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PARALLEL LLL-REDUCTION FOR BOUNDING THE INTEGRAL SOLUTIONS OF ELLIPTIC DIOPHANTINE EQUATIONS

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In this file we provide some extra examples for the paper. For the notation and background theory, see the paper.

Example 1. This example is from [10]. The problem is to find the integral points on the curve

$$C: \quad 30u^3 + 45u^2 + 15u = v^3 + 5v^2 + 6v.$$

The curve is birationally equivalent to

$$E: \quad y^2 = x^3 - 2041200x + 968403600.$$

The rank of E is $r = 4$, and an ST-basis of E (obtained by the method in [18]) is given by

$$P_1 = (-1080, -43740), P_2 = (-540, 43740), P_3 = (1080, 4860), P_4 = (540, 4860).$$

The final bound obtained for the coordinates of the images of the integral points of C on E is $N_{final} = 8$ in this basis (cf. [10]).

Strategy 1. Using the above explained methods, we get the following table.

i	j	bound for $\lvert 10n_i \pm n_j \rvert$	bound for $\lvert n_i \rvert$
1	2	(72,66)	6
2	1	(77,70)	7
3	4	(75,77)	7
4	3	(76,78)	7

Based upon the table, the improvement is given by

$$\frac{(2 \cdot 6 + 1)(2 \cdot 7 + 1)(2 \cdot 7 + 1)(2 \cdot 7 + 1)}{(2 \cdot 8 + 1)^4} = 0.525316.$$

Strategy 2. The basis transformation matrices (with respect to the ST-basis) of the best ten bases (obtained by the method of Stroeker and Tzanakis [18]) are given by

$$\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&1&0 \\ 1&0&-1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0 \\ 0&0&-1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ -1&-1&1&0 \\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&-1&0 \\ 0&0&1&0 \\ -1&1&1&0 \\ 0&0&0&1 \end{pmatrix},$$

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$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The corresponding λ values are

$$0.428207, 0.399243, 0.388631, 0.330698, 0.309021,$$

$$0.280895, 0.279407, 0.278964, 0.278013, 0.277034,$$

and the final bounds N_{final} obtained after reduction are

$$8, 8, 8, 9, 10, 10, 10, 10, 10, 10,$$

respectively. Combining these data, using the notation (**??**) (with respect to the ST-basis) we get the system of linear inequalities

$$\begin{pmatrix} -6 \\ -7 \\ -7 \\ -7 \\ -8 \\ -9 \\ -10 \\ -10 \\ -10 \\ -10 \end{pmatrix} \leq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} \leq \begin{pmatrix} 6 \\ 7 \\ 7 \\ 7 \\ 8 \\ 9 \\ 10 \\ 10 \\ 10 \\ 10 \end{pmatrix}. \tag{1}$$

Note that here we could already use the improved upper bounds obtained by *Strategy 1* for the $|n_i|$. Using Latte [11], we get that the above inequality (1) has precisely $N^* = 25825$ integral solutions in (n_1, n_2, n_3, n_4). Hence the "improvement ratio" is

$$N^*/(2N_{final} + 1)^4 = 25825/(2 \cdot 8 + 1)^4 = 0.309204,$$

where $N_{final} = 8$ corresponds to the ST-basis P_1, P_2, P_3, P_4.

Example 2. This example is also from [10]. The problem is to determine the integral points on the curve

$$C: \quad 2u^4 + 56u^3 + 504u^2 + 1440u = v^2 + v.$$

The curve is birationally equivalent to

$$E: \quad y^2 = x^3 - 4034x + 83056.$$

We have $r = \text{rank}(E) = 4$, and an ST-basis of E is

$$P_1 = (-72, 16), P_2 = (15, -161), P_3 = (-38, -426), P_4 = (24, 8).$$

The final bound obtained for the coordinates of the images of the integral points of C on E is $N_{final} = 7$ in this basis (cf. [10]).

