# ALGEBRAIC ASPECTS OF EMISSION TOMOGRAPHY WITH ABSORPTION

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Abstract. In a previous paper the authors analysed the classical discrete tomography problem to construct a $0-1$-matrix with given line sums in some given directions. One of the physical representations is that material at the lattice points corresponding to 1's emit units of radiation and that the radiation is measured along the given lines. In the present paper they extend their approach to the case that the intermediate material is absorbing the radiation. They generalise results obtained by Kuba and Nivat.

## 1. Introduction

Discrete tomography is a subject of current interest (see [2] for the most recent progress). The main problem is to reconstruct a function $f: A \to \{0,1\}$, where $A$ is a finite subset of $\mathbb{Z}^l$ $(l \geq 2)$, if the sums of the function values along all the lines in a finite number of directions are known. In this paper we deal with the mathematical problem of the reconstruction of the places from where radiation is emitted in case the radiation is partially absorbed by the medium.

To model the physical background of emission tomography with absorption, consider a ray (such as light or X-ray) transmitting through homogeneous material. Let $I_0$ and $I$ denote the initial and the detected intensities of the ray. Then

$$I = I_0 \cdot e^{-\mu x},$$

where $\mu \geq 0$ denotes the absorption coefficient of the material, and $x$ is the length of the path of the ray in the material. We put $\beta = e^{\mu}$, and we call $\beta$ the exponential absorption coefficient. We mention that as $\mu \geq 0$, we have $\beta \geq 1$. Note that by the absorption we have to work with directed line sums which do not only depend on the line, but also on the direction of the radiation through that line.

We further assume that $f$ represents (radio-active) material which is emitting radiation. If $f(i,j) = 1$, then there is a unit of radiating material at $(i,j)$, otherwise $f(i,j) = 0$ and there is no such material at $(i,j)$. Mathematically, in emission tomography with absorption for $l = 2$ we deal with a problem of the following type. (See also the DA2D($\beta$) reconstruction problem in [4].)

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**Problem 1.** *Let $m, n, D$ be positive integers. Let $A = \{(i,j) \in \mathbb{Z}^2 \ : \ 0 \le i < m, 0 \le j < n\}$ and $f : \ A \to \{0,1\}$. Let $S = \{(a_d, b_d, \beta_d) \ : \ d = 1, \dots, D\}$ be a set, where the pairs of coprime integers $(a_d, b_d)$ are distinct, and for the real numbers $\beta_d$ we have $\beta_d \ge 1$. Let $(i,j)$ be the $s_{(i,j,d)}$-th point of $A$ on the line $a_d y - b_d x = a_d j - b_d i$ counted with $a_d i$ decreasing when $a_d \ne 0$ and $b_d j$ decreasing otherwise. Suppose $f$ is unknown, but all the directed absorption line sums*

$$\sum_{\substack{a_d j = b_d i + t \\ (i,j) \in A}} f(i,j) \beta_d^{-s_{(i,j,d)}}$$

*corresponding to the triples $(a_d, b_d, \beta_d)$ in $S$ are given for $d = 1, \dots, D$ and $t \in \mathbb{Z}$. Construct a function $g : \ A \to \{0,1\}$ such that*

$$\sum_{\substack{a_d j = b_d i + t \\ (i,j) \in A}} \frac{f(i,j)}{\beta_d^{s_{(i,j,d)}}} = \sum_{\substack{a_d j = b_d i + t \\ (i,j) \in A}} \frac{g(i,j)}{\beta_d^{s_{(i,j,d)}}} \quad \text{for } d = 1, \dots, D \text{ and } t \in \mathbb{Z}. \tag{1}$$

If the absorption is independent of the direction, then $\beta_d = e^{\mu \sqrt{a_d^2 + b_d^2}}$, since $\sqrt{a_d^2 + b_d^2}$ is the distance between consecutive lattice points on the line $a_d y = b_d x + t$. However, we prefer to leave the possibility open that the absorption coefficient depends on the direction in which the medium is passed. In order to distinguish between two opposite directions, we assume that the points along the directed line are counted with $a_d i$ decreasing when $a_d \ne 0$ and $b_d j$ decreasing otherwise. Thus $a_d y = b_d x + t$ and $(-a_d) y = (-b_d) x - t$ represent the same line, but opposite directions.

