d, k, b, y) of (1) $x \geq C$ holds. For such a solution Saradha derived an upper bound $k_0(d)$

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for k, which varies between 18 and 314 as d ranges through the interval $[7, 22]$. It is not guaranteed that her method provides an upper bound $k_0(d)$ for an arbitrary value of d. Subsequently she proved that $4 \leq k \leq 6$ if $d = 7$, $4 \leq k \leq 8$ if $d \in \{11, 13, 17, 19\}$, respectively, and that (1) has no solutions for other values of d with $1 < d < 23$. The remaining cases were verified by numerical calculations.

In [1] Brindza, Hajdu and Ruzsa proved the following result.

Theorem A. *If (x, d, k, b, y) is a solution to* (1) *with $k \geq 8$, then $x < D$, where $D = 4d^4(\log d)^4$.*

This implies that we can take $k_0(d) = 8$ if $x \geq D$. This uniform bound makes it possible, at least in principle, to resolve equation (1) for any fixed d. This paper provides an algorithm to do so. We shall illustrate the algorithm by determining all solutions of (1) with $23 \leq d \leq 30$.

2. Result and description of the algorithm

The main steps of our method for the resolution of (1) with fixed d are the following. First we provide a simple search algorithm to find the solutions with small x. According to Theorem A we have $k \leq 7$ for the large solutions. We show that each such solution corresponds to a point on one among 16 elliptic curves. The elliptic equations can be resolved by a mathematical software package.

Theorem. *Suppose that $23 \leq d \leq 30$. The only solutions to equation* (1) *are the following ones:*
$d = 23$, $k = 3$: $(x, b, y) = (2, 6, 20)$, $(4, 6, 30)$, $(75, 6, 385)$, $(98, 2, 924)$, $(338, 3, 3952)$, $(3675, 6, 91805)$,
$d = 23$, $k = 4$: $(x, b, y) = (75, 6, 4620)$,
$d = 24$, $k = 3$: $(x, b, y) = (1, 1, 35)$.

Remark. The above theorem provides a solution to (1) with $k > 3$, namely $(x, d, k, b, y) = (75, 23, 4, 6, 4620)$. This is not surprising, as it was pointed out by F. Beukers that equation (1) has infinitely many solutions with $k = 4$.

Proof of the Theorem. Suppose first that (x, d, k, b, y) is a solution to (1) with $23 \leq d \leq 30$ and $x < D$, where D is defined in Theorem A. Using the estimate $k < 4d(\log d)^2$ due to Saradha [7], the left hand side of equation (1) is bounded by a constant depending only on d. Hence after fixing d, all solutions to (1) can be found by a simple search. However, as a huge amount of computation is needed, it is worth to be more economical.

Let d be fixed. A positive integer a is called a *bad number*, if some prime p with $p \geq 4d(\log d)^2$ occurs in the prime factorization of a on an odd exponent. Suppose that $x + id$ is a bad number for some i with $0 \leq i \leq k-1$, and choose a prime p with the above properties for $a = x+id$. Then by Saradha's result we have $p > k$. By the condition $(x, d) = 1$, there is no other factor $x + jd$ which is divisible by p. Hence p divides the left-hand side on an odd exponent, which yields a contradiction with $P(b) \leq k$. This argument shows that no factor $x + id$ is bad.

We work with the residue classes (mod d) separately. Let m be a positive integer with $(m, d) = 1$, $m < d$. We make a list L_3 consisting of all those positive integers $x' < D$ with $x' \equiv m$ (mod d) for which none of the numbers x', $x' + d$, $x' + 2d$ is bad. Then we make a list L_4 of all the numbers $x' \in L_3$ with $x' + d \in L_3$. Subsequently we make a list L_5 of all the numbers $x' \in L_4$ with $x' + d \in L_4$ and so on. For $23 \leq d \leq 30$ the process stops around L_{15}. Observe that $x' \in L_i$ if and only if none of the numbers $x', x' + d, \ldots, x' + (i-1)d$ is bad. Hence every solution (x, d, k, b, y) of (1) with $x < D$ satisfies $x \in L_k$. Finally, for each number $x' \in L_k$ we check if $x'(x' + d)\ldots(x' + (k-1)d)$ has a square-free part which has a greatest prime factor $\leq k$, for all lists L_k. The numbers which pass this last test provide all the solutions with $x \equiv m$ (mod d). Finally we take the union over all m to collect all solutions of (1) with $x < D$.

Now suppose that (x, d, k, b, y) is a solution to (1) with $x \geq D$. Then, by Theorem A, $k \leq 7$. Write now $x + id = a_i x_i^2$ ($i = 0, \ldots, k - 1$) with square-free a_i's and suppose that $P(a_i) > k$ for some i. By the assumption $(x, d) = 1$ this implies $P(b) > k$, which is a contradiction. This shows that $P(a_i) \leq k$. Hence we get

$$(2) \qquad\qquad x(x + d)(x + 2d) = cz^2,$$

where c and z are positive integers with $P(c) \leq k$, c square-free. Moreover, by the assumption $(x, d) = 1$ we get that $(c, d) = 1$ in (2). Hence $c \in \{1, 2, 3, 5, 6, 7, 10, 14,$
$15, 21, 30, 35, 42, 70, 105, 210\}$. Thus for each d we have to resolve 16 elliptic equations of the form

$$u^3 - c^2 d^2 u = v^2 \quad \text{in} \quad u, v \in \mathbb{Z},$$

where u and v are given by $u = c(x + d)$ and $v = c^2 z$, respectively. Using the program package SIMATH (cf. [12]) these elliptic equations can be resolved easily. For a detailed description of the algorithm implemented in SIMATH, see e.g. [3].

The simple search method already yielded all the solutions mentioned in the theorem. As in these solutions $k \leq 7$, all of them, but no more were also provided by the resolution of the elliptic equations. $\square$

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