REPRESENTATION OF FINITE GRAPHS AS DIFFERENCE GRAPHS OF S-UNITS, II

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ABSTRACT. In part I of the present paper the following problem was investigated. Let G be a finite simple graph, and S be a finite set of primes. We say that G is representable with S if it is possible to attach rational numbers to the vertices of G such that the vertices v_1, v_2 are connected by an edge if and only if the difference of the attached values is an S-unit. In part I we gave several results concerning the representability of graphs in the above sense.

In the present paper we extend the results from paper I to the algebraic number field case and make some of them effective. Besides we prove some new theorems: we prove that G is infinitely representable with S if and only if it has a degenerate representation with S, and we also deal with the representability with S of the union of two graphs of which at least one is finitely representable with S.

1. INTRODUCTION

In part I of the paper [13] we obtained a variety of theorems on graphs where the vertices have distinct rational values and two vertices are connected by an edge if and only if their values differ by an S-unit where S is a given finite set of primes. In this paper we generalize many of these results to the case when the underlying field is not necessarily $\mathbb{Q}$, but any algebraic number field K. Moreover, we give effective versions of most of the results. Besides we derive some new results. We study for which sets S a given graph has (infinitely many equivalence classes of) representations with S.

All our results in the present paper deal with finite graphs G where the vertices have distinct values from an algebraic number field K, and with finite sets S of prime ideals of K, such that two vertices of G

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are connected by an edge if and only if the difference of their values is an S-unit. We call such a graph G a difference graph of S-units. In Section 2 we introduce S-equivalence of difference graphs, finitely representable and infinitely representable difference graphs, degenerate and non-degenerate representations of difference graphs, and further notation.

In Section 3 we prove that in every number field K, for every graph G there is an effectively computable S such that G is representable with S. We further deal with graphs with more than one component and with connected graphs which are not doubly connected. In the rest of the paper we restrict ourselves to the case where G is doubly connected. In Section 4 we consider the cases that G is a cycle, a complete bipartite graph or a cubical graph. Section 5 contains the theorem that G is infinitely representable with S if and only if it has a degenerate representation with S. In Section 6 two theorems on the union of two graphs are formulated, one for the case that both are finitely representable with S, one for the case that only one of them is finitely representable with S. Finally, in Section 7, some effective results are stated for graphs G of which the complement or the attached triangle graph $G^{\triangle}$ satisfy some connectedness condition. In Sections 8-12 we give the proofs of the statements from Sections 3-7, respectively.

2. Notation and terminology

We introduce notation which will be used throughout the paper.

Let K be an algebraic number field with degree d and discriminant D_K, and S a finite (possibly empty) set of prime ideals of K. We recall that an $\alpha \in K$ is said to be an *S-integer* if in the prime ideal factorization of the ideal (α) generated by α no prime ideal from outside S has negative exponent. The S-integers in K form a ring, denoted by O_S, which is called the ring of S-integers. The units ε of O_S (when $\varepsilon, 1/\varepsilon \in O_S$) are called *$S$-units*. They form a multiplicative group, denoted by O_S^* and called the group of S-units. If in particular S is empty, then O_S and O_S^* are just the ring of integers O_K and the unit group O_K^* of K, respectively. Further, in the case $K = \mathbb{Q}$, we denote by $\mathbb{Z}_S$ the ring of S-integers. In the sequel we suppose that if S is empty, then K is not $\mathbb{Q}$ and not an imaginary quadratic field. Therefore O_S^* is infinite.

We write

$$N(S) := \max_{\mathfrak{p} \in S} N(\mathfrak{p}),$$

where $N(\mathfrak{p})$ stands for the norm of a prime ideal $\mathfrak{p}$. If $a \in K$, then write $h(a)$ for the logarithmic height of a, and for $A \subset O_S$ with $A =$

$\{a_1, \dots, a_n\}$ set

$$h(A) := \max_{i=1,\dots,n} h(a_i).$$

For any finite ordered subset $A = \{\alpha_1, \dots, \alpha_n\}$ of O_S, we denote by $\mathcal{G}_S(A)$ the graph whose vertices are $\alpha_1, \dots, \alpha_n$ and whose edges are the (unordered) pairs $\{\alpha_i, \alpha_j\}$ for which

$$\alpha_i - \alpha_j \in O_S^*;$$

cf. Győry [10] where mainly the complements of these graphs were studied. The ordered subsets A and A' of O_S are called *S-equivalent* if

$$A' = \varepsilon A + \beta$$

for some $\varepsilon \in O_S^*$ and $\beta \in O_S$. In this case the graphs $\mathcal{G}_S(A)$ and $\mathcal{G}_S(A')$ are obviously isomorphic.