Just as in [1], Problem 1 can be transformed into an extremal problem, i.e. instead of finding a function $g : \ A \to \{0,1\}$ satisfying (1), we may find a "smallest" solution of a related problem. To formulate this problem, for every $(i,j) \in A$ put $\rho_S(i,j) = \sum_{d=1}^{D} \beta_d^{-s_{(i,j,d)}}$, where $s_{(i,j,d)}$ is defined as above.

**Problem 2.** *Let $m, n, D, A, S$ be as in Problem 1. Suppose $f : \ A \to \mathbb{Z}$ is unknown, but all the directed absorption line sums*

$$\sum_{\substack{a_d j = b_d i + t \\ (i,j) \in A}} f(i,j) \beta_d^{-s_{(i,j,d)}}$$

*corresponding to the triples in $S$ are given for $d = 1, \dots, D$ and $t \in \mathbb{Z}$. Construct a function $g : \ A \to \mathbb{Z}$ such that (1) holds and*

$$\sum_{(i,j) \in A} g(i,j)^2 \rho_S(i,j) \quad \text{is minimal.}$$

In order to see that Problem 2 is more general than Problem 1, suppose $f : \ A \to \{0,1\}$. Then for every function $g : \ A \to \mathbb{Z}$

$$\sum_{(i,j) \in A} f(i,j)^2 \rho_S(i,j) = \sum_{(i,j) \in A} f(i,j) \rho_S(i,j) =$$

$$= \sum_{(i,j) \in A} g(i,j) \rho_S(i,j) \le \sum_{(i,j) \in A} g(i,j)^2 \rho_S(i,j)$$

with equality if and only if $g: A \to \{0,1\}$. Hence the minimum is realized if and only if $g$ is a solution to Problem 1. Thus it suffices to study Problem 2.

Note that $\rho_S$ is a weight function: we associate the weight $\rho_S(i,j)$ to the $(i,j)$-th point of $A$. As clearly

$$|g| = \sqrt{\sum_{(i,j)\in A} g(i,j)^2 \rho_S(i,j)}$$

is a norm function on the real vectorspace $\{g : A \to \mathbb{R}\}$, Problem 2 can be considered as searching for a solution among the solutions of (1), which is shortest with respect to this norm.

Kuba and Nivat in their papers [3] and [4] dealt with the special case $S = \{(-1,0,\beta),(0,1,\beta)\}$, where $\beta = (1+\sqrt{5})/2$. They gave a characterization of the solutions having the same directed line sums corresponding to the elements of $S$ in this case. Their approach is completely different from ours. In an example of Section 4 we show how their case fits into our treatment.

In the present paper we extend our investigations in [1] to the case of absorption. The functions $f : A \to \mathbb{Z}$ form a $\mathbb{Z}$-module $M$, and the subset $M_0$ consisting of the functions having zero directed absorption line sums corresponding to any finite set $S$, is a submodule of $M$. Thus the functions with the same directed absorption line sums form a coset in the factor module $M/M_0$. By determining a basis of $M_0$, the so-called switching elements, we describe the structure of the functions $f : A \to \mathbb{Z}$ having the same directed absorption line sums, corresponding to any finite set $S$.

To illustrate our results, in Section 4 we work out a few examples. The above problems could also be considered in higher dimensions, and our results would apply in this more general case as well. However, for the convenience of the reader, we work out the details only in the two dimensional case.

## 2. Definitions and notation

We note that if $\beta_d$ is transcendental, then $f$ is uniquely determined by its directed absorption line sums in the corresponding direction $(a_d, b_d)$. Thus throughout the paper we will work with algebraic exponential absorption coefficients.

In the presentation of our new results we use a more general setting than in the introduction. It requires only very little extra work to formulate and prove our theorem for an arbitrary unique factorization domain instead of $\mathbb{Z}$. We introduce some notation.

Let $a, b \in \mathbb{Z}$ with $\gcd(a,b) = 1$. We call $(a,b)$ a direction. From now on let $R$ denote a unique factorization domain, and $K$ its quotient field. Let $\beta$ be a nonzero algebraic element over $K$ of degree $k$, and let $P_\beta(z)$ be a defining polynomial of $\beta$ having coprime coefficients from $R$. Put

$$f_{(a,b,\beta)}(x,y) = P_\beta(x^a y^b) x^p y^q \quad \text{with} \quad p = \max\{-ka, 0\} \text{ and } q = \max\{-kb, 0\}.$$

Hence $f_{(a,b,\beta)}(x,y) \in R[x,y]$. By directed lines into the direction $(a,b)$ we mean lines of the form $ay = bx + t$ $(t \in \mathbb{Z})$ in the $(x,y)$ plane, passed through in the direction of increasing $ax$ (or if $a = 0$, of increasing $by$.).