Strategy 1. We get the following table.

i	j	bound for $\lvert 10n_i \pm n_j \rvert$	bound for $\lvert n_i \rvert$
1	4	(56,57)	5
2	3	(63,71)	6
3	2	(70,76)	7
4	1	(66,66)	6

The improvement is given by

$$\frac{(2 \cdot 5 + 1)(2 \cdot 6 + 1)(2 \cdot 7 + 1)(2 \cdot 6 + 1)}{(2 \cdot 7 + 1)^4} = 0.550815.$$

Strategy 2. The basis transformation matrices of the best ten bases are

$$\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&-1&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&1&1&1\\ 0&1&0&0\\ 0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ -1&-1&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&1&1&0\\ 0&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&0&0\\ -1&0&1&1\\ 0&0&1&1\\ 0&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&1&1&1\\ 0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\ 1&0&0&-1\\ 1&1&1&-1\\ 0&1&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 1&1&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&1&1&0\\ 0&0&0&1 \end{pmatrix}.$$

The corresponding λ values are

$$0.828603, 0.811000, 0.787555, 0.760924, 0.745728,$$

$$0.738958, 0.734586, 0.733220, 0.727657, 0.727082,$$

and the final bounds N_{final} obtained after reduction are

$$7, 8, 8, 8, 8, 8, 8, 8, 8, 8,$$

respectively. We get the system of linear inequalities

$$\begin{pmatrix} -5\\ -6\\ -7\\ -6\\ -8\\ -8\\ -8\\ -8\\ -8\\ -8 \end{pmatrix} \leq \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&1&-1&1\\ 1&1&-1&1\\ 0&-1&1&0\\ 0&0&-1&1\\ 1&1&0&0\\ -1&-1&1&0 \end{pmatrix} \begin{pmatrix} n_1\\ n_2\\ n_3\\ n_4 \end{pmatrix} \leq \begin{pmatrix} 5\\ 6\\ 7\\ 6\\ 8\\ 8\\ 8\\ 8\\ 8\\ 8 \end{pmatrix}.$$

By Latte [11] we get that the above system has $N^* = 14917$ integral solutions in (n_1, n_2, n_3, n_4). Hence the "improvement ratio" is

$$14917/(2 \cdot 7 + 1)^4 = 0.294657.$$

Example 3. This example is from [20]. The problem is to find the integral points on the curve

$$C: \quad 35u^4 - 350u^3 + 945u^2 - 630u + 11025 = v^2.$$

The curve is birationally equivalent to

$$E: \quad y^2 = x^3 - 1620675x + 385103250.$$

The rank of E is $r = 5$, and an ST-basis of E is given by

$$P_1 = (105, 14700), P_2 = (-4235/9, 872200/9), P_3 = (-315, -29400),$$

$$P_4 = (210, 7350), P_5 = (-1365, -7350).$$

The final bound obtained for the coordinates of the images of the integral point of C on E is $N_{final} = 10$ in this basis (cf. [20]).

Strategy 1. We obtain the following table.

i	j	bound for $\lvert 10n_i \pm n_j\rvert$	bound for $\lvert n_i\rvert$
1	2	(104,108)	10
2	1	(71,80)	7
3	2	(100,104)	10
4	5	(89,94)	9
5	4	(92,94)	9

The improvement is given by

$$\frac{(2\cdot 10+1)(2\cdot 7+1)(2\cdot 10+1)(2\cdot 9+1)(2\cdot 9+1)}{(2\cdot 10+1)^5} = 0.584710.$$