Throughout the paper, all graphs we consider are finite and simple. By the order of a graph G we mean the number of its vertices, denoted by $|G|$. We say that a graph G is *representable* over K with S if there is a subset A of O_S such that $\mathcal{G}_S(A)$ is isomorphic to G. Further, we say that G is *effectively representable* over K with S if a subset A of O_S can be effectively determined such that $\mathcal{G}_S(A)$ is isomorphic to G. A graph G is called *finitely representable* over K with S if, up to S-equivalence, there are at most finitely many subsets A of O_S for which G is isomorphic to $\mathcal{G}_S(A)$. Further, G is said to be *infinitely representable with* S if G is isomorphic to $\mathcal{G}_S(A)$ for infinitely many pairwise S-nonequivalent A. Note that in every representation the vertices have distinct values. In the sequel we omit 'over K' and 'with S' if it is obvious what K and S are.

We note that in all our results on infinite representability with S it suffices that there are more than a certain computable number of equivalence classes which provide representations (cf. Theorem C and the Remark after its proof in Section 10).

3. Basic representability theorems

The first theorem is an effective version of Theorem 2.1 of Part I and, at the same time, a generalization to the number field case.

Theorem 3.1. *Let G be a graph with $|G| = n$. Then there exist a finite set S of prime ideals of K and a set $A \subset O_S$ with $|A| = n$ such that G is isomorphic to $\mathcal{G}_S(A)$, and $N(S) \le c_1(n, d, D_K)$, $h(A) \le c_2(n, d, D_K)$ hold. Here the numbers $c_i(n, d, D_K)$ are effectively computable ($i = 1, 2$).*

Remark. A simple calculation shows that in case of $K = \mathbb{Q}$ (when $D = d = 1$) bounds of the form $c_i(n) = e^{e^{\cdot^{\cdot^{\cdot^{e^n}}}}}$ with n copies of e ($i = 1, 2$) apply in the previous theorem. However, since these bounds most probably are very far from the best possible ones, we do not calculate them. An important feature of our proof is that it is constructive: following our argument one can construct A and S with the required property. This is illustrated by an example after the proof of Theorem 3.1 in Section 8.

In what follows, we assume that K is *effectively given*, i.e. $K = \mathbb{Q}(\vartheta)$ and a minimal polynomial $P \in \mathbb{Q}[x]$ of ϑ is given. We may assume that $P \in \mathbb{Z}[x]$ and that ϑ is an algebraic integer. We say that $\alpha \in K$ is *effectively given / effectively determinable* if in the representation

$$\alpha = a_0 + a_1\vartheta + \ldots a_{d-1}\vartheta^{d-1}$$

of α the coefficients $a_0, \ldots, a_{d-1} \in \mathbb{Q}$ are given / effectively determinable.

We say that S is *effectively given / effectively determinable* if the prime ideals in S are effectively given / effectively determinable. This means that a finite set of generators for each prime ideal involved is effectively given / effectively determinable.

Corollary 3.1. *Let G be as in Theorem 3.1. Then there is a finite, effectively determinable set S of prime ideals of K such that G is representable with S, and some representation of G with S can be, at least in principle, effectively determined.*

Theorems 3.2 and 3.3 are effective versions and, at the same time, generalizations to the number field case of the corresponding results of Part I, obtained over $\mathbb{Q}$.

As usual, by a forest graph we mean a graph containing no cycles, i.e. it is a finite, disjoint union of trees.

