DISTINCT UNIT GENERATED TOTALLY COMPLEX QUARTIC FIELDS

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Abstract. The problem of characterization of rings whose elements can be expressed as sums of their units has a long history, and is also of current interest. In this paper we take up the question of describing totally complex quartic number fields K with the property that every algebraic integer in K is the sum of *distinct* units of K. In particular, we give a short list containing all such fields.

1. Introduction and main results

An algebraic number field K is called unit generated (in short UG) if the maximal order of K is generated by its units *additively*. Further, we call K distinct unit generated (in short DUG) if every algebraic integer of K can be written as a sum of distinct units of K.

In the 1960's Jacobson [4] observed that the number fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$ are DUG-fields. Moreover, Jacobson [4] conjectured that these are the only quadratic DUG-fields, which was proved by Śliwa [8]. Obviously the question arises for a complete characterization of DUG-fields (see e.g. [7, page 539, Problem 18]). Currently a complete characterization for all number fields seems to be far out of reach, but for certain families the problem was solved. E.g. Śliwa [8] showed that no pure cubic field, i.e. a field of the type $\mathbb{Q}(\sqrt[3]{d})$ is DUG by estimating sums of units. Belcher [1] proved that there are only seven complex cubic fields that are DUG. In fact, these seven fields are exactly those UG-fields with discriminant > -132. In particular, Belcher established his result by relating the solvability of certain unit equations to the property of being a DUG-field (for details see Proposition 1). A further important ingredient in Belcher's and Śliwa's proofs is to estimate sums of units by considering them as digit expansions. Recently Thuswaldner and Ziegler [10] generalized these methods by further developing the digit expansions approach. Unfortunately their approach works only for real fields.

In this paper we consider the case of totally complex quartic fields and we intend to further generalize and combine the mehtods of Śliwa [8], Belcher [1] and Thuswaldner and Ziegler [10]. The backbone of this paper is an algorithm which will be explained in Section 4. This algorithm will allow us to determine whether

2010 *Mathematics Subject Classification.* 11R16, 11R27, 11D85.

Key words and phrases. Unit sum number; additive unit structure; digit expansions.

1) Research supported in part by the OTKA grants K75566, K100339, NK101680, and by the TÁMOP 4.2.1./B-09/1/KONV-2010-0007 project. The project is implemented through the New Hungary Development Plan, cofinanced by the European Social Fund and the European Regional Development Fund.

2) The author was supported by the Austrian Sience Found (FWF) under the project J2886-NT.

a given algebraic integer α in a fixed number field is a sum of distinct units or not. This enables us to prove for several number fields that they are not DUG-fields and we will therefore obtain a finite list of possible DUG-fields.

Throughout the paper, let ζ_m be a primitive m-th root of unity; in particular, we have $\zeta_2 = -1$, and we write i for ζ_4. Our main result is the following.

Theorem 1. *Let α be a root of the polynomial $X^4 + X^2 - X + 1$. Then $K = \mathbb{Q}(\beta)$ with*

$$\beta \in \{\alpha, \zeta_5, \zeta_8, \zeta_{12}, \sqrt{-1-\sqrt{2}}, \sqrt{-\frac{1+\sqrt{5}}{2}}, \zeta_3 + \sqrt{5}, i + \sqrt{5}\}$$

is a DUG-field. Further, if K is a totally complex quartic DUG-field not given above, then K is of the form

- $\mathbb{Q}(\gamma)$ *where γ is the root of one of the polynomials $X^4 - X + 1$, $X^4 + 2X^2 - 2X + 1$, $X^4 - X^3 + X + 1$, $X^4 - X^3 + X^2 + X + 1$, $X^4 - X^3 + 2X^2 - X + 2$ or,*
- $\mathbb{Q}(\sqrt{a+ib})$, *with $(a, b) = (1, 1), (1, 2), (1, 4), (7, 4)$ or,*
- $\mathbb{Q}(\sqrt{a+\zeta_3 b})$, *with $(a, b) = (2, 1)$, $(4, 1)$, $(8, 1)$, $(3, 2)$, $(4, 3)$, $(7, 3)$, $(11, 3)$, $(5, 4)$, $(9, 4)$, $(13, 4)$, $(12, 5)$, $(11, 7)$, $(9, 8)$, $(15, 11)$, $(19, 11)$, $(17, 12)$, $(17, 16)$ or,*
- $\mathbb{Q}(\zeta_3, \sqrt{d})$, *with $d = 6$ or $d = 21$.*

Based upon our computations (see Section 8), we propose the following conjecture.

Conjecture 1. *All fields listed in Theorem 1 are DUG.*

In the next section we establish some bounds for the discriminants and regulators of number fields under consideration. In Section 3 we provide some results on the maximal orders of certain fields. The heart of the paper is in Section 4, where we present a method to test whether certain algebraic integers admit a representation as a sum of distinct units. The case of non-CM-fields (first with, then without non-trivial roots of unity) is considered in the following two sections. Since the bounds for the discriminant from Section 2 do not apply to CM-fields we have to consider this case separately in Section 7. In the last section we will discuss the implementation of our computer searches and give some indication for Conjecture 1.

Throughout the rest of the paper we will use the following notation. Let K be a quartic and totally complex number field. We denote by μ the maximal positive integer m such that $\zeta_m \in K$, i.e. μ is chosen such that ζ_μ generates the group of roots of unity in K. Moreover, from this point on if not specified otherwise by number fields we mean totally complex quartic number fields. Note that the unit rank of such fields is one.

2. Bounds for discriminants and regulators of totally complex quartic DUG-fields

First we state a simple test due to Belcher [1] which gives sufficient conditions for a field K to be DUG.

Proposition 1. *Let K be a number field and $\mathfrak{o}_K$ its maximal order. If the unit equation*

$$u + v = 2, \qquad u, v \in \mathfrak{o}_K^* \tag{1}$$

has a non-trivial solution, i.e a solution distinct from $1+1=2$*, then* K *is a DUG-field.*

Now we prove that cyclotomic fields are DUG. The reason for doing so is that these fields would cause some technical complications in formulating (and proving) our later statements. Note that a cyclotomic field $\mathbb{Q}(\zeta_\mu)$ is quartic if and only if $\varphi(\mu)=4$, that is, $\mu=5,8,10,12$. Note that since $\mathbb{Q}(\zeta_5)$ and $\mathbb{Q}(\zeta_{10})$ are isomorphic, we need to consider three cases only.

Lemma 1. *The cyclotomic fields* $\mathbb{Q}(\zeta_\mu)$ *with* $\mu=8,10,12$ *are DUG.*

Proof. We use Belcher's test (Proposition 1). By the help of MAGMA [2] we see that for each field in question equation (1) has a non-trivial solution. In particular we found the following identities:

$$\begin{aligned}(2+\zeta_5^2+\zeta_5^3)-(\zeta_5^2+\zeta_5^3)&=2,\\(1+\zeta_8-\zeta_8^3)+(1-\zeta_8+\zeta_8^3)&=2,\\(1-\zeta_{12})+(1+\zeta_{12})&=2,\end{aligned}$$

where the elements on the left hand sides are all in $\mathfrak{o}_K^*\setminus\{1\}$ with the corresponding number field K. Hence the statement follows. □

Next we give a result yielding a bound for the regulator R_K of a DUG-field K, or in other words, for a fundamental unit ϵ of K with $|\epsilon|>1$. Noting that $\mathbb{Q}(\zeta_3)$ and $\mathbb{Q}(\zeta_6)$ are isomorphic, by Lemma 1 we may clearly assume for the rest of the paper that $\mu=2,4,6$.