Let $m, n$ be positive integers and $A = \{(i,j) \in \mathbb{Z}^2 : 0 \le i < m, 0 \le j < n\}$. We call the elements of $A$ points. If $g : A \to R$ is a function, then the directed absorption line sums (or briefly line sums) of $g$ corresponding to the triple $(a, b, \beta)$ are defined as

$$\sum_{\substack{aj=bi+t\\(i,j)\in A}} g(i,j)\beta^{-s_{(i,j)}} \quad (t \in \mathbb{Z}),$$

where $(i, j)$ is the $s_{(i,j)}$-th point of $A$ on the line $ay - bx = aj - bi$ counted with $ai$ decreasing when $a \neq 0$ and $bj$ decreasing otherwise.

The following figure shows how line sums are interpreted.

$$\begin{array}{cccccc} [ & \longleftarrow & \bullet & \circ & \bullet & \circ \\ [ & \longleftarrow & \circ & \bullet & \circ & \bullet \\ [ & \longleftarrow & \circ & \bullet & \circ & \circ \end{array} \qquad g = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

**Fig. 1.** The symbols $\bullet$ denote particles emitting radiation, while $\circ$ show empty locations. The detectors are denoted by [. They measure the radiation in the direction $(-1,0)$. Further, $g$ is the corresponding function from $A$ to $\mathbb{Z}$. If $\beta$ is the exponential absorption coefficient in the direction $(-1,0)$, then $1/\beta+1/\beta^3$, $1/\beta^2+1/\beta^4$ and $1/\beta^2$ are the line sums (from top to bottom) corresponding to $(-1,0,\beta)$.

In the proof we shall make use of a fundamental correspondence between functions $f : A \to R$ and polynomials in two variables. Namely, to such a function $f$ we attach the polynomial $h(x, y) = \sum_{(i,j)\in A} f(i, j)x^i y^j$. Then into direction $(a, b)$ the line sums of $f$ are the coefficients of $h(x, y)$ modulo $f_{(a,b,\beta)}$. To combine the various directions, we use the Chinese Remainder Theorem for polynomials. The polynomials are pairwise coprime except for the case of opposite directions and reciprocally conjugate exponential absorption coefficients, in which case they are conjugate. Therefore the polynomial $F_S$ defined below represents the least common multiple of the polynomials $f_{(a_d,b_d,\beta_d)}$. The Chinese Remainder Theorem implies that there is exactly one solution modulo the least common multiple of the polynomials $f_{(a_d,b_d,\beta_d)}$. More precisely, let $S = \{(a_d, b_d, \beta_d) \ : \ d = 1, \dots, D\}$ be a set, where for each $d$, $(a_d, b_d)$ is a direction and $\beta_d$ is a nonzero algebraic element of degree $k_d$ over $K$. Let $S^*$ be the set of those triples $(a_d, b_d, \beta_d)$ of $S$ with $a_d < 0$ or $(a_d, b_d) = (0, -1)$, for which there exists another triple $(a_c, b_c, \beta_c) \in S$ such that $(a_d, b_d) = (-a_c, -b_c)$, and $\beta_c$ and $1/\beta_d$ are conjugated elements over $K$. Put

$$F_S(x, y) = \prod_{d=1}^{k} f^*_{(a_d,b_d,\beta_d)}(x, y),$$

where

$$f^*_{(a_d,b_d,\beta_d)}(x, y) = \begin{cases} 1, & \text{if } (a_d, b_d, \beta_d) \in S^*, \\ f_{(a_d,b_d,\beta_d)}(x, y), & \text{otherwise.} \end{cases}$$

We say that $S$ is valid for $A$, if $S_x = \deg_x(F_S(x, y)) < m$ and $S_y = \deg_y(F_S(x, y)) < n$. Throughout the paper we suppose that $S$ is a valid set for $A$. For $0 \le u < m - S_x$, $0 \le v < n - S_y$ set $F_{(u,v,S)}(x, y) = x^u y^v F_S(x, y)$, and define the functions $m_{(u,v,S)} : A \to R$ by

$$m_{(u,v,S)}(i, j) = \text{coeff}(x^i y^j) \text{ in } F_{(u,v,S)}(x, y) \quad \text{for } (i, j) \in A.$$

