Almost fifth powers in arithmetic progression

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Abstract

We prove that the product of k consecutive terms of a primitive arithmetic progression is never a perfect fifth power when $3 \leq k \leq 54$. We also provide a more precise statement, concerning the case where the product is an "almost" fifth power. Our theorems yield considerable improvements and extensions, in the fifth power case, of recent results due to Győry, Hajdu and Pintér. While the earlier results have been proved by classical (mainly algebraic number theoretical) methods, our proofs are based upon a new tool: we apply genus 2 curves and the Chabauty method (both the classical and the elliptic verison).

Key words: perfect powers, arithmetic progression, genus 2 curves, Chabauty method
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1 Introduction

Consider the Diophantine equation

$$x(x+d)\dots(x+(k-1)d) = by^n \tag{1}$$

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in non-zero integers x, d, k, b, y, n with $\gcd(x, d) = 1$, $d \geq 1$, $k \geq 3$, $n \geq 2$ and $P(b) \leq k$. Here $P(u)$ stands for the largest prime divisor of a non-zero integer u, with the convention $P(\pm 1) = 1$.

The equation has a very rich literature. For $d = 1$ and $b = 1$, equation (1) has been solved by Erdős and Selfridge [9]. This celebrated result can be reformulated as that the product of two or more consecutive positive integers is never a perfect power. The complete solution of (1) in case of $d = 1$ is due to Saradha [21] (case $k \geq 4$) and Győry [10] (case $k < 4$).

For an overview of the huge number of related results for $d > 1$ we refer to survey papers of Győry [11], Shorey [22], [23] and Tijdeman [25]. Now we mention only results which are closely related to the scope of the present paper, focusing on the complete solution of (1) when the number k of terms is fixed.

In case of $(k, n) = (3, 2)$ equation (1) has infinitely many solutions, already for $b = 1$ (c.f. [25]). Euler (see [8]) proved that (1) has no solutions with $b = 1$, and $(k, n) = (3, 3)$ or $(4, 2)$. Obláth [18], [19] obtained similar results for $(k, n) = (3, 4)$, $(3, 5)$ and $(5, 2)$.

By a conjecture of Erdős, equation (1) has no solutions in positive integers when $k > 3$ and $b = 1$. In other words, the product of k consecutive terms of a primitive positive arithmetic progression with $k > 3$ is never a perfect power. By primitive arithmetic progression we mean one of the form

$$x, x + d, \ldots, x + (k - 1)d,$$

with $\gcd(x, d) = 1$. The conjecture of Erdős has recently been verified for certain values of k in a more general form; see the papers [11], [12], [1], [13]. Since now we focus on the case $n = 5$, we give only the best known result for this particular exponent. (Though the results mentioned are valid for any $n \geq 2$.) The following statement is a combination of results from [11] (case $k = 3$), [12] (cases $k = 4, 5$), [1] (cases $k = 6, 7$) and [13] (cases $8 \leq k \leq 34$).

Theorem A. *The only solutions to equation* (1) *with* $n = 5$, $3 \leq k \leq 34$ *and* $P(b) \leq P_k$, *with*

$$P_k = \begin{cases} 2, & \text{if } k = 3, 4, \\ 3, & \text{if } k = 5, \\ 5, & \text{if } k = 6, 7, \\ 7, & \text{if } 8 \leq k \leq 22, \\ \frac{k-1}{2}, & \text{if } 23 \leq k \leq 34 \end{cases}$$

are given by

$$(k, d) = (8, 1),\ x \in \{-10, -9, -8, 1, 2, 3\}; \quad (k, d) = (8, 2),\ x \in \{-9, -7, -5\};$$

$$(k,d) = (9,1),\ x \in \{-10,-9,1,2\}; \quad (k,d) = (9,2),\ x \in \{-9,-7\};$$
$$(k,d) = (10,1),\ x \in \{-10,1\}; \quad (k,d,x) = (10,2,-9).$$

Note that knowing the values of k, d and x, all solutions (x, d, k, b, y, n) of (1) can be easily listed.

To explain why the case $n = 5$ in equation (1) is special, we need to give some insight into the method of solving (1) for fixed k, in the general case $n \geq 2$. One of the most important tools is the modular method, developed by Wiles [26]. In [11], [12], [1], [13] all three types of ternary equations (i.e. of signatures $(n, n, 2), (n, n, 3), (n, n, n)$) and related results of Wiles [26], Kraus [16], Darmon and Merel [7], Ribet [20], Bennett and Skinner [2], Bennett, Vatsal and Yazdani [3] and others are used. However, the modular technique works effectively only for "large" exponents, typically for $n \geq 7$. Thus the "small" exponents $n = 2, 3, 5$ must be handled separately. In fact these cases are considered in distinct sections, or are covered by separate theorems in the above mentioned papers.