Lemma 2. *Let* K *be a DUG-field with* $\mu=2,4,6$ *and let* ϵ *be a fundamental unit of* K *with* $|\epsilon|>1$*. Then we have* $|\epsilon|\le c_\mu$*, where* $c_2=3$*,* $c_4=2\sqrt{2}+1$ *and* $c_6=5$*.*

Let us note that the case $\mu=2$ and in particular the case of real fields was settled by Belcher [1, Lemma 4].

Proof. Let ϵ be a fundamental unit of a DUG-field K with $|\epsilon|>1$, and assume to the contrary that $|\epsilon|>c_\mu$. For any integer N put

$$U_N=\frac{c'_\mu|\epsilon|^{N+1}}{|\epsilon|-1},\qquad L_N=|\epsilon|^N-\frac{c'_\mu|\epsilon|^N}{|\epsilon|-1},$$

with $c'_2=1$, $c'_4=\sqrt{2}$, $c'_6=2$. Under our assumption we obtain

$$L_{N+1}>U_N>L_N>0. \tag{2}$$

Further note that the difference

$$L_{N+1}-U_N=|\epsilon|^{N+1}\overbrace{\left(1-\frac{2c'_\mu}{|\epsilon|-1}\right)}^{>1}$$

goes to infinity as N grows. Therefore we may choose rational integers s and α such that $L_{s+1}>\alpha>U_s$ and since K is DUG we may assume that α has a representation of the form

$$\alpha=\sum_{i=r}^{t}\sum_{j=1}^{\mu}d_{i,j}\zeta_\mu^j\epsilon^i, \tag{3}$$

where r and t are integers with $t \geq r$, $d_{i,j} \in \{0,1\}$ and $\sum_{j=1}^{\mu} d_{t,j}\zeta_\mu^j \neq 0$. A simple calculation shows that the non-zero coefficients of ϵ^i have absolute values between 1 and c'_μ, hence

$$\alpha = \sum_{i=r}^{t}\sum_{j=1}^{\mu} d_{i,j}\zeta_\mu^j\epsilon^i < \left|\sum_{j=1}^{\mu/2}\zeta_\mu^j\right| \sum_{i=-\infty}^{t}|\epsilon^i| = c'_\mu \frac{|\epsilon|^{t+1}}{|\epsilon|-1} = U_t.$$

On the other hand, we similarly obtain

$$\alpha = \sum_{i=r}^{t}\sum_{j=1}^{\mu} d_{i,j}\zeta_\mu^j\epsilon^i > |\epsilon|^t - c'_\mu \sum_{i=-\infty}^{t-1}|\epsilon^i| = |\epsilon|^t - c'_\mu \frac{|\epsilon|^{t}}{|\epsilon|-1} = L_t.$$

Therefore we obtain upper and lower bounds for α namely $U_t > \alpha > L_t$. But our previous choice $L_{s+1} > \alpha > U_s$ yields now $L_{s+1} > \alpha > L_t$, i.e. $s \geq t$, and $U_t > \alpha > U_s$, i.e. $s < t$. Therefore the assumption that α can be represented by a sum of the form (3) yields the desired contradiction.

□

We now state the main result of this section, which provides a bound for the discriminants of DUG-fields K which are not CM-fields. Note that a totally complex quartic field is CM if and only if it has a real quadratic subfield.

Proposition 2. *Let K be a totally complex quartic DUG-field which is not a CM-field. Then the discriminant D_K is bounded by*

- $D_K \leq 99887$ *if* $\mu = 2$,
- $D_K \leq 724732$ *if* $\mu = 4$,
- $D_K \leq 6210095$ *if* $\mu = 6$.

In order to prove this proposition we apply a result due to Nakamula [6, Proposition 1].

Lemma 3 (Nakamula [6])**.** *Let K be a totally complex quartic field with regulator R_K and discriminant D_K which is not a CM-field. Then we have*

$$R_K \geq l\left(\sqrt{\sqrt{D_K/16}+4}\right) \qquad \textit{provided that}\ \ D_K \geq 2^{14},$$

where $l(x) := \log(\frac{1}{2}(x+\sqrt{x^2-4}))$.

Proof of Proposition 2. Combining Lemmas 2 and 3, the statement follows by a simple calculation. □

Since there are only finitely many fields of fixed degree with bounded discriminant we immediately obtain the following result. Note that our Theorem 1 provides much more precise information.

Corollary 1. *There are only finitely many totally complex quartic non-CM DUG-fields.*

3. Maximal orders of certain families of quartic fields

In order to handle the case of CM-fields and also to exclude several fields of non-CM type we need some knowledge on the integral bases of certain number fields. We start with the known case where $K = \mathbb{Q}(i, \sqrt{d})$. The next result is due to Funakura [3].

Proposition 3 (Funakura [3]). *Let $d \geq 2$ be a square-free integer. An integral basis of $\mathbb{Q}(i, \sqrt{d})$ is given by*

$$\begin{array}{llll} 1,\ i, & \frac{1+\sqrt{d}}{2}, & \frac{i+\sqrt{-d}}{2}, & \text{if } d \equiv 1 \mod 4; \\ 1,\ i, & \frac{\sqrt{d}+\sqrt{-d}}{2}, & \frac{\sqrt{d}-\sqrt{-d}}{2}, & \text{if } d \equiv 2 \mod 4; \\ 1,\ i, & \frac{1+\sqrt{-d}}{2}, & \frac{i-\sqrt{d}}{2}, & \text{if } d \equiv 3 \mod 4. \end{array}$$

We also consider fields of the type $K = \mathbb{Q}(\zeta_3, \sqrt{d})$ with some square-free d. The integral bases for these fields are provided by the following proposition.

Proposition 4. *Let $d \geq 2$ be a square-free integer. An integral basis of $\mathbb{Q}(\zeta_3, \sqrt{d})$ is given by*

$$\begin{array}{llll} 1,\ \zeta_3, & \sqrt{d}, & \zeta_3\sqrt{d}, & \text{if } d \equiv 2,7,10,11 \mod 12; \\ 1,\ \zeta_3, & \frac{1+\sqrt{d}}{2}, & \zeta_3\frac{1+\sqrt{d}}{2}, & \text{if } d \equiv 1,5 \mod 12; \\ 1,\ \zeta_3, & \sqrt{-d/3}, & \zeta_3\sqrt{-d/3}, & \text{if } d \equiv 3,6 \mod 12; \\ 1,\ \zeta_3, & \frac{1+\sqrt{-d/3}}{2}, & \zeta_3\frac{1+\sqrt{-d/3}}{2}, & \text{if } d \equiv 9 \mod 12. \end{array}$$

Proof. First note that K is the compositum of the fields $E_1 = \mathbb{Q}(\zeta_3)$ and $E_2 = \mathbb{Q}(\sqrt{d})$. Since the discriminant of E_1 is $D_{E_1} = -3$ and the discriminant of E_2 is $D_{E_2} = d$ if $d \equiv 1 \mod 4$ and $D_{E_2} = 4d$ otherwise, we obtain that if $3 \nmid d$, then the discriminants are coprime. As is well-known, in such a case an integral base of the compositum is given by $\{v_i w_j\}$, where the v's and w's form integral bases of E_1 and E_2, respectively (cf. [5, Chapter III, Proposition 17]).