The functions $m_{(u,v,S)}$ are called the switching elements corresponding to the set $S$. By the bottom-left corner of the switching element $m_{(0,0,S)}$ we mean the point $(i^*, j^*)$ for which $m_{(0,0,S)}(i^*, j^*) \neq 0$, but $m_{(0,0,S)}(i,j) = 0$ whenever $i < i^*$, or $i = i^*$ and $j < j^*$. Observe that $(i^*, j^*)$ is lexicographically the first point of $A$ for which the function value of $m_{(0,0,S)}$ is nonzero. Since it corresponds with the bottom-left corner of $m_{(0,0,S)}$, for every $u$ and $v$ we define the bottom-left corner of $m_{(u,v,S)}$ as $(i^* + u, j^* + v)$. Again, the bottom-left corner of $m_{(u,v,S)}$ is lexicographically the first point of $A$ for which the function value of $m_{(u,v,S)}$ is nonzero.

## 3. New results

**Theorem.** *Let $m, n$ and $D$ be positive integers, $A = \{(i,j) \in \mathbb{Z}^2 \ : \ 0 \leq i < m, 0 \leq j < n\}$ and $S = \{(a_d, b_d, \beta_d) \ : \ d = 1, \ldots, D\}$ be a valid set for $A$. Let $R$ be an integral domain such that $R[x,y]$ is a unique factorization domain. Then any function $g : \ A \to R$ with zero line sums corresponding to the triples $(a_d, b_d, \beta_d)$ of $S$ can be uniquely written in the form*

$$g = \sum_{u=0}^{m-1-S_x} \sum_{v=0}^{n-1-S_y} c_{uv} m_{(u,v,S)}$$

*with $c_{uv} \in R$. Moreover, every such function $g$ has zero line sums corresponding to the elements of $S$.*

**Remark 1.** Let $M$ be the $R$-module of the functions $f : A \to R$. Then the functions having zero line sums corresponding to some finite set $S$, form a submodule $M_0$ of $M$. The above theorem establishes that the switching elements $m_{(u,v,S)}$ provide a basis of $M_0$.

**Remark 2.** In the theorem $R$ can be chosen as $\mathbb{Z}$ or any field.

**Remark 3.** As a simple consequence of the theorem we obtain that two functions $g_1 : \ A \to R$ and $g_2 : \ A \to R$ have the same line sums corresponding to the elements of $S$ if and only if

$$g_1 - g_2 = \sum_{u=0}^{m-1-S_x} \sum_{v=0}^{n-1-S_y} c_{uv} m_{(u,v,S)}$$

with some $c_{uv} \in R$. This simple observation provides a complete description of the structure of the functions having the same line sums.

**Remark 4.** If $S$ is not a valid set for $A$, then the only function $g : \ A \to R$ having zero line sums corresponding to the elements of $S$ is the identically zero function.

**Remark 5.** Using the weight function $\rho_S(i,j)$ and the norm function corresponding to it (see Problem 2), one could easily extend Theorem 2 of [1] to this more general case. However, we do not work out the details here.

## 4. Examples

In this section we give three examples to illustrate our method. For the convenience of the reader we restrict ourselves to functions with integer values and to real algebraic exponential absorption coefficients which are greater than 1.

**Example 1.** First we consider the same situation as Kuba and Nivat in [4]. Let $R = \mathbb{Z}$ and $S = \{(-1, 0, \beta), (0, 1, \beta)\}$, where $\beta = (1 + \sqrt{5})/2$. Using the notation of Section 2 we have $P_\beta(z) = z^2 - z - 1$ and

$$f_{(-1,0,\beta)}(x, y) = -x^2 - x + 1 \quad \text{and} \quad f_{(0,1,\beta)}(x, y) = y^2 - y - 1.$$

Thus we obtain $F_S(x, y) = -x^2y^2 + x^2y - xy^2 + x^2 + y^2 + xy + x - y - 1$ and $S_x = S_y = 2$. So if $A$ is of type $m \times n$ with $m, n \geq 3$, then $S$ is a valid set for $A$. We have

$$m_{(0,0,S)} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & -1 & -1 & 0 & \dots & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 \end{pmatrix}$$

and the switching elements $m_{(u,v,S)}$ $(0 \leq u \leq m - 3, 0 \leq v \leq n - 3)$ form a basis of the set of functions $g :\ A \to \mathbb{Z}$ having zero line sums corresponding to the two elements of $S$.

