DIOPHANTINE EQUATIONS FOR POLYNOMIALS WITH RESTRICTED COEFFICIENTS, I (POWER VALUES)

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Dedicated to the memory of Andrzej Schinzel.

Abstract. In this paper we give effective finiteness results for the power values of polynomials with coefficients composed of a fixed finite set of primes; in particular, of Littlewood polynomials.

1. Introduction

There is an extensive literature on polynomials with restricted coefficients, in particular, with coefficients belonging to one of the sets $\{-1, 1\}$, $\{0, 1\}$ or $\{-1, 0, 1\}$. Note that in the first case the polynomials are called Littlewood polynomials, while in the second case (assuming that the constant term is non-zero) the polynomials are the Newton polynomials. Here we mention only a few papers and directions; we suggest the interested reader to study these papers and their references. The zeroes (in particular, the number of real zeroes) of polynomials with coefficients belonging to $\{-1, 0, 1\}$ have been studied by Bloch and Pólya [1], Schur [14], Szegő [16], Erdős and Turán [8], Drungilas and Dubickas [6] (see also papers of Borwein and Erdélyi [2, 3]). A related question concerning the order of vanishing of such polynomials at 1 has been considered by Borwein and Mossinghoff [4]. Similar studies for Littlewood polynomials have also been done; see e.g. Peled, Sen and Zeitouni [12]. Finally, we also mention that divisibility properties of such polynomials are also of interest; see e.g. Dubickas and Jankauskas [7] for a question concerning Newton and Littlewood polynomials, and Mossinghoff [11] for the case of described cyclotomic factors.

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In this paper we initiate the study of Diophantine equations involving polynomials with restricted coefficients. As a generalization of Littlewood polynomials, we shall consider polynomials whose coefficients are composed of primes coming from a fixed finite set. We shall be interested in perfect power values of such polynomials - that is, in Schinzel-Tijdeman equations and hyper- and superelliptic equations related to them. We shall provide effective upper bounds for the solutions of such equations. For this, we need to combine the effective theory of such equations and the theory of S-unit equations with new assertions concerning the root structures of such polynomials. In view of the general interest (indicated above) in polynomials with restricted coefficients, we find the latter results (Lemmas 3.3, 3.4, 3.5, 3.6) of possible independent interest. In fact, this is one of the main reasons why we cut our research into parts: in this way we can present these 'background' results, as well. In the continuation of this paper, we take up the general problem of polynomial values of polynomials with restricted coefficients (which requires different techniques, and different background knowledge about the root structures and decomposability properties, as well).

2. Notation and main results

Let $S = \{p_1 < p_2 < \ldots < p_k\}$ be a finite set of primes, and write $\mathbb{Z}_S$ for the set of integers having no prime divisors outside S. Note that we have $\pm 1 \in \mathbb{Z}_S$ but $0 \notin \mathbb{Z}_S$ for any S. In particular, we have $\mathbb{Z}_S = \{-1, 1\}$ for $S = \emptyset$. Write P_S for the set of polynomials in $\mathbb{Z}[x]$ with coefficients belonging to $\mathbb{Z}_S$.

Now we give our main results. The first theorem shows that under a necessary condition, the polynomials in P_S may attain only power values with bounded exponents.

Theorem 2.1. *Let $f(x) \in P_S$ of degree d and b be a non-zero rational number. Then there exist effectively computable constants $C_1 = C_1(p_k)$ and $C_2 = C_2(b, d, p_k)$ depending only on p_k and on b, d, p_k, respectively, such that if $d > C_1$ then the equality*

$$f(x) = by^n \tag{1}$$

with $x, y, n \in \mathbb{Z}$ and $|y| > 1$ implies $n < C_2$.

Remark 1. The condition $d > C_1$ is necessary. Indeed, let d be arbitrary, and choose S such that $\binom{d}{i} \in \mathbb{Z}_S$ for all $i = 0, \ldots, d$. Then for any $a \in \mathbb{Z}_S$ we have $(x+a)^d \in P_S$, however, (1) clearly has infinitely many solutions x, y with any multiple n of d. So we see that we do need a lower bound for d in order to have the statement of Theorem 2.1 being valid.