Hence we are left with the possibility $3 \mid d$. In this case we have $K = \mathbb{Q}(\zeta_3, \sqrt{d}) = \mathbb{Q}(\zeta_3, \sqrt{-d/3})$. Now let $E_1 = \mathbb{Q}(\zeta_3)$ and $E_2 = \mathbb{Q}(\sqrt{-d/3})$. Since these fields have coprime discriminants, we immediately obtain an integral basis of K from the integral bases of E_1 and E_2 as above, and the statement follows. ☐

We also have to consider maximal orders of fields of the type $K = \mathbb{Q}(\sqrt{a + b\omega_d})$, where $d = -3, -1, 2, 5$, and $\omega_d = \sqrt{d}$ if $d = -1, 2$ and $\omega_d = \frac{1+\sqrt{d}}{2}$ if $d = -3, 5$.

Proposition 5. *With the notations above, let $K = \mathbb{Q}(\sqrt{a + b\omega_d})$ and write $d^* = a + b\omega_d$. Assume that d^* is square-free in the maximal order $A = \mathbb{Z}[\omega_d]$ of $E = \mathbb{Q}(\sqrt{d})$. Write $\mathfrak{o}$ for the maximal order of K. Then we have $\mathfrak{o} \subset \frac{1}{2}A[\sqrt{d^*}]$. Moreover, in every case Table 1 gives an integral basis of $\mathfrak{o}$.*

Proof. First we note that for all d under consideration A is a norm Euclidean ring. Therefore it makes sense speaking of the gcd-s and divisibility properties of certain elements of A. Using the above notation, let $B = A[\sqrt{d^*}]$. Then the relative discriminant of B is $D_{K/E}(B) = 4d^*$. Viewing B and $\mathfrak{o}$ as A-modules, we have $B \subset \mathfrak{o}$. Moreover, let X be a 2×2 matrix such that $X\mathfrak{o} = B$. It is well-known that

$$4d^* = D_{K/E}(B) = (\det X)^2 D_{K/E}(\mathfrak{o}).$$

Since d^* is square-free, we obtain $(\det X)|2$. Obviously, if $\det X \in A^*$, then X is an A-module isomorphism and hence $\mathfrak{o} = B$. Moreover, for any $Y \in \mathrm{GL}_2(A)$, i.e. $\det(Y) \in A^*$, we also have $YX\mathfrak{o} = B$.

We start with the case $\det X \in 2A^*$ and write

$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

TABLE 1. Integral bases for fields of the form $K = \mathbb{Q}(\sqrt{a + b\omega_d})$

(a, b) and d	Integral basis of $\mathfrak{o}$
$a \equiv 1 \mod 4$, $b \equiv 0 \mod 4$, $d = -3, -1, 2, 5$	$\{1, \omega_d, \frac{-1+\sqrt{d^*}}{2}, \omega_d \frac{-1+\sqrt{d^*}}{2}\}$
$a \equiv 3 \mod 4$, $b \equiv 3 \mod 4$, $d = -3$ $a \equiv 3 \mod 4$, $b \equiv 0 \mod 4$, $d = -1$ $a \equiv 2 \mod 4$, $b \equiv 0 \mod 4$, $d = 2$ $a \equiv 2 \mod 4$, $b \equiv 2 \mod 4$, $d = 5$	$\{1, \omega_d, \frac{-\omega_d+\sqrt{d^*}}{2}, \omega_d \frac{-\omega_d+\sqrt{d^*}}{2}\}$
$a \equiv 0 \mod 4$, $b \equiv 1 \mod 4$, $d = -3$ $a \equiv 0 \mod 4$, $b \equiv 2 \mod 4$, $d = -1$ $a \equiv 3 \mod 4$, $b \equiv 2 \mod 4$, $d = 2$ $a \equiv 0 \mod 4$, $b \equiv 2 \mod 4$, $d = 5$	$\{1, \omega_d, \frac{-(1+\omega_d)+\sqrt{d^*}}{2}, \omega_d \frac{-(1+\omega_d)+\sqrt{d^*}}{2}\}$
$a \equiv 1 \mod 4$, $b \equiv 2 \mod 4$, $d = 2$ $a \equiv 3 \mod 4$, $b \equiv 0 \mod 4$, $d = 2$	$\{1, \omega_d, \frac{-1+\sqrt{d^*}}{\sqrt{2}}, \omega_d \frac{-1+\sqrt{d^*}}{\sqrt{2}}\}$
$a \equiv 1 \mod 4$, $b \equiv 2 \mod 4$, $d = -1$ $a \equiv 3 \mod 4$, $b \equiv 2 \mod 4$, $d = -1$	$\{1, \omega_d, (1-i)\frac{-1+\sqrt{d^*}}{2}, \omega_d (1-i) \frac{-1+\sqrt{d^*}}{2}\}$
all other cases	$\{1, \omega_d, \sqrt{d^*}, \omega_d \sqrt{d^*}\}$

By the remark above we may apply the Euclidean algorithm to the pair α and γ. Since $(\det X)|2$, in case of $d = -3, 5$ we may assume that $\gcd(\alpha, \gamma) = 1, 2$. When $d = -1$ or 2, beside $\gcd(\alpha, \gamma) = 1, 2$, we also need to consider the possibilities $\gcd(\alpha, \gamma) = 1 + i$ and $\gcd(\alpha, \gamma) = \sqrt{2}$, respectively. Therefore, we may assume that X is of the form

$$X = \begin{pmatrix} \gcd(\alpha, \gamma) & \beta' \\ 0 & \delta' \end{pmatrix}.$$

Put $\alpha' = \gcd(\alpha, \gamma)$. Since $\alpha'\delta' = \det X \in 2A^*$ we may assume X is a matrix of the form

$$X = \begin{pmatrix} 2 & \beta'' \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad X = \begin{pmatrix} 1 & \beta'' \\ 0 & 2 \end{pmatrix},$$

or, in case of $d = 2$ or $d = -1$, of the form

$$X = \begin{pmatrix} \sqrt{2} & \beta'' \\ 0 & \sqrt{2} \end{pmatrix} \quad \text{or} \quad X = \begin{pmatrix} 1+i & \beta'' \\ 0 & 1-i \end{pmatrix},$$

respectively. Computing the residue obtained dividing β by 1 or 2 (in the case $d = -1$ or $d = 2$ we also consider division by $1 + i$ and $\sqrt{2}$, respectively) we are left with five possible matrices X:

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & \omega_d \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1+\omega_d \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Further, for $d = 2$ we also need to consider the matrices

$$\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \begin{pmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{pmatrix},$$

while in case of $d = -1$ the matrices

$$\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}, \begin{pmatrix} 1+i & 1 \\ 0 & 1-i \end{pmatrix}.$$

Computing the inverses of these matrices we obtain that the corresponding integral A-bases would be

$$\left\{1, \frac{-1+\sqrt{d^*}}{2}\right\}, \quad \left\{1, \frac{-\omega_d+\sqrt{d^*}}{2}\right\}, \quad \left\{1, \frac{-1-\omega_d+\sqrt{d^*}}{2}\right\},$$

(4)

$$\left\{1, \frac{\sqrt{d^*}}{2}\right\}, \quad \left\{\frac{1}{2}, \sqrt{d^*}\right\}.$$

In addition we get in case of $d = 2$ the A-bases

$$\left\{\frac{1}{\sqrt{2}}, \frac{\sqrt{d^*}}{\sqrt{2}}\right\}, \quad \left\{\frac{1}{\sqrt{2}}, \frac{-\frac{1}{\sqrt{2}}+\sqrt{d^*}}{\sqrt{2}}\right\}$$

and in case of $d = -1$ also

$$\left\{\frac{1-i}{2}, \frac{(1+i)\sqrt{d^*}}{2}\right\}, \quad \left\{\frac{1-i}{2}, \frac{-1+(1+i)\sqrt{d^*}}{2}\right\}.$$

Obviously, the last two cases in (4) and the additional cases do not yield integral bases. Therefore we need only to decide which of the three algebraic numbers

$$\frac{-1+\sqrt{d^*}}{2}, \quad \frac{-\omega_d+\sqrt{d^*}}{2}, \quad \frac{-1-\omega_d+\sqrt{d^*}}{2}$$

are algebraic integers.

A simple calculation shows that the minimal polynomial of $\frac{-\omega_d+\sqrt{d^*}}{2}$ over A is given by

$$X^2 + \omega_d X + \frac{\omega_d^2 - d^*}{4}.$$

Computing $c_0 = \frac{\omega_d^2-d^*}{4}$ for $d = -3, -1, 2, 5$ explicitly, it is easy to verify that $c_0 \in A$ if and only if

$$\begin{aligned}
a &\equiv 0 \mod 4, & b &\equiv 1 \mod 4, & d &= -3 \text{ or}\\
a &\equiv 0 \mod 4, & b &\equiv 2 \mod 4, & d &= -1 \text{ or}\\
a &\equiv 3 \mod 4, & b &\equiv 2 \mod 4, & d &= 2 \text{ or}\\
a &\equiv 0 \mod 4, & b &\equiv 2 \mod 4, & d &= 5.
\end{aligned}$$

The other two cases are similar and we obtain all the integral bases listed in Table 1, except the last two ones.

These two integral bases correspond to the cases $\det X \in \sqrt{2}A^*$, $d = 2$ and $\det X \in (1+i)A^*$, $d = -1$, respectively. Since the computation of the possible integral bases follows similar lines as before, we omit the details. □

4. A useful test

In this section we explain how we can test whether a fixed algebraic integer $\alpha \in K$ can be written as a sum of distinct units or not. Let ϵ be a fundamental unit of K with $|\epsilon| > 1$. Suppose that α can be represented as the sum of distinct units, i.e. we have

$$\alpha = \sum_{i=-\infty}^{n} \sigma_i \epsilon^i \qquad (5)$$

with $\sigma_i \in \Sigma_K$ $(i \leq n)$, where almost all $\sigma_i = 0$ and $\sigma_n \neq 0$. By Σ_K we denote the set of those algebraic integers in K which can be obtained as the sum of distinct roots of unity of K. Without loss of generality we may assume that here n is minimal. Thus for any $N_1, N_2 \geq 1$ we have

$$\left|\frac{\alpha}{\epsilon^n}\right| > f(N_1,N_2) := C_0(N_1) - C_1(N_2)|\epsilon|^{-N_1-1} - \frac{c'_\mu|\epsilon|^{-N_1-N_2-1}}{|\epsilon|-1}. \tag{6}$$

Here

$$C_0(N_1) = \min_{\sigma_0\neq 0}\left\{\left|\sum_{i=0}^{N_1}\sigma_i\epsilon^{-i}\right|\right\},$$

and

$$C_1(N_2) = \max\left\{\left|\sum_{i=0}^{N_2}\sigma_i\epsilon^{-i}\right|\right\},$$

where the minimum and maximum is taken over all non-vanishing sums with $\sigma_i \in \Sigma_K$. Further, the constants c'_μ are given by $c'_2 = 1, c'_4 = \sqrt{2}$ and $c'_6 = 2$ (cf. the proof of Lemma 2). Suppose that for some N_1 and N_2 we have $f(N_1,N_2) > 0$. Then (since α and ϵ are fixed) we get an upper bound for the largest exponent n in the representation (5) of α. Construct the set

$$T_n := \{\sigma\epsilon^n \,:\, \sigma \in \Sigma_K\}$$

and discard all elements $t \in T_n$ for which

$$|\alpha - t| > C_1(N_2)|\epsilon|^{n-1} + \frac{c'_\mu|\epsilon|^{n-1-N_2}}{|\epsilon|-1}.$$

If T_n is empty, we deduce that α cannot be the sum of distinct units and K is not DUG. On the other hand, if $\alpha \in T_n$ then we have found a sum of distinct units representing α. Otherwise, i.e. if T_n is non-empty but also $\alpha \notin T_n$, we form the set

$$T_{n-1} := \left\{t + \sigma\epsilon^{n-1} \,:\, \sigma \in \Sigma_K, t \in T_n\right\}$$

and discard all elements $t \in T_{n-1}$ such that

$$|\alpha - t| > C_1(N_2)|\epsilon|^{n-2} + \frac{c'_\mu|\epsilon|^{n-2-N_2}}{|\epsilon|-1}.$$

If T_{n-1} is empty we deduce that α cannot be the sum of distinct units and if $\alpha \in T_{n-1}$ we have found a representation of α. Continuing this procedure, and always constructing new sets

$$T_k := \left\{t + \sigma\epsilon^k \,:\, \sigma \in \Sigma_K, t \in T_{k+1}\right\},$$

discarding all elements $t \in T_k$ such that

$$|\alpha - t| > C_1(N_2)|\epsilon|^{k-1} + \frac{c'_\mu|\epsilon|^{k-1-N_2}}{|\epsilon|-1},$$

we ultimately obtain an empty set T_{k_0}, or a set T_{k_0} containing α. Thus we either conclude that α cannot be represented as a sum of distinct units, or we find a representation of α of this form.