**Example 2.** Now we consider an example which is similar to the previous one, but we include opposite directions and use different exponential absorption coefficients, too. Let $S = \{(-1, 0, \beta), (1, 0, \beta), (0, -1, \gamma), (0, 1, \delta)\}$ with $\beta = (1 + \sqrt{5})/2$, $\gamma = 2 + \sqrt{2}$ and $\delta = \gamma/2$. Now we obtain $P_\beta(z) = z^2 - z - 1$, $P_\gamma(z) = z^2 - 4z + 2$ and $P_\delta(z) = 2z^2 - 4z + 1$. We have

$$f_{(-1,0,\beta)}(x, y) = -x^2 - x + 1, \quad f_{(1,0,\beta)}(x, y) = x^2 - x - 1$$

and

$$f_{(0,-1,\gamma)}(x, y) = f_{(0,1,\delta)}(x, y) = 2y^2 - 4y + 1,$$

as $\gamma$ and $1/\delta$ are associated elements over $\mathbb{Q}$. We get $F_S(x, y) = -2x^4y^2 + 4x^4y - x^4 + 6x^2y^2 - 12x^2y + 3x^2 - 2y^2 + 4y - 1$ and $S_x = 4$, $S_y = 2$. So if $A$ is of type $m \times n$ with $m \geq 5$ and $n \geq 3$, then $S$ is a valid set for $A$. We have

$$m_{(0,0,S)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ -2 & 0 & 6 & 0 & -2 & 0 & \dots & 0 \\ 4 & 0 & -12 & 0 & 4 & 0 & \dots & 0 \\ -1 & 0 & 3 & 0 & -1 & 0 & \dots & 0 \end{pmatrix}$$

and the switching elements $m_{(u,v,S)}$ $(0 \leq u \leq m - 5, 0 \leq v \leq n - 3)$ form a basis of the set of functions $g :\ A \to \mathbb{Z}$ having zero line sums corresponding to the four elements of $S$.

**Example 3.** Finally we give a somewhat more complicated example with $|S| = 6$. Let $S = \{(-1, 0, \beta), (0, -1, \sqrt{2}), (-1, -1, \beta), (1, 1, \sqrt{3}), (-1, 1, \gamma), (1, -1, \delta)\}$ with the same $\beta, \gamma, \delta$ as in the previous example. Now for the new exponential absorption coefficients we have $P_{\sqrt{2}}(z) = z^2 - 2$ and $P_{\sqrt{3}}(z) = z^2 - 3$, whence

$$f_{(-1,0,\beta)}(x, y) = -x^2 - x + 1, \quad f_{(0,-1,\sqrt{2})}(x, y) = -2y^2 + 1,$$

$$f_{(-1,-1,\beta)}(x,y) = -x^2y^2 - xy + 1, \quad f_{(1,1,\sqrt{3})}(x,y) = x^2y^2 - 3,$$

and

$$f_{(-1,1,\gamma)}(x,y) = f_{(1,-1,\delta)}(x,y) = 2x^2 - 4xy + y^2.$$

We obtain $S_x = 8$, $S_y = 8$, and if $m \geq 8, n \geq 8$, then

$$m_{(0,0,S)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & -2 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 2 & -2 & -10 & 8 & 8 & 0 & 0 & \dots & 0 \\ 0 & 0 & -8 & 8 & -1 & 9 & 13 & -4 & -4 & 0 & \dots & 0 \\ 0 & -6 & 6 & 37 & -31 & -23 & -8 & -8 & 0 & 0 & \dots & 0 \\ 6 & -6 & 22 & -28 & -40 & 12 & 10 & 2 & 2 & 0 & \dots & 0 \\ 0 & -21 & 21 & -7 & 28 & 26 & 2 & 2 & 0 & 0 & \dots & 0 \\ -3 & 3 & 3 & 0 & 8 & -8 & -8 & 0 & 0 & 0 & \dots & 0 \\ 0 & 12 & -12 & -6 & -6 & -6 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -6 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and the switching elements $m_{(u,v,S)}$ $(0 \leq u \leq m-8, 0 \leq v \leq n-8)$ form a basis of the set of functions $g: A \to \mathbb{Z}$ having zero line sums corresponding to the six elements of $S$.

## 5. Proofs

To prove the theorem, we need some lemmas. Lemma 1 shows the correspondence between zero line sums and division of polynomials.