Strategy 2. The basis transformation matrices of the best ten bases:

$$\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&1&1\\ 0&0&-1&-1&-1\\ 0&1&1&1&1\\ 0&0&0&0&-1\\ 0&0&0&-1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&1&1\\ 0&0&-1&-1&-1\\ 0&1&1&1&1\\ 0&0&0&0&-1\\ 0&0&0&-1&0 \end{pmatrix}, \begin{pmatrix} 1&1&1&0&-1\\ 0&-1&-1&0&1\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0\\ 0&0&0&-1&0\\ 0&1&0&1&0\\ 0&0&0&0&1\\ 0&0&-1&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&-1&0&0\\ 0&0&1&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&-1&0&0\\ 0&0&1&0&0\\ 0&1&0&0&0\\ 0&0&0&0&1\\ 0&0&1&1&-1 \end{pmatrix}, \begin{pmatrix} 1&0&-1&0&0\\ 0&0&1&0&0\\ 0&1&0&0&0\\ 0&0&1&1&1\\ 0&0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0\\ 0&0&0&-1&0\\ 0&1&0&1&0\\ 0&0&0&0&1\\ 0&0&1&-1&-1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1\\ 0&0&0&0&-1\\ 0&1&0&0&1\\ 0&0&1&1&-1\\ 0&0&0&-1&0 \end{pmatrix}.$$

The corresponding λ values are

$$0.516652, 0.516646, 0.514401, 0.509521, 0.502631,$$

$$0.501732, 0.499272, 0.497399, 0.497375, 0.491591,$$

and the final bounds N_{final} obtained after reduction are

$$10, 10, 11, 11, 11, 11, 11, 11, 11, 11,$$

respectively. These yield the system of linear inequalities

$$\begin{pmatrix} -10\\ -7\\ -10\\ -9\\ -9\\ -10\\ -11\\ -11 \end{pmatrix} \leq \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&-1&0&1&1\\ 1&1&0&0&0\\ 0&1&1&0&0 \end{pmatrix} \begin{pmatrix} n_1\\ n_2\\ n_3\\ n_4\\ n_5 \end{pmatrix} \leq \begin{pmatrix} 10\\ 7\\ 10\\ 9\\ 9\\ 10\\ 11\\ 11 \end{pmatrix}.$$

Using Latte [11] we get that the above inequality has precisely $N^* = 1378657$ integral solutions in $(n_1, n_2, n_3, n_4, n_5)$. Hence the "improvement ratio" is

$$1378657/(2\cdot 10+1)^5 = 0.337567.$$

Example 4. This example is from [18]. We would like to determine the integral points on the curve

$$E: \quad y^2 = x^3 - 203472x + 18487440.$$

The rank of E is $r = 5$, and an ST-basis of E is

$$P_1 = (468, 5076), P_2 = (-216, 7236), P_3 = (432, 3348),$$

$$P_4 = (-36, 5076), P_5 = (36, 3348).$$

The final bound obtained for the coordinates of the integral points of E is $N_{final} = 9$ in this basis (see [18]).

Strategy 1. We get the table

i	j	bound for $\lvert 10n_i \pm n_j \rvert$	bound for $\lvert n_i \rvert$
1	4	(77,82)	7
2	1	(85,79)	8
3	5	(76,81)	7
4	5	(84,88)	8
5	1	(75,81)	7

The improvement is

$$\frac{(2\cdot 7+1)(2\cdot 8+1)(2\cdot 7+1)(2\cdot 8+1)(2\cdot 7+1)}{(2\cdot 9+1)^5} = 0.393916.$$

Strategy 2. The basis transformation matrices of the best ten bases are

$$\begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\-1&-1&1&1&1 \end{pmatrix}, \begin{pmatrix} 1&1&1&1&1\\0&0&0&0&-1\\1&0&0&0&0\\0&1&0&0&0\\0&0&0&-1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\1&1&-1&1&0\\0&0&0&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\1&1&-1&1&0\\1&1&-1&1&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&-1\\-1&-1&1&1&2\\0&-1&1&1&1\\-1&0&1&1&1\\0&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&1\\1&0&0&0&-1\\0&1&0&0&0\\1&-1&1&0&0\\1&-1&1&1&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&0&1\\1&0&0&0&-1\\1&1&1&1&0\\0&0&-1&-1&0\\0&0&0&-1&0 \end{pmatrix}, \begin{pmatrix} 1&1&1&0&1\\0&0&0&0&-1\\1&0&0&0&0\\0&1&0&0&0\\0&0&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\1&1&1&1&0\\0&0&0&-1&0\\0&0&0&0&1 \end{pmatrix}.$$