Theorem 3.2. *Let S be any fixed finite set of prime ideals in K, and let G be a forest graph with $|G| = n$. Then G is effectively representable with S. Further, such a representation $\mathcal{G}_S(A)$ can be, at least in principle, effectively constructed such that $h(A) \le c_3(n, N(S), d, D_K)$, where the bound is effectively computable.*

In fact, Theorem 3.2 is a simple consequence of the following result.

Theorem 3.3. *Let S be any fixed finite set of prime ideals in K, and let A be any fixed finite set of S-integers, with $|A| = n$.*

i) *There exist infinitely many $a' \in O_S$ outside A such that writing $A' = A \cup \{a'\}$, a' is an isolated vertex of $\mathcal{G}_S(A')$.*

ii) *For every* $a \in A$ *there exist infinitely many* $a' \in O_S$ *such that writing* $A' = A \cup \{a'\}$, *in* $\mathcal{G}_S(A')$ *the vertex* a' *is connected by an edge with* a *only.*

Further, one can effectively find an element a' *with either one of the above properties such that* $h(a') < c_4(n, N(S), d, D_K, h(A))$ *holds. Here the upper bound is effectively computable.*

The next result is a kind of extension of part ii) of Theorem 3.3. We say that a graph is *simply connected* if it is connected but not doubly connected, and that it is *at most simply connected* if it is disconnected, or simply connected.

Theorem 3.4. *Let* G *be a graph which is at most simply connected. If* G *is representable with some* S, *then it is infinitely representable with* S.

The following result, which is an extension of Theorem 2.4 of Part I to the number field case, shows that the investigations can be reduced to the components of a graph.

Theorem 3.5. *Let* S *be any fixed finite set of prime ideals in* K, *and suppose that every component of a graph* G *is representable with* S. *Then* G *is representable with* S.

4. Cyclic, bipartite and cubical graphs

In view of the results of the previous section, the question of representability of graphs which are not connected, or contain a bridge, is completely settled. So from this point on it is sufficient to consider only graphs not of these types. We say that a graph G having at least one edge is *doubly connected*, if after deleting any edge of G, the graph obtained is connected. If G is not doubly connected, then it is at most simply connected.

In this section we discuss those graphs which are always representable (i.e. representable for all K, with all S). Further, we study certain doubly connected graphs, namely cycles and complete bipartite graphs. The research upon cycles (over $\mathbb{Q}$) was initiated by Ruzsa [16]. His intention was, for given S, to study the graphs which can be represented with S. Besides providing related theorems of various types, Ruzsa also formulated some problems and conjectures. Some of them were solved in [3]. For details see [16, 3].

Let C_n denote the cyclic graph of order n, and write $K_{m,n}$ for the complete bipartite graph of type (m, n). The next theorem is an extension of Theorem 3.1 of [13] to the number field case.

Theorem 4.1. i) *The graphs C_{2n} $(n = 1, 2, \dots)$ and $K_{2,2}$ are infinitely representable with all S.*
ii) *The graphs C_3, C_5 and $K_{m,n}$ with $m > n > 1$ or $m = n \geq 3$ are finitely representable with any S.*

As one can easily check by examples, it depends on S whether C_{2n+1} for $n > 2$ is infinitely representable.

The following result extends Theorem 3.2 of [13] to the number field case.

Theorem 4.2. *If $m > 1$, $n > 1$ and*

$$m + n > 3 \cdot 2^{16(|S|+d)} \tag{1}$$

then $K_{m,n}$ is not representable with S.

Theorem 4.1 states that there exist graphs which are representable with all S, for example $G = K_{2,2}$. In the remaining part of the section we study such graphs.

As we shall see, in this context the so-called cubical graphs play an important role. The n-cube Q_n is defined in the following way. The vertices of Q_n are the n-tuples with coordinates 0 and 1, and two vertices are connected by an edge if and only if the vertices differ in exactly one coordinate. It follows that Q_n has 2^n vertices, and $n2^{n-1}$ edges. An embedding of a graph G into Q_n is an injective mapping of the vertices of G into the vertices of Q_n such that the edges of G are mapped into the edges of Q_n. A graph G is called *cubical*, if it can be embedded in Q_n for some n. Obviously, cubical graphs are bipartite.

The following theorem is a generalization of Theorem 6.1 of [13] to the number field case.