Further, the exponents $n = 2, 3$ have already been considered in separate papers. Equation (1) with $n = 2$ has a broad literature in itself; see e.g. [15] and the references given there. Here we focus only on the resolution of (1) with fixed k. For $n = 2$ and positive x, equation (1) has been completely solved (up to a few exceptional cases) by Hirata-Kohno, Laishram, Shorey and Tijdeman [15] for $k \leq 100$, and in case of $b = 1$, even for $k \leq 109$. Their main tools were elliptic curves and quadratic residues. Later, the exceptional remaining cases have been handled by Tengely [24], by the help of the Chabauty method. At this point we note that we shall refer to the Chabauty method frequently in this paper. For the description of the method, and in particular how to use it in the frame of the program package Magma [4], we refer to the papers of Bruin [5], [6] and the references given there.

When $n = 3$, working mainly with cubic residues, however making use of elliptic curves and the Chabauty method as well, Hajdu, Tengely and Tijdeman [14] obtained all solutions to equation (1) with $k < 32$ such that $P(b) \leq k$ if $4 \leq k \leq 12$ and $P(b) < k$ if $k = 3$ or $k \geq 13$. Further, if $b = 1$ then they could solve (1) for $k < 39$.

The case $n = 5$ has not yet been closely investigated. In this case (in the above mentioned papers considering equation (1) for general exponent n) mainly classical methods were used, due to Dirichlet and Lebesgue (see e.g. [13]). Apparently, for $n = 5$ elliptic curves are not applicable. In the present paper we show that in this case the Chabauty method (both the classical and the elliptic version) can be applied very efficiently. As we mentioned, the Chabauty method has been already used for the cases $n = 2, 3$ in [1], [24], [14]. However, it has been applied only for some particular cases and equations. To prove our results we solve a large number of genus 2 equations by Chabauty method, and then build a kind of sieve system based upon them.

2 New results

Our first theorem considerably extends Theorem A, in the most interesting case of $b = 1$ in equation (1). We call an arithmetic progression of the form $x, x + d, \ldots, x + (k - 1)d$ *primitive*, if $\gcd(x, d) = 1$.

Theorem 1 *The product of k consecutive non-zero terms in a primitive arithmetic progression with $3 \leq k \leq 54$ is never a fifth power.*

In fact Theorem 1 follows directly from the next result. To formulate it, we need to introduce a new concept. An arithmetic progression $x, x + d, \ldots, x + (k - 1)d$ is called *trivial* if $d \leq 5$ and $|x + id| \leq 15$ for some $i = 0, 1, \ldots, k - 1$. Further, a solution to equation (1) is also called *trivial*, if the terms $x, x + d, \ldots, x+(k-1)d$ on the left-hand side of (1) form a trivial arithmetic progression. This concept is needed because of the huge number of trivial solutions; on the other hand, such solutions of (1) can be listed easily for any fixed k.

Theorem 2 *Equation (1) with $n = 5$, $3 \leq k \leq 24$ and $P(b) \leq P_k$ has precisely the non-trivial solutions with*

$$(k, d) = (3, 7),\ x \in \{-16, -8, -6, 2\};$$

$$(k, d) = (4, 7),\ x \in \{-16, -15, -12, -9, -6, -5\};$$

$$(k, d) = (4, 11),\ x \in \{-27, -6\}; \quad (k, d) = (5, 7),\ x \in \{-16, -12\};$$

$$(k, d) = (5, 11),\ x \in \{-36, -32, -12, -8\};$$

$$(k, d) = (5, 13),\ x \in \{-40, -27, -25, -12\};$$

$$(k, d) = (6, 7),\ x \in \{-32, -25, -10, -3\};$$

$$(k, d) = (6, 9),\ x \in \{-25, -20\}; \quad (k, d) = (6, 13),\ x \in \{-40, -25\};$$

$$(k, d) = (7, 7),\ x \in \{-39, -32, -27, -22, -20, -15, -10, -3\};$$

$$(k, d) = (8, 7),\ x \in \{-39, -27, -22, -10\};$$

$$(k, d) = (9, 7),\ x \in \{-39, -34, -32, -24, -22, -17\};$$

$$(k, d) = (10, 7),\ x \in \{-39, -24\},$$

where the values of P_k are given by

k	3	4	5	6	7, 8
P_k	3	5	7	11	13
k	9, 10, 11, 12	13, 14, 15	16, 17	18, 19, 20, 21, 22, 23	24
P_k	17	19	23	29	31

Observe that $P_k > k$ for $k \geq 4$ in Theorem 2, which is a new feature about equation (1).

As a simple and immediate corollary of Theorem 2 we get the following statement, concerning the case $P(b) \leq k$. We mention that already this result yields considerable improvement of Theorem A, in particular with respect to the bound for $P(b)$.