5. Fields with only trivial roots of unity

In this section we only consider number fields K with trivial roots of unity, i.e. with $\mu = 2$. By Lemma 2 we may further assume that K has a fundamental unit ϵ with $|\epsilon| < 3$. By Proposition 2 we know that the discriminant of such a field is at most 10^5, provided that it is not a CM-field. The case of CM-fields is discussed separately later, so in this section we assume that the number fields in question are not CM-fields. In particular, we do not consider fields for which either $\frac{1+\sqrt{5}}{2}$ or $1 + \sqrt{2}$ is a fundamental unit, since these are obviously CM-fields. So using the number field tables provided by PARI [9], we are left with 132 fields. By computing relative discriminants we can easily find and discard all fields that are not UG. In particular, we compared the absolute discriminant of the field K with the discriminant of the order generated as a $\mathbb{Z}$-module by the units. It is easy to see that this order is identical with $\mathbb{Z}[\epsilon]$, provided $\mu = 2$ and ϵ is the fundamental unit of K. Then we are left with 115 fields. Using the notation from the previous section, we compute for all these fields a lower bound for $f(1, 7)$ defined in (6), and we obtain a positive lower bound for $f(1, 7)$ in 107 cases.

For these 107 cases we test whether $\alpha = 2$ can be written as a sum of distinct units or not, as explained in Section 4. Only in two cases we find a representation. These exceptional fields E_1 and E_2 have minimal polynomials $X^4 + 3X^2 - 2X + 1$ and $X^4 - X^3 + 3X^2 + X + 1$, respectively. However, by the same method as before we find that neither $\alpha = 5$ in E_1 nor $\alpha = 23$ in E_2 has a representation as a sum of distinct units. Therefore we are left with eight fields listed in Table 2.

Table 2. Fields to be investigated more closely.

The field K	Minimal polynomial defining K
K_1	$X^4 - X + 1$
K_2	$X^4 + X^2 - X + 1$
K_3	$X^4 + 2X^2 - 2X + 1$
K_4	$X^4 - 2X^3 + X^2 + 2X + 1$
K_5	$X^4 - X^3 + X + 1$
K_6	$X^4 - X^3 + X^2 + X + 1$
K_7	$X^4 - X^3 + X^2 + 2X + 1$
K_8	$X^4 - X^3 + 2X^2 - X + 2$

First, consider the field K_2. Note that ϵ, a root of the minimal polynomial of K_2 is a fundamental unit of K_2. As $\epsilon^3 + \epsilon^{-4} = 2$, by Proposition 1 we get that K_2 is DUG.

For K_3, K_4, K_6, K_7 and K_8 we find that $f(N, N)$ is positive for the values $N = 10, 10, 12, 12$ and 12, respectively. Then applying the test from the previous section, we get that in K_4 and K_7, $\alpha = 2$ and $\alpha = 16$ cannot be represented as a sum of distinct units, respectively.

However, in case of $K = K_3, K_6$ and K_8, as well as for the remaining cases $K = K_1$ and $K = K_5$, we tested several algebraic integers $\alpha \in K$, but for each tested algebraic integer α we found a sum of distinct units representing α. Thus we think that these fields are probably DUG. For details see Section 8.

6. Fields containing non-trivial roots of unity

First, assume that $\mu = 4$, i.e. $i \in K$. In this case we have $K = \mathbb{Q}(\sqrt{a+ib})$, where $a+ib$ is square-free in the ring of Gaussian integers $\mathbb{Z}[i]$. Write $d^* := a+ib$. Take the relative number field extension $K/\mathbb{Q}(i)$, and consider units of the form $\epsilon = (x_1+x_2 i)+(y_1+y_2 i)\sqrt{d^*}$ or $\epsilon = \frac{(x_1+x_2 i)+(y_1+y_2 i)\sqrt{d^*}}{2}$, where we choose the form of ϵ according to Table 1. Then taking relative norms and writing $x = x_1 + ix_2$ and $y = y_1 + iy_2$ we obtain one of the relative Pell equations

$$x^2 - Dy^2 = \pm 1, \pm i, \pm 4, \pm 4i.$$

As one can easily see, $|\epsilon| > 1$ implies $|\epsilon| > |\sqrt{d^*}/2|$. Thus by Lemma 2 we may assume that $|d^*| < 58.63$.

We compute now for all $|d^*| < 58.63$ the regulator R_K of K, and rule out all number fields for which $\exp(R_K/2) < \sqrt{2}+1$. This computation took only a few minutes using the computer algebra system PARI [9], and we could discard most number fields. Moreover we computed the discriminant of the order $\mathbb{Z}[\epsilon, i]$ (which is the order generated by the units of K) and compared it with the absolute discriminant of K. We discarded all number fields for which the discriminants did not match, since those are not generated by their units. Thus we are left with six pairwise non-isomorphic number fields, namely:

$$\mathbb{Q}(\sqrt{i}),\ \mathbb{Q}(\sqrt{1+i}),\ \mathbb{Q}(\sqrt{1+2i}),\ \mathbb{Q}(\sqrt{1+4i}),\ \mathbb{Q}(\sqrt{3+8i}),\ \mathbb{Q}(\sqrt{7+4i}).$$

The first of these fields is cyclotomic, and is dealt with in Lemma 1.

In case of $K = \mathbb{Q}(\sqrt{3+8i})$ the method described in Section 4 showed $f(5,5) > 0$ and by using the algorithm of Section 4 we are able to prove that $\alpha = 12$ is not the sum of distinct units.

For the other four cases we made an intensive computer search looking for an algebraic integer α that is not the sum of distinct units, but no such α was found. So it seems that these fields are DUG. For details see Section 8.

Now we consider the case $\mu = 6$, i.e. $\zeta_3 \in K$. Similarly as in the case of $i \in K$, we obtain $|d^*| \leq 100$ with $d^* \in \mathbb{Z}[\zeta_3]$. We perform again a computer search using PARI [9]. After the sieving, we are left with 33 fields of the form $K = \mathbb{Q}(\sqrt{d^*})$, where d^* is one of the following:

$$\begin{gathered}1+\zeta_3, 2+\zeta_3, 4+\zeta_3, 5+\zeta_3, 8+\zeta_3, 2+2\zeta_3, 3+2\zeta_3, 4+3\zeta_3, 7+3\zeta_3, 11+3\zeta_3,\\ 5+4\zeta_3, 9+4\zeta_3, 13+4\zeta_3, 25+4\zeta_3, 12+5\zeta_3, 20+5\zeta_3, 24+5\zeta_3, 7+7\zeta_3, 11+7\zeta_3,\\ 19+7\zeta_3, 9+8\zeta_3, 20+9\zeta_3, 28+9\zeta_3, 15+11\zeta_3, 19+11\zeta_3, 17+12\zeta_3, 24+13\zeta_3,\\ 28+13\zeta_3, 17+16\zeta_3, 25+16\zeta_3, 21+20\zeta_3, 25+20\zeta_3, 23+23\zeta_3.\end{gathered}$$

The first of these fields is the cyclotomic field $\mathbb{Q}(\zeta_{12})$ (and is DUG by Lemma 1). Moreover, one can easily check the identities

$$\begin{gathered}\mathbb{Q}(\sqrt{2+2\zeta_3}) = \mathbb{Q}(\sqrt{6}, \zeta_3), \quad \mathbb{Q}(\sqrt{7+7\zeta_3}) = \mathbb{Q}(\sqrt{21}, \zeta_3),\\ \mathbb{Q}(\sqrt{23+23\zeta_3}) = \mathbb{Q}(\sqrt{69}, \zeta_3).\end{gathered}$$

Therefore these fields are also CM-fields and are dealt within the next section.