Theorem 4.3. *A graph G is representable with all K and S if and only if G is cubical.*

For a survey on related results concerning cubical graphs, we refer to [13].

Remark. It is not true that for fixed K, only cubical graphs would be representable with every S. Indeed, if $K = \mathbb{Q}(\sqrt{5})$, then since $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$ are both units, a triangle (which is clearly not cubical) is representable with any S as $\{0, 1, (1+\sqrt{5})/2\}$.

The following result is an extension of Theorem 4.1 from [13]. It plays a crucial role in the proof of Theorem 4.3.

Theorem 4.4. *Suppose that a graph G with $|G| \geq 3$ is representable over $\mathbb{Q}$ for some S of the form $S = \{p\}$, where p is a (rational) prime* *larger than twice the number of edges of G. Then G is infinitely representable with all K and S.*

Note that Theorem 4.4 implies that if G is representable with all K and S, then G is infinitely representable with all K and S.

5. Infinite representability

Let K and S be as above. Let G be a graph and $\mathcal{G}_S(A)$ a representation of G where $A = \{a_1, \ldots, a_n\}$ is the set of vertex values. By a path in $\mathcal{G}_S(A)$ we mean a sequence of vertices $a_{i_1}, \ldots, a_{i_m}$ (repetitions permitted) such that a_{i_j} is connected with $a_{i_{j+1}}$ by an edge for $j = 1, \ldots, m-1$. We call a_{i_1} and a_{i_m} the *endpoints* of the path. We define its *path value* as $a_{i_m} - a_{i_1}$. Note that the path value of every closed path (i.e. with $a_{i_m} = a_{i_1}$) is 0. If a_i and a_j are connected by an edge, we call the path value from a_i to a_j the *arrow value* from a_i to a_j. Hence every edge in $\mathcal{G}_S(A)$ generates two arrow values with opposite signs, which we call the arrow values of that edge. Observe further that a path value is the sum of the composing arrow values, $a_{i_m} - a_{i_1} = \sum_{j=1}^{m-1}(a_{i_{j+1}} - a_{i_j})$. If S and A are fixed, we write $\mathcal{G}$ for $\mathcal{G}_S(A)$. Every representation is meant with respect to S.

We shall prove the following properties.

Lemma 5.1. *Let G be connected. The equivalence class to which a representation $\mathcal{G}$ of G belongs is determined by its arrow values.*

Lemma 5.2. *Let G be connected and let values from K be given to all arrows (directed edges) of G. Then these arrow values form a representation of G if and only if*
1. *a path length is* 0 *if and only if the path is closed and*
2. *the endpoints of a path are connected by an edge if and only if the path length is in O_S^*.*

Remark. Obviously two representations of G are in the same S-equivalence class if and only if the quotient of every two corresponding arrow values is the same constant.

Lemma 5.3. *Let $\mathcal{G}$ be a representation of a graph G. Then there exist only finitely many pairs $(\mathcal{E}, \varepsilon)$ of non-empty proper subsets $\mathcal{E}$ of the set of edges of $\mathcal{G}$ and S-units ε such that multiplying all the arrow values of $\mathcal{E}$ by ε and leaving all the other arrow values unchanged yields a representation $\mathcal{G}_{\mathcal{E},\varepsilon}$ of a graph $G_{\mathcal{E},\varepsilon}$ such that G and $G_{\mathcal{E},\varepsilon}$ are not isomorphic.*

These lemmas are used in the proofs of the following characterizations of graphs with infinitely many representations with S.

Theorem 5.1. *Let G be doubly connected and have at least one edge. Then the following statements are equivalent.*
a) *The graph G is infinitely representable with S.*
b) *There are a representation $\mathcal{G}$ of G, a non-empty proper subset $\mathcal{E}$ of the edges of $\mathcal{G}$ and an S-unit $\varepsilon_0 \neq 1$ such that multiplying the arrow values of $\mathcal{E}$ by ε_0 and leaving the other arrow values unchanged yields another representation of G.*
c) *There are a representation $\mathcal{G}$ of G, a non-empty proper subset $\mathcal{E}$ of the edges of $\mathcal{G}$ and infinitely many S-units ε such that multiplying the arrow values of $\mathcal{E}$ by ε and leaving the other arrow values unchanged yields another representation of G.*

We call a representation $\mathcal{G}$ of a graph G *degenerate* if a non-empty proper subset $\mathcal{E}$ of the edges of $\mathcal{G}$ and infinitely many S-units ε exist such that multiplying the arrow values of $\mathcal{E}$ by ε and leaving the other arrow values unchanged yields another representation of G. Thus Theorem 5.1 has the following consequence.