**Lemma 1.** *Let $A$ be as in the theorem, $a, b$ coprime integers, and $\beta$ a nonzero algebraic element over the quotient field $K$ of $R$. Let $L$ be any field containing the splitting field of a defining polynomial of $\beta$ over $K$. Put*

$$\tilde{f}_{(a,b,\beta)}(x,y) = \begin{cases} x^a y^b - \beta, & \text{if } a \geq 0, b \geq 0, \\ x^a - \beta y^{-b}, & \text{if } a \geq 0, b < 0, \\ y^b - \beta x^{-a}, & \text{if } a < 0, b \geq 0, \\ 1 - \beta x^{-a} y^{-b}, & \text{if } a < 0, b < 0. \end{cases}$$

*Then a function $g: A \to L$ has zero line sums corresponding to the triple $(a, b, \beta)$ if and only if $\tilde{f}_{(a,b,\beta)}(x,y)$ divides $\sum_{(i,j)\in A} g(i,j)x^i y^j$ in $L[x,y]$.*

*Proof.* We prove the lemma only with $a > 0, b > 0$, all the other cases can be treated similarly.

Let $T = \{t \in \mathbb{Z} : aj = bi + t \text{ for some } (i,j) \in A\}$. For every $t \in T$ let $(i_t, j_t)$ be the index pair of the first entry of $A$, which is on the directed line $ax = by + t$, and let $I_t$ be the number of the points of $A$ on this line. Set

$$h(x,y) = \sum_{(i,j)\in A} g(i,j)x^i y^j.$$

Observe that we may write

$$h(x,y) = \sum_{t\in T}\sum_{s=0}^{I_t-1} c_t(s)x^{i_t+sa}y^{j_t+sb},$$

where $c_t(s) = g(i_t + sa, j_t + sb)$. This equality can be reformulated as

$$h(x,y) = \sum_{t\in T} x^{i_t}y^{j_t}\sum_{s=0}^{I_t-1} c_t(s)\left(x^a y^b\right)^s.$$

If $(x^a y^b - \beta)$ divides $h(x, y)$ in $L[x, y]$, then after substituting $x^a y^b \leftarrow \beta$, or more precisely $x \leftarrow \left(\beta y^{-b}\right)^{1/a}$, $h(x, y)$ becomes identically zero. However, this yields that $\sum_{s=0}^{I_t-1} c_t(s)\beta^s = \beta^{I_t}\sum_{s=0}^{I_t-1} c_t(I_t - 1 - s)\beta^{-s-1}$ must vanish for every $t \in T$, i.e. $g$ has zero line sums corresponding to $(a, b, \beta)$. This proves the 'if' part of the statement.

To prove the 'only if' part, suppose that all the line sums

$$\sum_{s=0}^{I_t-1} c_t(I_t - 1 - s)\beta^{-s-1} = \beta^{-I_t}\sum_{s=0}^{I_t-1} c_t(s)\beta^s \quad (t \in T)$$

corresponding to $(a, b, \beta)$ of $g$ vanish. This means that $\beta$ is a root of the polynomials $P_t(z) = \sum_{s=0}^{I_t-1} c_t(s)z^s$ for each $t \in T$. This yields that for every $t \in T$ we have

$$P_t(x^a y^b) = (x^a y^b - \beta)Q_t(x, y) \text{ for some } Q_t(x, y) \in L[x, y].$$

Hence $(x^a y^b - \beta)$ divides $h(x, y)$ in $L[x, y]$, and the lemma follows in case $a > 0, b > 0$. In the other cases the proofs are similar. $\square$

**Lemma 2.** *Use the notation of Lemma 1, and write $k$ for the degree and $\beta^{(r)}$ $(1 \le r \le k)$ for the conjugates of $\beta$ over $K$. Then the polynomials $\tilde{f}_{(a,b,\beta^{(r)})}(x, y)$ $(1 \le r \le k)$ defined in Lemma 1 are pairwise non-associated irreducible elements in $L[x, y]$.*

*Proof.* To prove the irreducibility of these polynomials, suppose that for some $r$ with $1 \le r \le k$

$$\tilde{f}_{(a,b,\beta^{(r)})}(x, y) = Q_1(x, y)Q_2(x, y) \text{ with some } Q_1(x, y), Q_2(x, y) \in L[x, y].$$

In this case under the substitution $x^a y^b \leftarrow \beta$ if $a > 0, b > 0$, or under a similar substitution for the other choices of these parameters, one of the polynomials $Q_1, Q_2$ must identically vanish. However, by Lemma 1 this yields that either $Q_1$ or $Q_2$ is divisible by $\tilde{f}_{(a,b,\beta^{(r)})}(x, y)$, so the latter polynomial is an irreducible element in $L[x, y]$. The statement that the polynomials are pairwise non-associated, is trivial. $\square$

Lemma 3 shows that the coefficients of the quotients in Lemma 1 are from $R$.