Now we would like to bound also the solutions x, y of (1). However, for this we need to switch to $S = \emptyset$, i.e. to the case of Littlewood polynomials. This is in fact necessary; a condition as in Theorem 2.1 saying that the degree of the polynomial should be large enough is not sufficient - for the reason see part i) of Remark 2 after the statement.

Theorem 2.2. *Let $f(x) \in P_S$ with $S = \emptyset$ (i.e. $f(x)$ is a Littlewood polynomial, with all coefficients being ± 1). Assume further that $\deg f \geq 3$, and let b be a non-zero rational number. Then all solutions $x, y, n \in \mathbb{Z}$ of the equation*

$$f(x) = by^n \tag{2}$$

with $n \geq 2$, satisfy

$$\max(|x|, |y|, n) \leq C_4,$$

except when $n = 2$ and f is one of the forms

$$f(x) = \pm(x^{2k+1} + \ldots + x^{k+1} - x^k - \ldots - 1),$$
$$\pm\,(x^{2k+1} - x^{2k} + \ldots + (-1)^{k+2}x^{k+1} + (-1)^k x^k + \cdots + 1)$$

with some $k \geq 1$. Here $C_4 = C_4(b, d)$ is an effectively computable constant depending only on b and the degree d of f.

Remark 2. Here we mention two things.

i) The statement is not valid for arbitrary S, the restriction $S = \emptyset$ cannot be omitted - even if we would restrict to polynomials of degrees 'large enough'. To see this, let $f_1(x) \in \mathbb{Z}[x]$ be any not identically zero polynomial. Let S be an arbitrary, finite set of primes, containing 2, such that both $f_1(x)$ and $(f_1(x))^2$ belong to P_S. Then inductively define

$$f_{i+1}(x) = \left((x^{d_i+1} + 1) f_i(x)\right)^2 \quad (i \geq 1),$$

where $d_i = \deg(f_i)$ $(i \geq 1)$. Observe that then all the polynomials $f_i(x)$ $(i \geq 1)$ are full squares in P_S, and clearly, the sequence $d_1, d_2, d_3, \ldots$ of their degrees is unbounded. That is, for this set S there exists a polynomial of arbitrarily large degree in P_S which is a square. Hence clearly, equation (2) with $b = 1$ and $n = 2$ has infinitely many solutions in $x, y \in \mathbb{Z}$.

ii) In the exceptional cases equation (2) (with appropriate choices of b) have infinitely many solutions with $n = 2$ in $x, y \in \mathbb{Z}$. Indeed, for any $k \geq 1$ we have

$$\pm(x^{2k+1} + \ldots + x^{k+1} - x^k - \ldots - 1) = \pm(x - 1)(x^k + \cdots + x + 1)^2,$$

which gives a square value whenever $\pm(x-1)$ is a square. Similarly, for any $k \geq 1$ we have

$$\begin{aligned}\pm\,(x^{2k+1} - x^{2k} + \ldots + (-1)^{k+2}x^{k+1} + (-1)^k x^k + \cdots + 1) = \\ = \pm(x+1)(x^k - x^{k-1} + \cdots + (-1)^k)^2,\end{aligned}$$

which gives a square value whenever $\pm(x+1)$ is a square.

3. Lemmas, auxiliary results and proofs

To prove Theorem 2.1, we need two lemmas. Here and later on, by the height $H(F(x))$ of a polynomial $F(x)$ with integer coefficients we mean the maximum of the absolute values of its coefficients.

Lemma 3.1. *Let $F(x) \in \mathbb{Z}[x]$ having two distinct (complex) roots of degree D and height H, and B be a non-zero rational number. Then the equality*

$$F(x) = By^n$$

with $x, y \in \mathbb{Z}$, $|y| > 1$ implies that $n < C_4$, where $C_4 = C_4(B, D, H)$ is an effectively computable constant depending only on B, D and H.