Corollary 3 *For $n = 5$ and $3 \leq k \leq 36$ all non-trivial solutions of equation (1) with $P(b) \leq k$ are given by*

$$(k,d) = (3,7),\ x \in \{-16,-8,-6,2\}; \quad (k,d) = (5,7),\ x \in \{-16,-12\}.$$

Our last theorem provides the key to the proof of Theorem 2 in case of $k \geq 4$. It has been proved by a kind of sieving procedure, based upon genus 2 equations and the Chabauty method. Note that having an increasing arithmetic progression $z_1 < \ldots < z_l$, by symmetry we obtain that $-z_l < \ldots < -z_1$ is also an increasing arithmetic progression. Hence dealing with such arithmetic progressions it is sufficient to give only one progression from each symmetric pair.

Theorem 4 *Let $4 \leq t \leq 8$ and $z_0 < z_1 < \ldots < z_{t-1}$ be a non-trivial primitive arithmetic progression. Suppose that*

$$z_0 = b_0 x_0^5, z_{i_1} = b_{i_1} x_{i_1}^5, z_{i_2} = b_{i_2} x_{i_2}^5, z_{t-1} = b_{t-1} x_{t-1}^5,$$

with some indices $0 < i_1 < i_2 < t-1$ such that $P(b_0 b_{i_1} b_{i_2} b_{t-1}) \leq 5$. Then the initial term z_0 and common difference $z_1 - z_0$ of the arithmetic progression $z_0, \ldots, z_{t-1}$ for the separate values of $t = 4, \ldots, 8$ up to symmetry is one of

$t = 4 : (-9,7), (-6,7), (-6,11), (-5,7);$

$t = 5 : (-32,17), (-25,13), (-20,11), (-16,13), (-12,7), (-12,11), (-12,13), (-10,7), (-8,7), (-8,11), (-4,7), (-3,7), (-1,7), (2,7), (4,7), (4,23);$

$t=6:(-125,61),(-81,17),(-30,31),(-25,8),(-25,11),(-25,13),(-25,17),$ $(-20,9),(-20,13),(-20,19),(-20,29),(-15,7),(-15,11),(-15,13),(-15,23),$ $(-10,7),(-10,11),(-8,7),(-5,7),(-3,7),(-1,11),(-1,13),(1,7),(5,11);$

$t=7:(-54,19),(-54,29),(-48,23),(-30,11),(-30,13),(-27,17),(-24,13),$ $(-18,7),(-18,11),(-18,13),(-18,19),(-16,11),(-15,7),(-12,7),(-12,11),$ $(-10,7),(-6,7),(-6,11),(-4,9),(-3,13),(-2,7),(-2,17),(2,13),(3,7),(6,7),$ $(8,7),(9,11),(18,7);$

$t=8:(-405,131),(-125,41),(-100,49),(-32,11),(-27,11),(-27,13),$ $(-25,19),(-24,7),(-16,13),(-10,13),(-9,7),(-5,11),(-4,7),(-2,11),$ $(-1,13),(-1,7),(1,7),(3,11),(4,11),(5,7),(6,17).$

3 Preliminaries

Before giving the proofs of our results, we explain some principles and techniques which shall be used rather frequently later on. We present these tools separately because in this way the structure of our proofs will be more transparent.

3.1 Reducing equation (1) *to arithmetic progressions of "almost" fifth powers*

In a standard way, as $\gcd(x,d)=1$ and $n=5$, any solution of equation (1) can be written as

$$x+id=a_ix_i^5 \quad (i=0,1,\dots,k-1) \tag{2}$$

where x_i is a non-zero integer and a_i is a fifth power free positive integer with $P(a_i)\le k$. This observation justifies the title of the paper, as well: the members of the arithmetic progression $x,x+d,\dots,x+(k-1)d$ are "almost" n-th powers.

3.2 Listing the possible coefficient tuples

Suppose that

$$a_{i_1}x_{i_1}^5<a_{i_2}x_{i_2}^5<\cdots<a_{i_t}x_{i_t}^5 \tag{3}$$

are t (not necessarily consecutive) nonzero terms of a primitive arithmetic progression, with a_{i_j} as in (2). In this subsection we explain a method to list all the possible coefficient t-tuples $(a_{i_1},a_{i_2},\dots,a_{i_t})$ corresponding to (3).

Observe that knowing a_{i_j} is equivalent to knowing the exponents $\nu_p(a_{i_j})$ of the primes $p \leq k$ in the factorization of a_{i_j}. Take an arbitrary prime $p \leq k$ dividing one of the terms $a_{i_j} x_{i_j}^5$, and suppose that i_{j_0} is such an index that

$$\nu_p(a_{i_{j_0}} x_{i_{j_0}}^5) \geq \nu_p(a_{i_j} x_{i_j}^5) \quad \text{for all } j = 1, \ldots, t.$$

Since the arithmetic progression is assumed to be primitive, one can easily check that then for all $j = 1, \ldots, t$ with $j \neq j_0$ we have $\nu_p(a_{i_j} x_{i_j}^5) = \nu_p(j - j_0)$. As we have $\nu_p(a_{i_{j_0}}) < 5$, we can simply list all possibilities for the exponents of the prime p in the coefficients $a_{i_1}, a_{i_2}, \ldots, a_{i_t}$. Then combining these possibilities for all primes $p \leq k$, we can list all the possible coefficient t-tuples $(a_{i_1}, a_{i_2}, \ldots, a_{i_t})$ which may occur in (3).