Corollary 5.1. *A doubly connected graph is infinitely representable if and only if it has a degenerate representation.*

6. Finite representability

Let in the above notation K, S and G be given. Again representability means representability with S. We state some results which can help to establish the finite representability of G.

Let G_1, G_2 be induced subgraphs of G. We define $G_1 \cup G_2$ as the minimal graph which has all the vertices of G_1 and G_2 as vertices and all the edges of G_1 and G_2 as edges.

We first treat the case that both G_1 and G_2 are finitely representable. Suppose $G = G_1 \cup G_2$ and G_1 and G_2 have at most one vertex in common. If either G_1 or G_2 is not representable, then G is not representable. If both G_1 and G_2 are representable and both have an edge which the other does not have, then G is infinitely representable. Therefore the interesting case is that the intersection of G_1 and G_2 consists of at least two vertices. In that case we have the following result.

Theorem 6.1. *Suppose $G = G_1 \cup G_2$ and G_1 and G_2 are both finitely representable. If G_1 and G_2 have at least two vertices in common, then G is finitely representable.*

By Theorem 5.1 we know that if G is infinitely representable, then there is a degenerate representation. The following theorem says that if G_1 is finitely representable and both G_2 and $G_1 \cup G_2$ are infinitely representable, then the degeneracy is entirely in $G_2 \setminus G_1$.

Theorem 6.2. *Suppose $G = G_1 \cup G_2$ such that G_1 is finitely representable and G_2 is infinitely representable. If G is infinitely representable, then there are a representation $\mathcal{G}$ of G, a non-empty subset $\mathcal{E}$ of the edges of $\mathcal{G}$ belonging to the edges of $G \setminus G_1$, and infinitely many S-units ε such that multiplying the arrow values of $\mathcal{E}$ by such an ε and leaving the other arrow values unchanged yields another representation of G.*

Remark. The remaining case is that both G_1 and G_2 are infinitely representable. In that case their union both may be finitely representable and infinitely representable and we do not know a simple criterion to distinguish them. For example, let $K = \mathbb{Q}$ and S consist of odd primes. Consider a cycle G_1 with 8 vertices, successively $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$, and a cycle G_2 with 8 vertices, successively $w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8$, such that G_1 and G_2 have no edge in common. If $v_1 = w_1, v_5 = w_4$, and G_1 and G_2 have no other vertices in common, then the union contains a 7-cycle $(v_1, v_2, v_3, v_4, v_5 = w_4, w_3, w_2)$ and therefore there is no representation of $G_1 \cup G_2$ at all. However, if $v_1 = w_1, v_5 = w_5$ and G_1 and G_2 have no other vertices in common, then $G_1 \cup G_2$ has two vertices $v_1 = w_1$ and $v_5 = w_5$ which are connected by four disjoint paths of length 4, and their arrows can be given values $1, p, p^2, p^r$ and $p, p^2, p^r, 1$ and $p^2, p^r, 1, p$ and $p^r, 1, p, p^2$, respectively, where r is any integer ≥ 3. Hence $G_1 \cup G_2$ is infinitely representable.

Theorem 6.1 implies two theorems of [13]. The theorems are partial counterparts of Theorem 3.4. For a graph G we denote by $G^{\triangle}$ the graph whose vertices are the edges of G, and where two vertices e_1 and e_2 of $G^{\triangle}$ are connected by an edge if and only if G contains a triangle having e_1 and e_2 as sides. Further, if both G and $G^{\triangle}$ are connected then we say that G is $\triangle$-*connected*. The $\triangle$-graphs of tree and forest graphs have only isolated vertices. Observe that if both G and $G^{\triangle}$ are connected, then G is doubly connected. For a graph G we denote by G^{∇} the graph whose vertices are the triangles of G where two vertices of G^{∇} are connected by an edge if and only if the triangles in G have a common side.