**Lemma 3.** *Let $a, b$ and $\beta$ be as in Lemma 1. Using the notation of Section 2, a function $g: A \to R$ has zero line sums corresponding to the triple $(a, b, \beta)$ if and only if $P_\beta(x^a y^b)x^p y^q$ divides $\sum_{(i,j)\in A} g(i,j)x^i y^j$ in $R[x, y]$.*

*Proof.* The 'if' part of the statement easily follows from Lemma 1. We prove the 'only if' part only for $a > 0, b > 0$, the other cases can be handled similarly. In this case observe that by Lemma 1, $(x^a y^b - \beta)$ divides $h(x, y) = \sum_{(i,j)\in A} g(i,j)x^i y^j$ over any field $L$ which contains the splitting field of $P_\beta(z)$ over $K$. However, by conjugation, for every conjugate $\beta^{(r)}$ of $\beta$, $(x^a y^b - \beta^{(r)})$ also divides $h(x, y)$ in $L[x, y]$. By Lemma 2 this assertion immediately implies the statement. □

It follows from Lemmas 4 and 5 that the division polynomials in non-opposite directions are coprime and in opposite directions are coprime or associated.

**Lemma 4.** *The polynomials $P_\beta(x^a y^b)x^p y^q$ are irreducible in $R[x, y]$.*

*Proof.* We prove the lemma only for $a > 0, b > 0$, the other cases are similar. In this case we have $p = q = 0$.

Suppose the contrary, that is $P_\beta(x^a y^b)$ is reducible. For $1 \le r \le k$ let $\beta^{(r)}$ be the conjugates of $\beta$ over $K$, and $L$ any field containing $\beta^{(r)}$ $(1 \le r \le k)$. By Lemma 2 there is a set $N \subseteq \{1, \dots, k\}$ and a nonzero $\varepsilon_N \in L$ such that the polynomial

$$Q_N(x, y) = \varepsilon_N \prod_{r\in N} (x^a y^b - \beta^{(r)})$$

has coefficients from $R$. However, by conjugation, this yields that for every $r$ with $1 \le r \le k$, $(x^a y^b - \beta^{(r)})$ divides $Q_N(x, y)$ over $L$. By Lemma 2 we immediately get that $Q_N$ and $P_\beta(x^a y^b)$ are associated elements in $R[x, y]$, and the lemma follows for $a > 0, b > 0$. The proofs in the other cases are similar. □

**Lemma 5.** *Let $\beta$ and $\beta^*$ be nonzero algebraic elements over $K$, and $(a, b)$ and $(a^*, b^*)$ two distinct pairs of coprime integers. Then the polynomials $P_\beta(x^a y^b)x^p y^q$ and $P_{\beta^*}(x^{a^*} y^{b^*})x^{p^*} y^{q^*}$ are associated in $R[x, y]$ if and only if $(a, b) = (-a^*, -b^*)$, and $\beta$ and $1/\beta^*$ are conjugated elements over $K$.*

*Proof.* The 'if' part of the statement is trivial. Suppose that $P_\beta(x^a y^b)x^p y^q$ and $P_{\beta^*}(x^{a^*} y^{b^*})x^{p^*} y^{q^*}$ are associated. Then the degrees of $\beta$ and $\beta^*$ must be equal. Denote by $k$ this number, and for $1 \le r \le k$ let $\beta^{(r)}$ and $\beta^{*(r)}$ be the conjugates of $\beta$ and $\beta^*$ over $K$, respectively. Let $L$ be any field which contains the splitting fields of both $P_\beta$ and $P_{\beta^*}$ over $K$. Then we have the factorizations

$$P_\beta(x^a y^b)x^p y^q = B \prod_{r=1}^{k} \tilde{f}_{(a,b,\beta^{(r)})}(x, y)$$

and

$$P_{\beta^*}(x^{a^*} y^{b^*})x^{p^*} y^{q^*} = B^* \prod_{r=1}^{k} \tilde{f}_{(a^*,b^*,\beta^{*(r)})}(x, y)$$