In case of 12 of these number fields K we found algebraic integers $\alpha \in K$ that are not the sum of distinct units. For details see the table below.

TABLE 3. Fields K that are not DUG, i.e. some algebraic integer $\alpha \in K$ is not the sum of distinct units.

K	α	K	α
$\mathbb{Q}(\sqrt{5+\zeta_3})$	55	$\mathbb{Q}(\sqrt{28+9\zeta_3})$	7
$\mathbb{Q}(\sqrt{25+4\zeta_3})$	16	$\mathbb{Q}(\sqrt{24+13\zeta_3})$	7
$\mathbb{Q}(\sqrt{20+5\zeta_3})$	7	$\mathbb{Q}(\sqrt{28+13\zeta_3})$	7
$\mathbb{Q}(\sqrt{24+5\zeta_3})$	7	$\mathbb{Q}(\sqrt{25+16\zeta_3})$	10
$\mathbb{Q}(\sqrt{19+7\zeta_3})$	50	$\mathbb{Q}(\sqrt{21+20\zeta_3})$	10
$\mathbb{Q}(\sqrt{20+9\zeta_3})$	7	$\mathbb{Q}(\sqrt{25+20\zeta_3})$	10

For the remaining 17 fields we performed a computer search for algebraic integers that are not the sum of distinct units, but no counterexamples were found. Thus we conjecture that all of them are DUG-fields (for details see Section 8).

7. K is a CM-field

When K is a CM-field, we distinguish two cases, according as K contains non-trivial roots of unity, or not. The following well-known result (e.g. see [11, Theorem 4.12]) will be very helpful.

Lemma 4. *Let K be a CM-field and E its maximal real subfield. Let U_E and U_K be the group of units of the maximal orders of E and K respectively and let W be the set of roots of unity of K. Then $[U_K : U_E W] = 1$ or 2.*

We consider the case of $\mu = 2$ first. Let $E = \mathbb{Q}(\sqrt{d})$ be the unique real quadratic subfield of K. Let $\epsilon > 1$ be the fundamental unit of E. Then either ϵ is the fundamental unit of K, or the fundamental unit η of K satisfies $\eta^2 = -\epsilon$. (This assertion is an immediate consequence of Lemma 4). In the case $\eta = \epsilon$ or $\eta = \epsilon^2$ it is impossible for K to be a DUG-field since η is real. When $\eta^2 = -\epsilon$, we deduce that η is purely imaginary. Therefore K can be a DUG-field only if E is a DUG-field. However, due to Śliwa [8] this happens if and only if $E = \mathbb{Q}(\sqrt{2})$ or $E = \mathbb{Q}(\sqrt{5})$. Thus we only have to consider fields of the type $K = \mathbb{Q}\left(\sqrt{a + b\sqrt{2}}\right)$ and $\mathbb{Q}\left(\sqrt{a + b\frac{1+\sqrt{5}}{2}}\right)$. As we also have $\eta^2 = -\epsilon$, the following result provides a complete answer in this case.

Proposition 6. *Let K be a quartic CM-field with real subfield $E = \mathbb{Q}(\sqrt{2})$ or $E = \mathbb{Q}(\sqrt{5})$ and let $\epsilon > 1$ be the fundamental unit of E. Then $-\epsilon$ is not a square in K, except $K = \mathbb{Q}\left(\sqrt{-1-\sqrt{2}}\right)$ and $K = \mathbb{Q}\left(\sqrt{-\frac{1+\sqrt{5}}{2}}\right)$. Moreover, the fields $\mathbb{Q}\left(\sqrt{-1-\sqrt{2}}\right)$ and $\mathbb{Q}\left(\sqrt{-\frac{1+\sqrt{5}}{2}}\right)$ are DUG.*

Proof. According to Proposition 5 we have to consider several cases. Since every case is similarly dealt with, we only consider two cases. In one case we find a solution to the equation $\eta^2 = -\epsilon$ and in another case we prove that no solution exists. Note that all other cases are very similar to either of the two cases treated below.

We start with the case $K = \mathbb{Q}\left(\sqrt{a+b\sqrt{2}}\right)$. Let us assume that neither $b \equiv 0 \mod 4$ and $a \equiv 1,2,3 \mod 4$ nor $b \equiv 2 \mod 4$ and $a \equiv 2 \mod 4$. Then an integral basis of K is given by

$$\left\{1, \sqrt{2}, \sqrt{a+b\sqrt{2}}, \sqrt{2}\sqrt{a+b\sqrt{2}}\right\}.$$

Write

$$\eta = x + y\sqrt{2} + z\sqrt{a+b\sqrt{2}} + w\sqrt{2}\sqrt{a+b\sqrt{2}} \qquad (x,y,z,w \in \mathbb{Z})$$

and assume η is a solution to $\eta^2 = -\epsilon = -1-\sqrt{2}$. Comparing coefficients we obtain the system of equations

$$\begin{aligned} 2wy + xz &= 0, \\ wx + yz &= 0, \\ 1 + 2aw^2 + x^2 + 2y^2 + 4bwz + az^2 &= 0, \\ 1 + 2bw^2 + 2xy + 2awz + bz^2 &= 0. \end{aligned}$$

Computing the Groebner basis of the corresponding ideal I generated by the polynomials on the left hand sides, with respect to the lexicographic term order $x \prec y \prec z \prec w$, we obtain

$$32x^9 + 48x^7 + 30x^5 + 6x^3 - x$$

as lowest element. The only integral zero of this polynomial is $x = 0$. Now computing the Groebner basis of I after the substitution $x = 0$, with respect to the lexicographic term order $y \prec z \prec w$, the lowest element is y. Hence we obtain that

$$\eta = (z + w\sqrt{2})\sqrt{a+b\sqrt{2}}.$$

Since by assumption η is a unit, we deduce that also $a + b\sqrt{2}$ is a unit. We may assume that $a + b\sqrt{2} = -1 - \sqrt{2}$, and therefore $K = \mathbb{Q}\left(\sqrt{-1-\sqrt{2}}\right)$.