The corresponding λ values are

$$0.46493, 0.45844, 0.45792, 0.44837, 0.44736,$$

$$0.42425, 0.41358, 0.41295, 0.41229, 0.41173,$$

and the final bounds N_{final} obtained after the reduction are

$$9, 9, 9, 9, 9, 10, 10, 10, 10, 10,$$

respectively. Combining these data, we get

$$\begin{pmatrix} -7\\-8\\-7\\-8\\-7\\-9\\-9\\-9\\-10\\-10\\-10\\-10 \end{pmatrix} \le \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\1&1&-1&-1&1\\-1&-1&1&1&0\\0&0&0&-1&1\\-1&-1&1&0&0\\-1&-1&0&1&0\\0&-1&1&1&-1\\1&1&0&0&0 \end{pmatrix} \begin{pmatrix} n_1\\n_2\\n_3\\n_4\\n_5 \end{pmatrix} \le \begin{pmatrix} 7\\8\\7\\8\\7\\9\\9\\9\\10\\10\\10\\10 \end{pmatrix}.$$

Latte [11] gives that the above system has $N^* = 396785$ integral solutions in $(n_1, n_2, n_3, n_4, n_5)$. Hence the "improvement ratio" is

$$396785/(2\cdot 9+1)^5 = 0.160246.$$

Example 5. This example is from [18]. The problem is to find the integral points on the curve

$$E: \quad y^2 = x^3 - 879984x + 319138704.$$

The rank of E is $r = 5$, and an ST-basis of E is given by

$$P_1 = (468, 3132), P_2 = (-684, -24516), P_3 = (720, -7668),$$

$$P_4 = (432, -4428), P_5 = (540, -1188).$$

The final bound obtained for the coordinates of the integral points of E is $N_{final} = 9$ in this basis (cf. [18]).

Strategy 1. We obtain the table

Hence the improvement is given by

$$\frac{(2\cdot 8+1)(2\cdot 7+1)(2\cdot 7+1)(2\cdot 9+1)(2\cdot 7+1)}{(2\cdot 9+1)^5} = 0.440259.$$

i	j	bound for $\lvert 10n_i \pm n_j \rvert$	bound for $\lvert n_i \rvert$
1	5	(83,79)	8
2	1	(76,82)	7
3	5	(77,78)	7
4	3	(94,89)	9
5	1	(79,77)	7

Strategy 2. The basis transformation matrices of the best ten bases:

$$\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&1&1\\ 0&0&1&0&0\\ 0&0&0&0&-1\\ 0&0&0&-1&0\\ 0&-1&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&1&1&0\\ 0&0&0&0&1\\ 1&0&0&0&0\\ -1&0&-1&-1&1\\ 0&0&0&-1&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0\\ 0&0&0&1&0\\ -1&1&1&0&1\\ 0&-1&0&1&-1\\ 0&0&1&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ -1&0&1&1&-1\\ 0&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&0\\ 0&0&0&0&1\\ 1&0&0&0&0\\ 0&1&-1&0&1\\ 0&0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0\\ -1&1&0&1&1\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&1&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&1&0\\ 0&0&0&0&1\\ 1&0&0&0&0\\ 0&1&0&-1&1\\ 1&1&1&-1&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0\\ 0&0&0&1&0\\ 1&-1&0&1&1\\ 0&0&0&0&-1\\ 0&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 1&-1&0&1&-1\\ 0&0&1&0&0 \end{pmatrix}.$$

The corresponding λ values are

$$0.492063, 0.462853, 0.457636, 0.454803, 0.454749,$$

$$0.453727, 0.451024, 0.450503, 0.448775, 0.431040,$$

and the final bounds N_{final} obtained after reduction are

$$9, 9, 9, 9, 9, 9, 9, 9, 9, 9,$$

respectively. Thus we get the system of linear inequalities

$$\begin{pmatrix} -8\\ -7\\ -7\\ -9\\ -7\\ -9\\ -9\\ -9 \end{pmatrix} \leq \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 1&-1&1&1&0\\ 1&-1&0&1&0\\ 0&1&-1&-1&1 \end{pmatrix} \begin{pmatrix} n_1\\ n_2\\ n_3\\ n_4\\ n_5 \end{pmatrix} \leq \begin{pmatrix} 8\\ 7\\ 7\\ 9\\ 7\\ 9\\ 9\\ 9 \end{pmatrix}.$$