3.3 Local testing of coefficient tuples

As we will see, some of the coefficient tuples listed in the previous subsection in fact cannot occur as coefficients of fifth powers in arithmetic progressions. In many cases this can be shown already modulo m with some appropriate choice of m. We shall use the moduli $m = 11, 25$.

Let $0 \leq i_1 < i_2 < \cdots < i_t \leq k - 1$ be t indices, and consider a coefficient t-tuple $(a_{i_1}, a_{i_2}, \ldots, a_{i_t})$, which in fact we would like to exclude - that is, we would like to show that no corresponding subsequence

$$a_{i_1} x_{i_1}^5, \ldots, a_{i_t} x_{i_t}^5 \tag{4}$$

of any appropriate arithmetic progression exists. For this purpose, consider (4) modulo m (with $m = 11$ or 25). Observe that to have such a sequence, we should find appropriate fifth powers modulo m. We check all the possibilities. (Since we work with $m = 11$ and $m = 25$, the fifth powers modulo m are only $\{0, \pm 1\}$ and $\{0, \pm 1, \pm 7\}$, respectively.) Observe that by coprimality, we know that $m \mid a_{i_{j_1}}, a_{i_{j_2}}$ yields that $m \mid j_1 - j_2$. If we find that no fifth powers modulo m exist having also the previous property, then the actual coefficient tuple $(a_{i_1}, \ldots, a_{i_t})$ is not valid in the sense that no underlying subsequence (4) exists. We shall illustrate how to use this test later on.

3.4 Reducing the problem to genus 2 equations

We found two ways to get access to genus 2 equations.

3.4.1 Reduction method I

Suppose that $a_0x_0^5, a_1x_1^5, a_2x_2^5$ is an arithmetic progression with nonzero terms, and with common difference d. Then we have

$$(a_1x_1^5)^2 - a_0x_0^5 \cdot a_2x_2^5 = d^2$$

which after the substitutions $X = -x_0x_2/x_1^2$, $Y = d/x_1^5$ and $A = a_0a_2$, $B = a_1^2$ yields the genus 2 equation

$$AX^5 + B = Y^2$$

in $X, Y \in \mathbb{Q}$.

3.4.2 Reduction method II

Suppose that

$$a_ix_i^5, a_jx_j^5, a_ux_u^5, a_vx_v^5$$

are four terms of an arithmetic progression. Then we have

$$(j-u)a_ix_i^5 + (u-i)a_jx_j^5 = (j-i)a_ux_u^5$$

and

$$(j-v)a_ix_i^5 + (v-i)a_jx_j^5 = (j-i)a_vx_v^5.$$

Multiplying these identities we get an equation of the form

$$AX^{10} + BX^5Y^5 + CY^{10} = DZ^5, \tag{5}$$

where $A = (j-u)(j-v)a_i^2$, $B = ((j-u)(v-i) + (u-i)(j-v))a_ia_j$, $C = (u-i)(v-i)a_j^2$, $D = (j-i)^2a_ua_v$ and $X = x_i$, $Y = x_j$, $Z = x_ux_v$. Then from (5) we can easily get a pair of genus 2 curves over $\mathbb{Q}$

$$A_1Z_1^5 + B_1 = X_1^2 \quad \text{and} \quad A_2Z_2^5 + B_2 = X_2^2$$

with the notation $A_1 = 4AD$, $B_1 = B^2 - 4AC$, $X_1 = 2AX^5/Y^5 + B$, $Z_1 = Z/Y^2$ and $A_2 = 4CD$, $B_2 = B^2 - 4AC$, $X_2 = 2CY^5/X^5 + B$, $Z_2 = Z/X^2$, respectively.

The rational points on the genus 2 curves obtained by both methods (under suitable assumptions) can be determined by the Chabauty method. Then, following the corresponding substitutions backwards we can determine the actual members of the original arithmetic progressions.

Note that in fact in case of $k = 3$ in the proof of Theorem 2 we also use genus 1 curves over some number fields, which can be treated by the elliptic Chabauty method. However, since these are particular cases, we do not include them in this "general" discussion.