Proof. The statement immediately follows from the Schinzel-Tijdeman theorem (the main result of [13]; see also [17] and Chapter 9 of [15]). □

Lemma 3.2. *Let S be as above, and A, B be non-zero rational numbers. Then the solutions $x, y \in \mathbb{Z}_S$ of the equation*

$$Ax - By = 1$$

satisfy

$$\max(|x|, |y|) < C_5,$$

where $C_5 = C_5(A, B, p_k)$ is an effectively computable constant depending only on A, B and p_k.

Proof. The statement is an immediate consequence of a classical result of Győry [10]; see also Chapter 4 of [9]. □

Proof of Theorem 2.1. The statement immediately follows by Lemma 3.1, as soon as $f(x)$ has two distinct roots. Thus we can assume that $f(x)$ is of the form $f(x) = u(x+v)^d$, with some $u \in \mathbb{Z}$ and $v \in \mathbb{Q}$. Writing $v = v_1/v_2$ in its primitive form, we clearly have $v_2^d \mid u$ and $u, v_1, v_2 \in \mathbb{Z}_S$. Then checking the coefficients of x^{d-1} and x^{d-2} in f, we easily see that $d, \binom{d}{2} \in \mathbb{Z}_S$, as well. Hence we obtain that either $d-1 \in \mathbb{Z}_S$, or $d-1$ is even and $(d-1)/2 \in \mathbb{Z}_S$. In the first case $d, d-1 \in \mathbb{Z}_S$ satisfy the equation

$$w_1 - w_2 = 1, \tag{3}$$

while in the second case $d, (d-1)/2 \in \mathbb{Z}_S$ are solution to the equation

$$w_1 - 2w_2 = 1 \tag{4}$$

in $w_1, w_2 \in \mathbb{Z}_S$. However, by Lemma 3.2 we get that for the solutions of the above equations

$$\max(|w_1|, |w_2|) < C_6$$

holds, where $C_6 = C_6(p_k)$ is an effectively computable constant depending only on p_k. So if $d > C_6$, then d cannot come from a solution of either (3) and (4), which implies that $f(x)$ is not of the form $u(x+v)^d$. Hence taking $C_1 = C_6$, the statement follows. $\square$

Now we turn to the proof of Theorem 2.2. For this, we shall need five further lemmas. The first four of them are new, and we find them of possible independent interest.

Lemma 3.3. *Let m be a non-negative integer and let*

$$G(x) = b_0x^t + b_1x^{t-1} + \ldots + b_{t-1}x + b_t \tag{5}$$

with $b_0, b_1, \ldots, b_t \in \mathbb{Z}$, such that all the coefficients of the polynomial $(x-1)^mG(x)$ belong to $\{-1, 1\}$. Then $b_1 = 0$ implies $m = 1$.

Proof. Clearly, without loss of generality we may assume that $b_0 = 1$. We shall do so during the proof, without any further mentioning. Write B_m ($m \geq 0$) for the set of coefficients b_1 which occur in some polynomial $G(x)$ (being monic, of arbitrary degree) satisfying the conditions of the statement. We prove the lemma by describing the sets B_m inductively.

Obviously, we have $B_0 = \{-1, 1\}$. Assume that we already described the set B_m for some $m \geq 0$, and consider a polynomial $G(x)$ given by (5), such that all the coefficients of $(x-1)^{m+1}G(x)$ belong to $\{-1, 1\}$. Then of course, the same is true for the polynomial

$$\begin{aligned}(x-1)^{m+1}G(x) &= (x-1)^m((x-1)G(x)) = \\ &= (x-1)^m(x^{t+1} + (b_1-1)x^t + \ldots + (b_t - b_{t-1})x - b_t).\end{aligned}$$

Thus we obtain that $b_1 - 1 \in B_m$. From this, we immediately get that

$$B_{m+1} = B_m + \{1\} (= \{h+1 : h \in B_m\}) \quad (m \geq 0),$$

which inductively gives

$$B_m = \{m-1, m+1\} \quad (m \geq 0).$$

Hence the statement follows. $\square$

The next lemma describes how the coefficients of a polynomial $G(x)$ can 'spread' if it holds the property that $(x-1)^m G(x)$ is a Littlewood polynomial. In fact, for our present purposes we only need a specific consequence of this statement (namely, for the coefficient of the second largest power of x), but we find it interesting to describe this phenomenon completely.