By Latte [11] we obtain that the above inequality has precisely $N^* = 513939$ integral solutions in $(n_1, n_2, n_3, n_4, n_5)$. Hence the "improvement ratio" is

$$513939/(2 \cdot 9 + 1)^5 = 0.207560.$$

Example 6. This example is from [10]. The original problem translates to find the integral points on the curve

$$C: \quad u^4 + 14u^3 + 63u^2 + 90u = 315v^2 + 630v.$$

The curve is birationally equivalent to

$$E: \quad y^2 = x^3 - 1620675x + 385103250.$$

The rank of E is $r = 5$, and an ST-basis of E is

$$P_1 = (-4235/9, -872200/27), P_2 = (-315, 29400), P_3 = (105, -14700),$$

$$P_4 = (210, 7350), P_5 = (-1365, 7350).$$

The final bound obtained for the coordinates of the images of the integral points of C on E is $N_{final} = 11$ (see [10]).

Strategy 1. We get the following table:

i	j	bound for $\lvert 10n_i \pm n_j \rvert$	bound for $\lvert n_i \rvert$
1	2	(85,79)	8
2	1	(114,109)	11
3	1	(117,110)	11
4	5	(99,106)	10
5	4	(101,110)	10

So the improvement in this case is given by

$$\frac{(2 \cdot 8+1)(2 \cdot 11+1)(2 \cdot 11+1)(2 \cdot 10+1)(2 \cdot 10+1)}{(2 \cdot 11+1)^5} = 0.616174.$$

Strategy 2. The basis transformation matrices (with respect to the ST-basis) of the ten best basis:

$$\begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&1&1&0\\-1&0&0&0&0\\0&0&0&0&1\\0&-1&0&0&0\\0&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&1&0\\1&-1&-1&-1&0\\0&-1&-1&-1&1\\0&-1&0&0&0\\0&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&1&0\\-1&0&0&0&0\\0&-1&-1&-1&1\\0&-1&0&0&0\\0&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0\\1&-1&0&0&0\\0&-1&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\-1&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\-1&0&1&0&0\\0&0&0&1&0\\1&0&0&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\-1&0&0&1&0\\-1&0&1&0&1\\0&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0\\1&-1&0&0&0\\0&-1&1&0&0\\0&0&0&1&0\\0&1&0&1&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0\\1&-1&0&0&0\\0&-1&0&1&0\\0&-1&1&0&1\\0&0&1&0&0 \end{pmatrix}.$$

The corresponding λ values are

$$0.516652, 0.516646, 0.514401, 0.509521, 0.502631,$$

$$0.501732, 0.499272, 0.497399, 0.497375, 0.491591,$$

and the final bounds N_{final} obtained after reduction are

$$11, 12, 12, 12, 12, 12, 12, 12, 12, 12,$$

respectively. Hence we get the system of linear inequalities

$$\begin{pmatrix} -8\\-11\\-11\\-10\\-10\\-11\\-11\\-11 \end{pmatrix} \leq \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\1&0&0&1&-1\\1&1&0&0&0\\1&0&1&0&0 \end{pmatrix} \begin{pmatrix} n_1\\n_2\\n_3\\n_4\\n_5 \end{pmatrix} \leq \begin{pmatrix} 8\\11\\11\\10\\10\\11\\11\\11 \end{pmatrix}.$$