4 Proofs

We give the proofs of our results in a specific order. First we prove the case $k = 3$ of Theorem 2. We do so because this result is needed in the proof of Theorem 4, which is the next step. The latter result gives the key to derive Theorem 2 for $k \geq 4$. Then we continue by proving the cases $k \geq 4$ of Theorem 2 and its corollary. Finally, we give the proof of Theorem 1, which easily follows from Theorem 2.

In the proof of case $k = 3$ of Theorem 2 we shall make use of two lemmas. The first one is due to Bennett, Bruin, Győry, Hajdu [1].

Lemma 5 *Let C be a positive integer with $P(C) \leq 5$. If the Diophantine equation*

$$X^5 + Y^5 = CZ^5$$

has solutions in nonzero coprime integers X, Y and Z, then $C = 2$ and $X = Y = \pm 1$.

The second lemma is a result of Kraus [16].

Lemma 6 *Let A and B be coprime positive integers with $AB = 2^\alpha 3^\beta$ for nonnegative integers α and β with $\alpha \geq 4$. Then the Diophantine equation*

$$AX^5 + BY^5 = Z^5$$

has no solutions in coprime nonzero integers X, Y and Z.

Proof of the case $k = 3$ of Theorem 2. First list all the possible coefficient triples (a_0, a_1, a_2) as in (2). This can be done by the method explained in Subsection 3.2. Altogether we obtain 182 such triples. Observe that $a_2x_2^5, a_1x_1^5, a_0x_0^5$ is also an arithmetic progression. Hence by symmetry it is sufficient to consider those 106 triples for which $a_0 \leq a_2$. (It will be clear from our method that we can do so without loss of generality indeed.)

Clearly, $a_0x_0^5, a_1x_1^5, a_2x_2^5$ is also an arithmetic progression modulo 11 and 25. So we can test the coefficient triples modulo 11 and 25, as explained in Subsection 3.3. After the modulo 11 test we are left with 88 triples; for example $(1, 1, 6)$ gets excluded by this method. The test modulo 25 excludes 6 more triples (e.g. $(1, 4, 3)$), and we are left with 82 ones.

Then we apply Lemmas 5 and 6, in this order, for the remaining set of triples. As an example for the application of Lemma 5 consider $(a_0, a_1, a_2) = (2, 1, 4)$. The identity $a_0x_0^5 + a_2x_2^5 = 2a_1x_1^5$ gives an equation of the shape

$$X^5 + Y^5 = 2Z^5,$$

with $X = -x_0$, $Y = x_2$, $Z = x_1$, hence with X, Y, Z coprime. Then Lemma 5 gives that the only solutions are given by $(X, Y, Z) = \pm(1, 1, 1)$. In view of our assumption that the arithmetic progression on the left hand side of (1) has a positive common difference, we get that in this case the progression must be given by $(x, d) = (-2, 3)$, i.e. $x_0 = -1$, $x_1 = x_2 = 1$. Note that here we can automatically handle the "symmetric" case $(a_0, a_1, a_2) = (4, 1, 2)$. For this triple we get the only arithmetic progression is defined by $(x, d) = (-4, 3)$, belonging to $x_0 = x_1 = -1$, $x_2 = 1$. By the help of Lemma 5 we can exclude 58 triples. (Note that from this step, as we have seen, some solutions are obtained.) To see an example also for the application of Lemma 6, take $(a_0, a_1, a_2) = (1, 1, 54)$. As one can easily check, this triple has not been excluded so far, by any of our previous filters. Observe that since $a_2x_2^5$ is even, $a_0x_0^5$ also must be even, i.e. $2 \mid x_0$. Thus using the identity $a_0x_0^5 + a_2x_2^5 = 2a_1x_1^5$ once again, we get an equation of the form

$$16X^5 + 27Y^5 = Z^5$$

with $X = x_0/2$, $Y = x_2$, $Z = x_1$, and $\gcd(X, Y, Z) = 1$. Then Lemma 6 shows that this equation has no solutions, so there is no arithmetic progression with coefficient triple $(1, 1, 54)$. By Lemma 6 we can exclude 6 more triples, so at this stage we are left with 18 ones.

Now we apply our Reduction method I explained in Subsection 3.4.1 to handle the remaining triples. Note that the Chabauty method for determining the rational points on a genus 2 curve is applicable only if the rank of the Jacobian of the curve is at most one. We find that in 16 out of the 18 triples this is just the case. For example, when $(a_0, a_1, a_2) = (4, 1, 18)$ we get the curve

$$72X^5 + 1 = Y^2,$$

where the rank of the Jacobian of the curve is 0. The rational points on this curve (and two more curves where the ranks of the Jacobians are zero) can be determined by the procedure `Chabauty0` of Magma. It turns out that the above equation has the only rational solutions $(X, Y) = (0, \pm 1)$. Since there is no corresponding arithmetic progression on the left hand side of (1), this triple is simply excluded. In case of $(a_0, a_1, a_2) = (1, 2, 3)$ the corresponding genus 2 curve is given by