Now let us consider the case $K = \mathbb{Q}\left(\sqrt{a+b\sqrt{2}}\right)$, with $4|a-1$ and $4|b$. This might be the case with the most technical difficulties. However, in this case we may assume

$$\eta = x + y\sqrt{2} + z\frac{1+\sqrt{a+b\sqrt{2}}}{2} + w\frac{1+\sqrt{a+b\sqrt{2}}}{\sqrt{2}} \qquad (x,y,z,w \in \mathbb{Z}).$$

Write $a = 4A + 1$ and $b = 4B$. By comparing the corresponding coefficients we obtain the following system of equations:

$$\begin{aligned} w^2 + 2wy + xz + z^2/2 &= 0, \\ yz + w(x+z) &= 0, \\ 1 + w^2 + 2Aw^2 + x^2 + 2wy + 2y^2 + 4Bwz + xz + z^2/2 + Az^2 &= 0, \\ 1 + y(2x+z) + w(x+z+2Az) + B(2w^2+z^2) &= 0. \end{aligned} \tag{7}$$

Computing the Groebner basis of the corresponding ideal I with respect to the lexicographic term order $w \prec z \prec y \prec x$, we get

$$w \times \big(2(1+4A-4B)^2 + 2(1+4A-8B)((1+4A)^2-32B^2)w^2 \\ +((1+4A)^2-32B^2)^2w^4\big) \\ \times \big(-1+4w^2(1+4A-8B+((1+4A)^2-32B^2)w^2)\big)$$

as lowest element. Therefore one of the three factors has to be zero. Assume first that $w=0$. We substitute $w=0$ in I, obtaining I', and compute its Groebner basis with respect to the lexicographic term order $z \prec y \prec x$. The lowest element of this Groebner basis is now $z(1+4A-4B)$, whence $z=0$. Substituting $z=0$ in I' we obtain the ideal I'', and the minimal element of its Groebner basis with respect to the lexicographic term order $y \prec x$ is $1+4y^2+8y^4$. As this polynomial is irreducible over $\mathbb{Q}$, $w=0$ yields no solution.

Consider now the case when the second factor vanishes. If we introduce the notation $W=w^2$, this factor becomes a quadratic polynomial in W, with discriminant

$$32(1+4A-4B)^2.$$

This is obviously not a square in $\mathbb{Z}$, hence the quadratic polynomial has no integral solution too. Thus also the system (7) has no solution.

Finally, if the third factor is zero, then we obtain again a quadratic polynomial after substituting $W=w^2$, with discriminant

$$4\left(32B^2-(1+4A)^2\right)^3.$$

Since $32B^2-(1+4A)^2 \equiv -1 \mod 8$, the discriminant cannot be a square. Thus again we obtain no solution and this case is completely solved. As we already mentioned above, the other cases work analogously.

So we only have to prove that the fields $\mathbb{Q}\left(\sqrt{-1-\sqrt{2}}\right)$ and $\mathbb{Q}\left(\sqrt{-\frac{1+\sqrt{5}}{2}}\right)$ are DUG, indeed. Using Belcher's theorem (Proposition 1) we see that by

$$(1+\sqrt{2})+(1-\sqrt{2})=2,$$

hence $\mathbb{Q}\left(\sqrt{-1-\sqrt{2}}\right)$ is DUG.

In case of $K=\mathbb{Q}\left(\sqrt{-\frac{1+\sqrt{5}}{2}}\right)$ we have by Proposition 5 that the maximal order of K is

$$\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]+i\sqrt{\frac{1+\sqrt{5}}{2}}\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right].$$

Since $\mathbb{Q}\left(\frac{1+\sqrt{5}}{2}\right)$ is DUG, from this assertion we get that $\mathbb{Q}\left(\sqrt{-\frac{1+\sqrt{5}}{2}}\right)$ is DUG, as well. □

Therefore we are left with the cases of $\mu=4$ and $\mu=6$.

7.1. **The case of $\mu=4$.** In this case we know that there is a real quadratic subfield of K and we deduce that $K=\mathbb{Q}(i,\sqrt{d})$ for some square-free d.

As explained before, by Lemma 2 we only have to consider those square-free numbers d for which the fundamental unit $\epsilon>1$ of $\mathbb{Q}(\sqrt{d})$ is at most $(1+2\sqrt{2})^2$.

Since for all $d > d_0^2+1$ we have $\epsilon > d_0+\sqrt{d_0^2-1}$ for $d \not\equiv 1 \mod 4$ and $\epsilon > \frac{d_0+\sqrt{d_0^2-4}}{2}$ otherwise, we deduce that $1 < d < 229$. We compute for all fields of the form $K = \mathbb{Q}(i, \sqrt{d})$ the fundamental unit and according to Lemma 2 we discard all fields with fundamental unit $|\eta| > 1 + 2\sqrt{2}$. Therefore we are left with the five fields $K = \mathbb{Q}(i, \sqrt{d})$ $(d = 2, 3, 5, 6, 13)$.

In case of $d = 2$ and $d = 3$ we immediately see that $K = \mathbb{Q}(\zeta_8)$ and $K = \mathbb{Q}(\zeta_{12})$, which have been treated already. If $d = 5$, the maximal order $\mathfrak{o}_K$ is of the form $\mathbb{Z}[\omega_5] + i\mathbb{Z}[\omega_5]$ and since $\mathbb{Z}[\omega_5]$ is the maximal order of the DUG-field $\mathbb{Q}(\sqrt{5})$ we deduce that K is DUG, as well. If $d = 13$ then $\eta = \frac{3+\sqrt{13}}{2}$ is a fundamental unit of K and the maximal order of $\mathfrak{o}_K$ is of the form $\mathbb{Z}[\omega_{13}] + i\mathbb{Z}[\omega_{13}]$, i.e. K is DUG if and only if $\mathbb{Q}(\sqrt{13})$ is DUG. But, $\mathbb{Q}(\sqrt{13})$ is not DUG hence also K is not DUG.

Therefore only the case $d = 6$ remains. We apply the test explained in Section 4 to $\alpha = 2$ with $N_1 = N_2 = 5$, and obtain that α is not the sum of distinct units. Thus $K = \mathbb{Q}(i, \sqrt{6})$ is not DUG.

7.2. **The case of $\mu = 6$.** Let K be of the form $K = \mathbb{Q}(\zeta_3, \sqrt{d})$. Similarly as before, we only have to consider those values of d for which the fundamental unit ϵ with $\epsilon > 1$ of $\mathbb{Q}(\sqrt{d})$ is at most $5^2 = 25$. Therefore we can restrict our attention to the case $1 < d < 680$. We compute for all fields of the form $K = \mathbb{Q}(\zeta_3, \sqrt{d})$ the fundamental unit, and according to Lemma 2 we discard all fields with fundamental unit $|\eta| > 5$. Therefore we are left with the seven fields $K = \mathbb{Q}(\zeta_3, \sqrt{d})$ $(d = 2, 3, 5, 6, 13, 21, 69)$.

In case of $d = 2$ and $d = 3$ we obtain again cyclotomic fields. If $d = 5$ and $d = 13$ we find that $\mathfrak{o}_K$ is of the form $\mathbb{Z}[\omega_5] + \zeta_3\mathbb{Z}[\omega_5]$ and $\mathbb{Z}[\omega_{13}] + \zeta_3\mathbb{Z}[\omega_{13}]$, hence $K = \mathbb{Q}(\zeta_3, \sqrt{5})$ is DUG and $K = \mathbb{Q}(\zeta_3, \sqrt{13})$ is not DUG, since the fundamental unit of K is also the fundamental unit of $\mathbb{Q}(\sqrt{13})$.