Lemma 3.4. *Let $G(x) \in \mathbb{Z}[x]$ and m be a non-negative integer. If all the coefficients of $(x-1)^m G(x)$ belong to $\{-1, 1\}$ then, writing*

$$G(x) = b_0 x^t + b_1 x^{t-1} + \ldots + b_{t-1} x + b_t,$$

for all $i = 0, 1, \ldots, t$ we have

$$-\min\left(\binom{m+i}{m}, \binom{m+t-i}{m}\right) \le b_i \le \min\left(\binom{m+i}{m}, \binom{m+t-i}{m}\right).$$

Here we use the convention $\binom{0}{0} = 1$.

Remark 3. We note that, as one can easily check (e.g. by following the proof), the bounds given for the coefficients of $G(x)$ are sharp.

Proof. First we show that

$$-\binom{m+i}{m} \le b_i \le \binom{m+i}{m} \quad (i = 0, 1, \ldots, t), \tag{6}$$

by induction on m. For $m = 0$ the statement is obvious. Assume that (6) holds for some $m \ge 0$, and assume that all the coefficients of $(x-1)^{m+1} G(x)$ belong to $\{-1, 1\}$. Then in view of

$$\begin{aligned}(x-1)^{m+1} G(x) &= (x-1)^m ((x-1) G(x)) = \\ &= (x-1)^m (b_0 x^{t+1} + (b_1 - b_0) x^t + \ldots + (b_t - b_{t-1}) x - b_t)\end{aligned}$$

the induction hypothesis by $b_i = (b_i - b_{i-1}) + b_{i-1}$ $(i \ge 1)$ successively yields

$$\begin{gathered}-1 \le b_0 \le 1, \quad -1 - \binom{m+1}{1} \le b_1 \le 1 + \binom{m+1}{1}, \\ -1 - \binom{m+1}{1} - \binom{m+2}{2} \le b_2 \le 1 + \binom{m+1}{1} + \binom{m+2}{2}, \ldots .\end{gathered}$$

This by a well-known identity gives

$$-\binom{m+1+i}{m+1} \le b_i \le \binom{m+1+i}{m+1} \quad (0 \le i \le t),$$

that is, (6) is valid also with m replaced by $m+1$. So (6) holds for all m. Now observing that $b_t = \pm 1$ and starting the argument at the constant term and going backwards (or alternatively, working with reciprocal polynomials), a similar argument gives

$$-\binom{m+t-i}{m} \le b_i \le \binom{m+t-i}{m} \quad (i = 0, 1, \ldots, t).$$

Hence the lemma follows. □

Lemma 3.5. *Let $n \ge 2$ and $g(x) \in \mathbb{Z}[x]$ be non-zero polynomial. If all the coefficients of $(x-1)^{n-1}g^n(x)$ belong to $\{-1, 1\}$ then we have $n = 2$.*

Proof. Write

$$F(x) = (x-1)^{n-1}g^n(x),$$

and assume that all the coefficients of $F(x)$ belong to $\{-1, 1\}$. Then

$$(x-1)F(x) = ((x-1)g(x))^n,$$

whence

$$H\left(((x-1)g(x))^n\right) = H\left((x-1)F(x)\right) \le 2. \tag{7}$$

Put

$$G(x) = (x-1)g(x).$$

Obviously, the constant term of $G(x)$ is ± 1. Let ℓ be the smallest positive exponent for which the coefficient of x^ℓ in $G(x)$ is non-zero. (Clearly, such an ℓ exists because $\deg G \ge 1$.) Write u_ℓ for the coefficient of x^ℓ in $G(x)$. Then the coefficient of x^ℓ in

$$((x-1)g(x))^n = (G(x))^n$$

is $\pm n u_\ell$. However, then by (7) we get $|\pm n u_\ell| \le 2$. This implies $n \le 2$, in fact $n = 2$, and the statement follows. □

Lemma 3.6. *Let $g(x) \in \mathbb{Z}[x]$ be a non-constant polynomial and m, n be integers with $0 \le m < n$. If all the coefficients of the polynomial $(x-1)^m(g(x))^n$ belong to $\{-1, 1\}$ then $n = 2$, $m = 1$ and $g(x)$ is of the form*

$$g(x) = \pm(x^\ell + \ldots + x + 1)$$

with some $\ell \ge 1$.