$$3X^5 + 4 = Y^2,$$

where the rank of the Jacobian of the curve is one. Then we use the procedure `Chabauty` of Magma (as well as in case of 12 alike curves) to get the rational points on the curve. We get that the above curve has the only rational points $(X,Y) = (-1,\pm1),(0,\pm2),(2,\pm10)$. These points yield the only arithmetic progression given by

$$(x,d) = (1,1).$$

(In the "symmetric" case $(a_0,a_1,a_2) = (3,2,1)$ we get the same curve, and the rational points yield the only arithmetic progression $(x,d) = (-3,1)$.) Only in the cases $(a_0,a_1,a_2) = (1,1,3),(2,9,16)$ do we get genus 2 curves where the ranks of the Jacobians are > 1 (namely, equal to 2 in both cases). We handle these triples by the elliptic Chabauty method, and the procedure `Chabauty` of Magma. We give details only for the triple $(1,1,3)$, the other one can be handled similarly. In this case, using the identity $(x+d)^2 - x(x+2d) = d^2$, we get the equation

$$X^5 - 3Y^5 = Z^2 \tag{6}$$

with $X = x_1^2$, $Y = x_0x_2$, $Z = d$. Further, the coprimality property yields $\gcd(X,Y,Z) = 1$. Finally, we may also assume that XY is odd. Indeed, $2 \mid Y$ would easily imply that both x_0 and x_2 are even, which would violate the coprimality property. Further, $2 \mid X$ would mean that $2 \mid x_1$. Then the identity $a_0x_0^5 + a_2x_2^5 = 2a_1x_1^5$ would give rise to

$$64(x_1/2)^5 - 3x_2^5 = x_0^5,$$

which is a contradiction by Lemma 6. Let K be the number field generated by $\alpha = \sqrt[5]{3}$ over $\mathbb{Q}$. Using the procedure `pSelmerGroup` of Magma, following the method of Bruin [6] we get that (6) can be factorized as

$$X^4 + \alpha XY^3 + \alpha^2X^2Y^2 + \alpha^3XY^3 + \alpha^4Y^4 = \delta U^2 \tag{7}$$

and

$$X - \alpha Y = \delta^{-1}V^2 \tag{8}$$

where U,V are some algebraic integers in K, and

$$\delta \in \{1, 7+6\alpha+5\alpha^2+4\alpha^3+3\alpha^4, 1+\alpha+\alpha^3, 4+2\alpha+\alpha^4\}.$$

Note that δ is a unit in K, so δ and δ^{-1} are algebraic integers in K. In case of $\delta = 1+\alpha+\alpha^3$ or $4+2\alpha+\alpha^4$, write

$$V = b_0 + b_1\alpha + b_2\alpha^2 + b_3\alpha^3 + b_4\alpha^4$$

with some integers b_0,b_1,b_2,b_3,b_4 (using that $1,\alpha,\alpha^2,\alpha^3,\alpha^4$ is an integral basis for K). Expanding equation (8) in both choices for δ and using that XY is odd, we easily get a contradiction modulo 2 or 4, respectively. Assume next that $\delta = 1$. Then equation (7) yields the elliptic curve

$$E_1:\ u^4 + \alpha u^3 + \alpha^2u^2 + \alpha^3u + \alpha^4 = v^2$$

over K, with $u = X/Y$ and $v = U/Y^2$. Using the point $(0,\alpha^2)$ of E_1, one can apply the elliptic Chabauty method and the procedure `Chabauty` of Magma to find the points of E_1 with $(u,v) \in \mathbb{Q} \times K$. In the present case the only such points are given by $(u, \pm v) = (0, \alpha^2)$. However, this point yields $x_1 = 0$ which is impossible. Finally, assume that $\delta = 7 + 6\alpha + 5\alpha^2 + 4\alpha^3 + 3\alpha^4$. Then (7) gives rise to the elliptic curve

$$E_2 : \; u^4 + \alpha u^3 + \alpha^2 u^2 + \alpha^3 u + \alpha^4 = (7 + 6\alpha + 5\alpha^2 + 4\alpha^3 + 3\alpha^4)v^2$$

over K, again with $u = X/Y$ and $v = U/Y^2$. Using the point $(-1, 1 + \alpha - \alpha^2 + \alpha^3 - \alpha^4)$ of E_2, by a similar procedure as in case of E_1 we get that the only points $(u, \pm v) \in \mathbb{Q} \times K$ of E_2 are $(-1, 1 + \alpha - \alpha^2 + \alpha^3 - \alpha^4)$ and $(3, 3 - 3\alpha + 7\alpha^2 - 3\alpha^3 - \alpha^4)$. These points yield the only arithmetic progression given by $(x,d) = (-1, 2)$, and the triple $(1,1,3)$ is completely discussed. Note that obviously, in case of the coefficient triple $(3,1,1)$ we get the only progression $(x,d) = (-3,2)$.