Corollary 6.1. *Let G be a graph such that G^{∇} is non-empty and both G and G^{∇} are connected. Then G is finitely representable with every S.*

The following corollary is a generalization of Theorem 5.1 of Part I to the number field case.

Corollary 6.2. *Let G be a graph such that $G^{\triangle}$ has an edge and both G and $G^{\triangle}$ are connected. Then G is finitely representable with every S.*

We denote by $\mathcal{H}(G)$ the graph whose vertices are the $\triangle$-connected components of G, and two vertices of $\mathcal{H}(G)$ are connected if the corresponding $\triangle$-connected components of G have at least two vertices in common in G. This graph $\mathcal{H}(G)$ is called the $\mathcal{H}(G)$-*graph* of G. The next corollary generalizes Theorem 5.2 of Part I to the number field case.

Corollary 6.3. *Let G be a graph such that G contains a triangle and both G and $\mathcal{H}(G)$ are connected. Then G is finitely representable with every S.*

7. Effective results for $\triangle$-connected graphs

We give effective versions of Corollaries 6.2 and 6.3.

Theorem 7.1. *Let G be a graph of order $n \geq 3$ such that both G and $G^{\triangle}$ are connected. Then, for effectively given K and S, all representations of G with S can be effectively determined.*

Theorem 7.1 can be generalized in the following way.

Theorem 7.2. *Let G be a graph of order $n \geq 3$. Suppose that both G and $\mathcal{H}(G)$ are connected. Then for effectively given K and S, all representations of G with S can be effectively determined.*

If $G^{\triangle}$ is connected, then $\mathcal{H}(G)$ consists of one vertex and is therefore also connected. Hence Theorem 7.1 is a special case of Theorem 7.2.

The following theorem is a generalization to the number field case and, for $d = 1$, an improvement of Theorem 5.3 of Part I. We denote the complement of G by $\overline{G}$.

Theorem 7.3. *Let $n \geq 3$ be an integer, and fix S. Then for all but at most*

$$\left(n \cdot 5^{114(|S|+d)}\right)^{4(n-1)}$$

S-equivalence classes of ordered n-term subsets A from O_S, one of the following cases holds:

i) $\overline{\mathcal{G}_S(A)}$ *is connected and at least one of $\mathcal{G}_S(A)$ and $\mathcal{G}_S(A)^{\triangle}$ is not connected;*

ii) $\overline{\mathcal{G}_S(A)}$ *has exactly two components, $\overline{\mathcal{G}_1}$ and $\overline{\mathcal{G}_2}$, say, such that $|\overline{\mathcal{G}_1}| = 1$, and $\mathcal{G}_2$ is not connected;*

iii) $n = 4$ *and* $\mathcal{G}_S(A) = K_{2,2}$.

As it is pointed out in [10], for each of i), ii), iii), one can choose S such that there are infinitely many S-equivalence classes of ordered n-term subsets A in O_S with the property i), ii), and iii), respectively.

The following corollary is a consequence of Theorem 7.3. This is a generalization and, for $d = 1$, an improvement of Theorem 5.4 of Part I to the number field case.

Corollary 7.1. *Let G be a graph of order $n \geq 3$ and suppose that G is more than*

$$\left(n \cdot 5^{114(|S|+d)}\right)^{4(n-1)}$$

times representable for some S. Then at least one of G and $G^{\triangle}$ is not connected.

The final theorem is concerned with the situation where no representation is possible.

Theorem 7.4. *Let G be a graph of order n such that $\overline{G}$ has either at least three components, or two components of order ≥ 2. If*

$$n > 3 \cdot 2^{16(|S|+d)}$$

then G is not representable with any S.

This is a generalization to the number field case of Theorem 5.5 of Part I.

Question. *Does there exist a criterion/algorithm to decide the representability of a graph G for fixed K and S?*

In case of graphs G for which G and $G^{\triangle}$ are connected, Theorem 7.1 gives a positive answer to the algorithmic part of the above question.