Proof. Write

$$g(x) = u_0 x^\ell + u_1 x^{\ell-1} + \ldots + u_{\ell-1}x + u_\ell$$

and

$$G(x) = (g(x))^n = b_0 x^t + b_1 x^{t-1} + \ldots + b_{t-1}x + b_t.$$

Clearly, $u_0 = \pm 1$ and $b_0 = \pm 1$. Further, we have $b_1 = \pm n u_1$. However, Lemma 3.4 implies that

$$|b_1| \le m + 1.$$

Hence either $u_1 = b_1 = 0$, or by $m < n$ we obtain $m = n - 1$. In the former case, by Lemma 3.3 we get that $m = 1$. Then the property that $(x-1)G(x)$ has only ± 1 coefficients, easily implies that $b_2 = \pm 1$. However, on the other hand (as $t \geq 2$ and $u_1 = 0$) we also have

$$b_2 = \pm n u_2,$$

which by $n \geq 2$ is not possible. Hence $u_1 = 0$ cannot hold, and we are left with the case $m = n - 1$. Then by Lemma 3.5 we obtain that $n = 2$. Thus we have $b_0 = 1$. Further, without loss of generality we may also assume that $u_0 = 1$, as well. (Then, having described the possible polynomials $g(x)$, we only need to insert a $\pm$ sign in front of them.) So we can write

$$\begin{aligned} x^t + b_1 x^{t-1} + \ldots + b_{t-1}x + b_t &= \left(x^\ell + u_1 x^{\ell-1} + \ldots + u_{\ell-1}x + u_\ell\right)^2 = \\ &= x^{2\ell} + 2u_1 x^{2\ell-1} + \left(u_1^2 + 2u_2\right) x^{2\ell-2} + \left(2u_3 + 2u_1u_2\right) x^{2\ell-3} + \ldots. \end{aligned} \tag{8}$$

We show that here we necessarily have $u_i = 1$ $(i = 1, \ldots, \ell)$. For this, recall that all the coefficients of $(x-1)(g(x))^2$, that is of

$$(x-1)(x^t + b_1x^{t-1} + \ldots + b_{t-1}x + b_t) = (x^{t+1} + (b_1 - 1)x^t + \ldots + (b_t - b_{t-1})x - b_t)$$

are ± 1. That is,

$$b_1 - 1, b_t, b_i - b_{i-1} \in \{-1, 1\} \quad (2 \leq i \leq t). \tag{9}$$

In particular, b_i is even if i is odd, and b_i is odd if i is even. Since $b_1 \neq 0$, $b_1 - 1 = 1$ whence $b_1 = 2$. Comparing the coefficients of x^{t-1} in (8), this gives $u_1 = 1$. Inductively assume that $b_i = i + 1$ and $u_i = 1$ for some i with $1 \leq i < \ell$. Then (9) yields that $b_{i+1} \in \{i, i+2\}$, while (8) gives

$$b_{i+1} = u_0 u_{i+1} + u_1 u_i + \ldots + u_{i+1} u_0.$$

This shows that $u_{i+1} \in \{0, 1\}$. However, $u_i = 0$ is impossible. Indeed, otherwise again by (8) we get that $b_{2(i+1)}$ is even, since in the coefficient of $x^{2(i+1)}$ all products $u_j u_{2(i+1)-j}$ occurs twice except for u_i^2 - which is assumed to be zero. However, $b_{2(i+1)}$ is known to be odd. Thus we see that $u_{i+1} = 1$ (and $b_{i+1} = i + 2$) must be valid. So we get that the only possibility is given by

$$g(x) = \pm(x^\ell + \ldots + x + 1).$$

Finally, we have to check that the polynomial $g(x)$ given above satisfies the requirements of the lemma (with $n = 2$ and $m = 1$). Indeed, we have

$$(x-1)(g(x))^2 = (x-1)\left(\frac{x^{\ell+1}-1}{x-1}\right)^2 =$$
$$= \left(x^{\ell+1}-1\right)\frac{x^{\ell+1}-1}{x-1} = x^{2\ell+1}+\ldots+x^{\ell+1}-x^{\ell}-\ldots-x-1.$$