In case of the triple $(a_0, a_1, a_2) = (2, 9, 16)$ by a similar method we obtain that the only underlying arithmetic progression is $(x,d) = (2,7)$ (and in case of $(a_0, a_1, a_2) = (16, 9, 2)$ it is $(x,d) = (-16, 7)$), and the proof of the case of $k = 3$ of Theorem 2 is complete.

Proof of Theorem 4. We work inductively on t. Assume first that $t = 4$. Then the four terms $b_0 x_0^5, b_{i_1} x_{i_1}^5, b_{i_2} x_{i_2}^5, b_3 x_3^5$ in fact are consecutive ones of an arithmetic progression, that is, $i_1 = 1, i_2 = 2$. Then by case $k = 3$ of Theorem 2 (which has already been proved) we may assume that $5 \mid b_1 b_2$. Using symmetry (just as before) we may further suppose that $b_0 \leq b_3$. Now following the method explained in Subsection 3.2 we can list all such coefficient quadruples (b_0, b_1, b_2, b_3), which further have the properties as the coefficients in (2). Then we check the remaining quadruples modulo 11, modulo 25, then by Lemmas 5, 6. Since these checks go along the same lines as in the proof of the case of $k = 3$ of Theorem 2 above, we suppress the details.

Then for the case of the quadruples still remaining, we choose two arbitrary indices out of $\{0, 1, 2, 3\}$ as i, j (the remaining two indices will play the role of u, v), and apply Reduction method II as explained in Subsection 3.4.2 to construct two genus 2 curves C_1 and C_2. If for either of these curves we have that the rank of the Jacobian is ≤ 1, then by applying the Chabauty method (using Magma) its rational points can be determined. Then we get all arithmetic progressions corresponding to the actual coefficient quadruple. If the choice of i, j and u, v yields curves where the ranks of the Jacobians are ≥ 2, then we make another choice for i, j and u, v, etc. Since we can construct $2 \cdot \binom{4}{2} = 12$ such curves (which apparently are "independent"), we have a good chance to handle all coefficient quadruples. In fact, this is just what happens indeed. For example, let $(b_0, b_1, b_2, b_3) = (3, 10, 1, 162)$. Then choosing $(i,j) = (0,1)$ and

$(u, v) = (2, 3)$ in Reduction method II, we have

$$-3x_0^5 + 20x_1^5 = x_2^5$$

and

$$-6x_0^5 + 30x_1^5 = 162x_3^5.$$

Multiplying these identities we get the equation

$$18x_0^{10} - 210(x_0x_1)^5 + 600x_1^{10} = 162(x_2x_3)^5.$$

Introducing the new variables $X = x_2x_3/x_1^2$ and $Y = 6x_0^5/x_1^5 - 35$, the previous equation yields

$$Y^2 = 324X^5 + 25.$$

This equation is of genus 2 where the rank of the Jacobian is 0. Using the procedure `Chabauty0` of Magma, we get that the only rational solutions of this equation are $(X, Y) = (0, \pm 5)$. Following the substitutions backwards, we obtain no solution for x_0, x_1, x_2, x_3.

We handled all the possible coefficient quadruples remaining after the above explained tests similarly. We get that the only non-trivial possibilities in case of $t = 4$ are those given in the theorem.

Now assume that the statement is proved for some $t \in \{4, 5, 6, 7\}$, and consider the value $t+1$. The indices i_1, i_2 may take only $(t-1)(t-2)/2$ values altogether. From this point on we just repeat the same steps as with $t = 4$. For instance, suppose we have already finished with the case $t = 7$ and consider the case of $t + 1 = 8$ terms. Then we have 15 possibilities for the pair of indices (i_1, i_2), given by $0 < i_1 < i_2 < 7$. As an example, take $(i_1, i_2) = (2, 3)$ and consider the tuple $(b_0, b_2, b_3, b_7) = (24, 10, 3, 25)$. As it cannot be excluded neither modulo 11, modulo 25, nor by Lemmas 5, 6, we use Reduction method II, again. Choosing $(i, j) = (0, 7)$ and $(u, v) = (2, 3)$, after simplifying by 10 and 3, respectively, we obtain

$$12x_0^5 + 5x_7^5 = 7x_2^5$$
$$32x_0^5 + 25x_7^5 = 7x_3^5.$$

Multiplying these identities we get

$$384x_0^{10} + 460(x_0x_7)^5 + 125x_7^{10} = 49(x_2x_3)^5.$$

After some calculations we are left with the equation

$$Y^2 = 3X^5 + 25,$$

where $X = 2x_2x_3/x_7^2$ and $Y = (192x_0^5/x_7^5 + 115)/7$. This equation is of genus 2 and the rank of the Jacobian of the curve is 1. Using the procedure `Chabauty` of Magma again, we conclude that its rational solutions are $(X, Y) = (0, \pm 5), (2, \pm 11)$. Following the substitutions backwards, we find the only solution for the tuple $(x_0, x_2, x_3, x_7) = (-1, -1, -1, 1)$ and the arithmetic progression $(-24, -17, -10, -3, 4, 11, 18, 25)$.