8. Proofs of the results stated in Section 3

In the proofs below we shall work with finite subsets A of O_K. In every S-equivalence class of ordered subsets A from O_S there is a set consisting of integers of K. Such a subset can be obtained from A by multiplying it by an appropriate element of $O_S^* \cap O_K$. Hence for Theorems 3.1-3.3 it suffices to study the graphs $\mathcal{G}_S(A)$ with subsets A having elements from O_K. In this case, $a, b \in A$ are connected by an edge if and only if $a - b \in O_S^* \cap O_K = O_K^*$.

Proof of Theorem 3.1. Let G be a fixed graph with $|G| = n$. Let N_0 be the second smallest norm of prime ideals in O_K. Note that we certainly have $N_0 \leq 3^d$. Write $n' := \max\{n, N_0\}$ and

$$S_0 := \{\mathfrak{p} \text{ prime ideal } : N(\mathfrak{p}) < n'\}.$$

We prove by induction on k that for any graph G' with $|G'| = k \leq n$ there exists a finite set S' of prime ideals with $S_0 \subset S'$ and a finite set $A' \subset O_K$ with $|A'| = k$ such that $\mathcal{G}_{S'}(A')$ is isomorphic to G'. We shall indicate how one can bound the sets S' and A' in terms of n, d, D_K.

Let $k = 1$. Then G' is a graph with one vertex (and without edges). Taking $S' = S_0$ and $A' = \{0\}$, since $N(S') < n'$ and $h(A') = 0$, we are obviously done in this case.

Let now G' be a graph such that $|G'| = k$ with $2 \leq k \leq n$. Write $v_1, \ldots, v_k$ for the vertices of G'. Let G'' be the graph obtained from G' by omitting the vertex v_k, together with the corresponding edges. By induction we may assume that there exists a set S'' of prime ideals containing all elements of S_0 and a set $A'' = \{a_1, \ldots, a_{k-1}\}$ of integers in K such that $\mathcal{G}_{S''}(A'')$ is isomorphic to G'', by an isomorphism $\varphi : \mathcal{G}_{S''}(A'') \to G''$. Further, here we may also suppose that $N(S'') < c''(n, d, D_K)$ and $h(A'') < c''(n, d, D_K)$ with some effectively computable constant $c''(n, d, D_K)$ depending only on n, d, D_K. Without loss of generality we may assume that $\varphi(a_i) = v_i$ $(i = 1, \ldots, k-1)$. Write T'' for the set of indices of those vertices of G'' which are **not** connected with v_k by an edge in G'. Further, put

$$\mathfrak{D} := \{\mathfrak{d} \text{ prime ideal } : \ \mathfrak{d} \notin S'', \ \mathfrak{d} \mid a - b \text{ for some distinct } a, b \in A''\}.$$

For later use, observe that for all $\mathfrak{d} \in \mathfrak{D}$ we have $N(\mathfrak{d}) \geq n' > k - 1$.

If $T'' \neq \emptyset$, write $T'' = \{t_1, \ldots, t_\ell\}$, and choose distinct prime ideals $\mathfrak{q}_{t_1}, \ldots, \mathfrak{q}_{t_\ell}$ such that for all $t_j \in T''$ we have

- $\mathfrak{q}_{t_j} \notin S''$,
- $\mathfrak{q}_{t_j} \notin \mathfrak{D}$.

Observe that having the upper bounds for $N(\mathfrak{q})$ with $\mathfrak{q} \in S'' \cup \mathfrak{D}$, the prime ideals $\mathfrak{q}_{t_1}, \ldots, \mathfrak{q}_{t_\ell}$ can be chosen in a way that their norms are bounded in terms of n, d, D_K. By the above properties, for any distinct $i_1, i_2 \in \{1, \ldots, k-1\}$ we have $a_{i_1} \not\equiv a_{i_2} \pmod{\mathfrak{q}_{t_j}}$. For each prime ideal $\mathfrak{d} \in \mathfrak{D}$ choose an $x_\mathfrak{d} \in O_K$ such that for all $i = 1, \ldots, k-1$ we have