Thus the lemma is proved. □

The following lemma is a theorem of Brindza [5]. To its formulation, we need some further notation. For any finite set S of primes, write $\mathbb{Q}_S$ for those rationals whose denominators (in their primitive forms) are composed exclusively from the primes in S. By the height $h(s)$ of a rational number s we mean the maximum of the absolute values of the numerator and the denominator of s (written again in primitive form).

Lemma 3.7. *Let $F(x) \in \mathbb{Z}[x]$ of degree D and height H, and write*

$$F(x) = A\prod_{i=1}^{\ell}(x-\gamma_i)^{r_i},$$

where A is the leading coefficient of F, and $\gamma_1,\ldots,\gamma_\ell$ are the distinct complex roots of $F(x)$, with multiplicities $r_1,\ldots,r_\ell$, respectively. Further, let n be an integer with $n\geq 2$, and put

$$q_i = \frac{n}{(n,r_i)} \quad (i=1,\ldots,\ell).$$

Suppose that $(q_1,\ldots,q_\ell)$ is not a permutation of any of the ℓ-tuples

$$(q,1,\ldots,1)\ (q\geq 1), \quad (2,2,1,\ldots,1).$$

Then for any finite set S of primes and non-zero rational B, the solutions $x,y\in\mathbb{Q}_S$ of the equation

$$F(x) = By^n$$

satisfy

$$\max\left(h(x),h(y)\right) < C_7(B,n,D,H,S),$$

where $C_7(B,n,D,H,S)$ is an effectively computable constant depending only on B,n,D,H,S.

Proof of Theorem 2.2. First we show that n can be bounded in the required way. Following the lines of the proof of Theorem 2.1, we see that it is sufficient to exclude the case when $f(x)$ is of the form $(x\pm 1)^d$. (In the notation of the proof of Theorem 2.1, here we need to have $u=\pm 1$ and $v=\pm 1$.) However, this is clearly impossible.

Hence from this point on we may suppose that $n \geq 2$ is fixed. Thus our statement immediately follows from Lemma 3.7, except in the following two cases:

i) $n = 2$ and $f(x) = h(x)(g(x))^2$ where $\deg h = 2$ and $h(x), g(x) \in \mathbb{Z}[x]$;
ii) n is arbitrary and $f(x) = (h(x))^m(g(x))^n$, where $\deg h \leq 1$, $0 \leq m < n$ and $h(x), g(x) \in \mathbb{Z}[x]$.

Clearly, without loss of generality we may assume that all the polynomials f, g, h are monic.

In the case i) write $h(x) = x^2 + v_1x + v_2$ and

$$g(x) = x^\ell + u_1x^{\ell-1} + \ldots + u_\ell.$$

Clearly, $v_2 = \pm 1$. Further, we have

$$g^2(x) \equiv x^{2\ell} + u_1^2x^{2\ell-2} + \ldots + u_\ell^2 \pmod 2.$$

Thus if v_1 is even, then all odd coefficients of $h(x)g^2(x)$ are even which is a contradiction. So v_1 must be odd. However, then

$$h(x)g^2(x) \equiv x^{2\ell+2} + u_1^2x^{2\ell+1} + (u_1^2 + 1)x^{2\ell} + \ldots \pmod 2.$$

Thus either the coefficient of $x^{2\ell+1}$, or that of $x^{2\ell}$ is even, but this is impossible. So case i) cannot hold.

Now consider the case ii). We can be suppose that the polynomials f, g, h are monic and $h(x) = x - 1$. (Indeed, if $h(x) = x + 1$, which is the only other possibility, then after the substitution $x \to -x$ and multiplying equation (2) by an appropriate power of -1, we are easily back to this case.) Thus the statement follows from Lemma 3.6. (The second possibility for $f(x)$ comes from the case $h(x) = x + 1$.) $\square$

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