Altogether we get the only possibilities listed in the statement, and the proof of the theorem is complete.

Proof of the case $k \geq 4$ of Theorem 2. Clearly, the case $k = 4$ is an immediate consequence of Theorem 4. Further, observe that the cases $k = 8, 10, 11, 12, 14, 15, 17, 19, 20, 21, 22, 23$ trivially follow from the corresponding cases for $k-1$. Hence it is sufficient to consider the values $k = 5, 6, 7, 9, 13, 16, 18, 24$. In each case we make the following steps. We list all the possible coefficient k-tuples $(a_0, a_1, \ldots, a_{k-1})$ by the method given in Subsection 3.2. As previously, by symmetry we may assume that $a_0 \leq a_{k-1}$. In the generation process we consider only those placements of primes which cannot be automatically excluded by induction. For example, let $k = 13$; then $P_k = 19$. If $19 \nmid a_4 a_5 a_6 a_7 a_8$ then by coprimality we have that either $P(a_0 a_1 \ldots a_8) \leq 17$ or $P(a_4 a_5 \ldots a_{12}) \leq 17$, and we can apply induction based upon the case $k = 9$. Further, if say $19 \mid a_8$ but $17 \nmid a_1 a_2 \ldots a_6$ then one of $P(a_0 a_1 \ldots a_6) \leq 13$, $P(a_1 \ldots a_7) \leq 13$ holds, and we can use the case $k = 7$, and so on. Then for the remaining tuples try to find indices $j_1, j_2, j_3, j_4 \in \{0, 1, \ldots, k-1\}$ which are (not necessarily consecutive) terms of an arithmetic progression of length t with $4 \leq t \leq 8$, such that $P(a_{j_1} a_{j_2} a_{j_3} a_{j_4}) \leq 5$. It turns out that it is possible to find such indices in case of all the remaining k-tuples. Having four such indices, we can simply apply Theorem 4 to handle the actual coefficient tuple. For example, let $k = 6$ and consider the tuple

$$(a_0, a_1, \ldots, a_5) = (20, 11, 2, 7, 16, 25).$$

Note that this tuple cannot be excluded by induction. Take the indices $(j_1, j_2, j_3, j_4) = (0, 2, 4, 5)$, and observe that $P(a_0 a_2 a_4 a_5) \leq 5$ holds. Applying Theorem 4 with $t = 6$, $b_0 = a_0, b_{i_1} = a_2, b_{i_2} = a_4, b_5 = a_5$, we find that the only non-trivial primitive increasing arithmetic progressions corresponding to this tuple are $-20, -11, -2, 7, 16, 25$ and its symmetric pair $-25, -16, -7, 2, 11, 20$. These progressions are listed in the statement.

Considering another example, let $k = 18$ and take the tuple

$$\begin{aligned}(a_0, a_1, \ldots, a_{17}) &= \\ &= (2, 125, 132, 13, 14, 57, 40, 29, 54, 1, 68, 105, 46, 11, 48, 1, 130, 9).\end{aligned}$$

This tuple cannot be excluded using induction. However, we find four appropriate indices again, namely $(j_1, j_2, j_3, j_4) = (8, 9, 14, 15)$ for which $P(a_8 a_9 a_{14} a_{15}) \leq 5$ holds. Applying Theorem 4 with $t = 8$, $b_0 = a_8, b_{i_1} = a_9, b_{i_2} = a_{14}, b_{t-1} = a_{15}$, we find that the only possible underlying 8-tuple is $(a_8, \ldots, a_{15}) = (54, 1, 68, 105, 46, 11, 48, 1)$. However, there is no arithmetic progression having the appropriate property corresponding to this tuple. Therefore we have no solution with the original 18-tuple $(a_0, a_1, \ldots, a_{17})$.

By this process we have found all the non-trivial arithmetic progressions, which are just the ones listed in the statement.

Proof of Corollary 3. Since the next prime after 31 is 37, the statement is an immediate consequence of Theorem 2.

Proof of Theorem 1. For $k \leq 24$ the statement is a simple consequence of Theorem 4. In case of $25 \leq k \leq 54$, observe that in (2) the product $A := a_0 a_1 \ldots a_{k-1}$ must be a full fifth power. Thus any prime $p \mid A$ must divide at least two coefficients a_i. Hence one can easily check that for these values of k there always exists an index i with $0 \leq i < k - 24$ such that $P(a_i a_{i+1} \ldots a_{i+23}) \leq 31$. So the statement follows from Theorem 4 also in this case.

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