in $L[x, y]$, where the polynomials on the right hand sides are defined in Lemma 1, and $B$ and $B^*$ are the leading coefficients of $P_\beta(z)$ and $P_{\beta^*}(z)$, respectively. By our assumption and Lemma 2 we obtain that for each $r_1$ with $1 \leq r_1 \leq k$ there exists an $r_2$ also with $1 \leq r_2 \leq k$, such that $\tilde{f}_{(a,b,\beta^{(r_1)})}(x,y)$ and $\tilde{f}_{(a^*,b^*,\beta^{*(r_2)})}(x,y)$ are associated elements in $L[x,y]$. By comparing the exponents of $x$ and $y$ in these polynomials, we get that $(a,b) = (-a^*, -b^*)$ holds, and for the corresponding pairs $(r_1, r_2)$, $\beta^{(r_1)}\beta^{*(r_2)} = 1$ must also be valid. This yields that the sets $\{\beta^{(r)} : 1 \leq r \leq k\}$ and $\{1/\beta^{*(r)} : 1 \leq r \leq k\}$ coincide, which verifies the 'only if' part of the statement. The proof of the lemma is now complete. □

*Proof of the theorem.* By definition, for every $u$ and $v$ the function $F_{(u,v,S)}$ is divisible by $f_{(a_d,b_d,\beta_d)}$ for any $d$ with $1 \leq d \leq D$. Hence by Lemma 1 $m_{(u,v,S)}$ has zero line sums corresponding to the triples in $S$. This proves the second statement of the theorem.

The first part of the theorem is now clearly equivalent to saying that the switching elements

$$m_{(u,v,S)} \quad (0 \leq u < m - S_x,\ 0 \leq v < n - S_y)$$

form a basis of the module

$$M_0 = \{e : A \to R \ : \ e \text{ has zero line sums corresponding to the elements of } S\}$$

over $R$. To show this, we first prove that the switching elements generate $M_0$. Suppose $g \in M_0$ and put

$$h(x,y) = \sum_{(i,j)\in A} g(i,j)x^i y^j.$$

Combining Lemmas 3, 4 and 5, we obtain

$$F_S(x,y) \mid h(x,y) \quad \text{in} \quad R[x,y].$$

Hence there exists a polynomial $Q(x,y) = \sum_{u=0}^{m-1-S_x} \sum_{v=0}^{n-1-S_y} c_{uv}x^u y^v$ in $R[x,y]$ such that $Q(x,y)F_S(x,y) = h(x,y)$. We rewrite this equation as

$$h(x,y) = \sum_{u=0}^{m-1-S_x} \sum_{v=0}^{n-1-S_y} c_{uv} F_{(u,v,S)}(x,y).$$

Now by the definitions of $h(x,y)$ and the switching elements $m_{(u,v,S)}$ we immediately obtain

$$g = \sum_{u=0}^{m-1-S_x} \sum_{v=0}^{n-1-S_y} c_{uv} m_{(u,v,S)},$$

which proves that the functions $m_{(u,v,S)}$ generate $M_0$.

Suppose now that for some coefficients $c_{uv} \in R$ we have

$$\sum_{u=0}^{m-1-S_x} \sum_{v=0}^{n-1-S_y} c_{uv} m_{(u,v,S)}(i,j) = 0 \quad \text{for} \quad 0 \leq i < m,\ 0 \leq j < n. \tag{2}$$

By the definitions of the switching elements, at the bottom-left corner of $m_{(0,0,S)}$ all the other switching elements vanish. This immediately implies $c_{00} = 0$. Considering now $m_{(0,1,S)}$ and using the same argument we obtain $c_{01} = 0$. Continuing this process (taking the switching elements $m_{(u,v,S)}$ in increasing lexicographical order in $(u,v)$ for $0 \leq u < m - S_x$, $0 \leq v < n - S_y$), we easily conclude that all the coefficients $c_{uv}$ must be zero in (2). This shows that the switching elements are linearly independent, which completes the proof of the theorem. □

## 6. Conclusion

In the present paper we extend our investigations in [1] to the case that there is a homogeneous medium absorbing the emitted radiation. A complication compared with [1] is that we have to distinguish opposite directions. The integer-valued functions with the same directed absorption line sums form a coset in the factor module $M/M_0$, where $M$ is the module of all integer-valued functions and $M_0$ the submodule of functions with vanishing line sums in the given directions. We show in which sense the $0-1$-solutions are the shortest vectors in the cosets. In the Theorem we present a basis for $M_0$, thereby characterizing the structure of the coset. We add some examples to illustrate the theory.

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