Now let us consider the case $d = 69$. In this case we apply again the test explained in Section 4. Since $f(3, 3) > 0$ we were able to prove that $\alpha = 3$ is not a sum of distinct units. In particular, $K = \mathbb{Q}(\zeta_3, \sqrt{69})$ is not DUG.

Therefore we are left with the two cases $d = 6, 21$. We conjecture that these fields are DUG; for details see the next section.

8. Some numeric results to Conjecture 1

In this section we describe our computer search for algebraic integers $\alpha \in K$ that are not the sum of distinct units. For all the remaining 28 fields (see the tables below) for which we do not know whether they are DUG or not, we did the following. First we computed for each field K depending on μ the value $f(N, N)$ (see Section 4) with $N = 12$ if $\mu = 2$, $N = 6$ if $\mu = 4$ and $N = 4$ if $\mu = 6$. In 13 cases (see the tables below) we obtained a positive value for $f(N, N)$. In these cases we can assure that our algorithm terminates either by proving that some algebraic integer has no representation as a sum of distinct units, or by finding such a representation. In case of the other 15 fields, for which $f(N, N)$ was negative we performed the same algorithm by pretending that $f(N, N) = 0.01$ but left $C_0(N)$ and $C_1(N)$ unchanged. In order to force the algorithm to terminate we set an upper bound (10^6) for the number of elements in the sets T_k described in Section 4. However, our algorithm always terminated because we always found a representation as a sum of distinct units and therefore the upper bound was never reached.

The running times to test one algebraic integer $\alpha \in K$ strongly varied with K. Therefore we implemented for a fixed number field K the following search algorithm. We started with $\alpha = 2$ and used our modified algorithm described in the paragraph above to find a representation of α as a sum of distinct units. If such a representation was found we went on to $\alpha + 1$ and searched again for a representation. If our algorithm could prove (this could only happen if $f(N, N) > 0$) that no such representation exists for α, we terminated the search in the field K. If $f(N, N) < 0$ and no representation of α as the sum of distinct units was found, our algorithm made an appropriate output and continued with $\alpha + 1$. As mentioned above this case never happened. However, we stopped the computer search for each field after 24 hours. In most cases the algorithm tested from 10^4 up to $5 \cdot 10^5$ instances. However, in case of 7 fields less than 10^4 instances were tested and we decided to extend the computer search for these cases. For three cases we performed the search one day more, and in two cases we performed the computer search for 10 and 20 days more, respectively. Unluckily the algorithm was very slow for the field $K = \mathbb{Q}(\sqrt{4 + \zeta_3})$ and we could cover only positive rational integers α up to 690 in 21 days.

After this we repeated our computations but this time we started with $n = 2$ and computed for n a complete set of all non-associate integers α with $|N_{K/\mathbb{Q}}(\alpha)| = n$. For each such algebraic integer we used our algorithm to find a representation in the same way described above and then went on to $n + 1$. In view of the different run-times of the algorithm we ran this test for each number field as long as we did in the case of rational α.

In Tables 4, 5, 6 and 7 the column "days" refers to the number of days searching for representations of small rational integers and algebraic integers with small norm respectively.

In case of non-CM-fields with $\mu = 2$ we had to test 5 fields $K = \mathbb{Q}(\alpha)$, with $P(X)$ being the minimal polynomial of α. We tested in each case all positive rational integers α with $2 \le \alpha \le B_1$ and all non-associated algebraic integers with $2 \le |N_{K/\mathbb{Q}}(\alpha)| \le B_2$. For details see Table 4.

TABLE 4. Computer search for non-CM-fields, with $\mu = 2$.

$P(X)$	$f(12, 12)$	B_1	B_2	days
$X^4 - X + 1$	-0.497326	10814	78063	11
$X^4 + 2X^2 - 2X + 1$	0.273211	218933	1194918	1
$X^4 - X^3 + X + 1$	-0.028583	23807	147536	1
$X^4 - X^3 + X^2 + X + 1$	-0.009071	72432	423723	1
$X^4 - X^3 + 2X^2 - X + 2$	0.355400	395131	1804167	1

In case of non-CM-fields with $\mu = 4$ there were four fields $K = \mathbb{Q}(\sqrt{a + ib})$ left. We tested all positive rational integers α with $2 \le \alpha \le B_1$ and all non-associated algebraic integers with $2 \le |N_{K/\mathbb{Q}}(\alpha)| \le B_2$ (see Table 5).

We were left with 17 instances in case of non-CM-fields and with $\mu = 6$, i.e. fields of the form $K = \mathbb{Q}(\sqrt{a + \zeta_3 b})$. We tested all positive rational integers α with $2 \le \alpha \le B_1$ and non-associated algebraic integers with $2 \le |N_{K/\mathbb{Q}}(\alpha)| \le B_2$ again (see Table 6).

In the case of CM-fields we could not decide whether K is DUG or not only in two cases. These were the fields $K = \mathbb{Q}(\zeta_3, \sqrt{d})$, with $d = 6, 21$. Again we tested every positive rational integers α with $2 \le \alpha \le B_1$ and non-associated algebraic integers with $2 \le |N_{K/\mathbb{Q}}(\alpha)| \le B_2$ (see Table 7).

Table 5. Computer search for non-CM-fields, with $\mu = 4$.

a	b	$f(6,6)$	B_1	B_2	days
1	1	−0.000181	71866	430625	1
1	2	−0.055514	22400	128965	2
1	4	−0.305867	16458	126353	11
7	4	0.242712	265950	1811285	1

Table 6. Computer search for non-CM-fields, with $\mu = 6$.

a	b	$f(4,4)$	B_1	B_2	days
2	1	−0.040340	12390	77017	1
4	1	−2.210628	690	4459	21
8	1	−0.016158	27526	145876	1
3	2	0.242186	121198	861363	1
4	3	0.319059	279862	1600893	1
7	3	−0.074705	12283	58147	2
11	3	0.207147	144049	763969	1
5	4	−0.636733	6652	38619	21
9	4	−0.002228	52851	235237	1
13	4	0.117106	130385	794224	1
12	5	0.040640	79467	463603	1
11	7	0.209913	112542	947203	1
9	8	−0.001369	38777	296107	1
15	11	0.299628	194737	1841733	1
19	11	0.338532	384730	2618463	1
17	12	0.431460	485717	2916139	1
17	16	0.445732	576517	3123268	1

Table 7. Computer search for CM-fields, with $\mu = 6$.

d	$f(4,4)$	B	B_2	days
6	0.162084	72175	588025	1
21	−0.053039	20841	146668	2

Summarizing the results of our computations, we can say that we were able to test large sets of integers in all fields listed in the second part of Theorem 1. Since we could get a representation of the desired form in all cases, we think that our calculations strongly support Conjecture 1.

Acknowledgment

We thank the anonymous referee for his valuable suggestions. In particular, his suggestions concerning the proof of Lemma 2 and Section 8 significantly improved the quality of the paper.

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