By Latte [11] we get that this inequality has $N^* = 2023095$ integral solutions in $(n_1, n_2, n_3, n_4, n_5)$. Thus the "improvement ratio" is

$$2023095/(2 \cdot 11+1)^5 = 0.314324.$$

Example 7. This example is from [10]. The original problem is to find the integral points on the curve

$$C: \quad 2u^3 + 3u^2 + u = 6v^3 + 60v^2 + 144v.$$

The curve is birationally equivalent to

$$E: \quad y^2 = x^3 - 1008x + 2985993.$$

The rank of E is $r = 6$, and an ST-basis of E is

$$P_1 = (-36, 1725), P_2 = (298, 5399), P_3 = (243, 4134),$$

$$P_4 = (-138, -705), P_5 = (24, 1725), P_6 = (-41, 1720).$$

The final bound obtained for the coordinates of the images of the integral points of C on E is $N_{final} = 7$ in this basis (see [10]).

Strategy 1. We get the table

i	j	bound for $\lvert 10n_i \pm n_j \rvert$	bound for $\lvert n_i \rvert$
1	3	(70,68)	6
2	6	(69,64)	6
3	4	(64,61)	6
4	3	(64,60)	6
5	6	(68,71)	6
6	5	(59,63)	6

Hence the improvement is given by

$$\frac{(2\cdot 6+1)(2\cdot 6+1)(2\cdot 6+1)(2\cdot 6+1)(2\cdot 6+1)(2\cdot 6+1)}{(2\cdot 7+1)^6} = 0.423753.$$

Strategy 2. The basis transformation matrices of the best ten bases:

$$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} -1&-2&1&-1&1&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&-1\\ 0&-1&0&0&1&0\\ 0&-1&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&-1\\ -1&0&0&0&1&0\\ -1&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} -1&-1&1&1&0&0\\ 0&0&0&-1&1&0\\ 0&0&0&0&0&1\\ 1&0&-1&-1&1&-1\\ 1&0&0&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&-1&1&1&1&0\\ -1&0&0&-1&-1&1\\ 0&0&0&0&-1&0\\ 0&0&0&-1&0&0\\ 1&0&0&0&0&0\\ 1&0&1&1&1&-1 \end{pmatrix}, \begin{pmatrix} -1&0&1&-1&1&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&-1\\ 0&0&0&0&1&0\\ 0&1&0&0&0&0 \end{pmatrix}, \begin{pmatrix} -1&1&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&1&0&-1&1&1\\ 0&0&1&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} -1&-1&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&-1&1&-1&1&1\\ 0&1&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 1&1&-1&1&1&1\\ 0&0&0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ -2&-1&1&-1&-1&-1\\ -1&0&0&0&-1&0\\ -1&0&0&0&0&0 \end{pmatrix}.$$

The corresponding λ values are

$$0.640325, 0.627020, 0.603695, 0.603010, 0.599688,$$

$$0.595452, 0.587593, 0.586898, 0.586647, 0.586371,$$

and the final bounds N_{final} obtained after reduction are

$$8, 8, 8, 8, 8, 8, 8, 8, 8, 8,$$

respectively. So we get the following system of linear inequalities

$$\begin{pmatrix} -6\\-6\\-6\\-6\\-6\\-6\\-8\\-8\\-8\\-8\\-8\\-8\\-8 \end{pmatrix} \leq \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ -1&1&-1&-1&1&1\\ 0&0&0&0&1&-1\\ -1&1&-1&-1&0&1\\ 0&1&-1&-1&1&0\\ 0&1&0&0&0&1\\ -1&1&-1&-1&1&0\\ 0&1&-1&-1&1&1 \end{pmatrix} \begin{pmatrix} n_1\\n_2\\n_3\\n_4\\n_5\\n_6 \end{pmatrix} \leq \begin{pmatrix} 6\\6\\6\\6\\6\\6\\8\\8\\8\\8\\8\\8\\8 \end{pmatrix}.$$

Latte [11] gives that the above system has precisely $N^* = 1801039$ integral solutions in $(n_1, n_2, n_3, n_4, n_5, n_6)$. Thus the "improvement ratio" is

$$1801039/(2\cdot 7+1)^6 = 0.158116.$$

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