$$a_i \not\equiv x_\mathfrak{d} \pmod{\mathfrak{d}}. \tag{2}$$

Since $N(\mathfrak{d}) > k - 1$ for all $\mathfrak{d} \in \mathfrak{D}$, such an $x_\mathfrak{d}$ exists. Consider now the following linear system of congruences:

$$\begin{cases} a \equiv x_\mathfrak{d} \pmod{\mathfrak{d}} & (\mathfrak{d} \in \mathfrak{D}), \\ a \equiv a_{t_j} \pmod{\mathfrak{q}_{t_j}} & (t_j \in T''). \end{cases} \tag{3}$$

By the Chinese Remainder Theorem, this system has infinitely many solutions a. Choose a_k to be a solution, and let $A' = A'' \cup \{a_k\}$. Here using the information concerning the ideals $\mathfrak{d}$ and $\mathfrak{q}_{t_1}, \ldots, \mathfrak{q}_{t_\ell}$, we may assume that $h(a_k)$ is bounded in terms of n, d, D_K. Further, put $T' = \{1, \dots, k-1\} \setminus T''$ and set

$$S' = S'' \cup \{\mathfrak{p} \text{ prime ideal } : \mathfrak{p} \mid a_k - a_i \text{ for some } i \in T'\}.$$

We claim that by these choices the graph $\mathcal{G}_{S'}(A')$ is isomorphic to G'. More precisely, an isomorphism is given by $\varphi^* : \mathcal{G}_{S'}(A') \to G'$ with $\varphi^*(a_i) = v_i$ $(i = 1, \dots, k)$.

Let $i \in \{1, \dots, k-1\}$. If $i \in T'$ then on the one hand, v_i and v_k are connected by an edge in G', and on the other hand, by the definition of S' we have that a_i and a_k are connected in $\mathcal{G}_{S'}(A')$. Assume now that $i \in T''$. Then v_i and v_k are not connected in G'. However, writing $i = t_j$, in view of $\mathfrak{q}_{t_j} \notin S''$ and $\mathfrak{q}_{t_j} \mid a_k - a_i$, we have that $\mathfrak{q}_{t_j} \notin S'$. Indeed, otherwise $\mathfrak{q}_{t_j} \mid a_k - a_{i'}$ for some $i' \in T'$, whence $\mathfrak{q}_{t_j} \mid a_i - a_{i'}$ with distinct $i, i' \in \{1, \dots, k-1\}$. This means that $\mathfrak{q}_{t_j} \in S'' \cup \mathfrak{D}$, which contradicts its definition. Thus $\mathfrak{q}_{t_j} \mid a_k - a_i$ implies that a_i and a_k are not connected by an edge in $\mathcal{G}_{S'}(A')$.

Finally, we need to check that for any $i, j \in \{1, \dots, k-1\}$, a_i and a_j are connected by an edge in $\mathcal{G}_{S'}(A')$ if and only if they are connected by an edge in $\mathcal{G}_{S''}(A'')$. If a_i and a_j are connected by an edge in $\mathcal{G}_{S''}(A'')$ then by $S'' \subset S'$, obviously they are connected by an edge in $\mathcal{G}_{S'}(A')$. Assume now that a_i and a_j are not connected in $\mathcal{G}_{S''}(A'')$. Then there is a prime ideal $\mathfrak{d} \in \mathfrak{D} \setminus S''$ dividing $a_i - a_j$. Observe that, by (3) and (2), $\mathfrak{d} \mid a_k - x_{\mathfrak{d}}$ and $\mathfrak{d} \nmid a_\ell - x_{\mathfrak{d}}$, whence $\mathfrak{d} \nmid a_k - a_\ell$ for $\ell = 1, \dots, k-1$. This implies that $\mathfrak{d} \notin S'$. Hence a_i and a_j are not connected by an edge in $\mathcal{G}_{S'}(A')$ either.

As one can see from the construction, $N(S')$, as well as $h(A')$ can be bounded effectively in terms of n, d, D_K. Hence the statement follows. □

An example. Here we illustrate (in fact by providing a detailed explanation) the construction given in the above proof through an example. For simplicity, we shall work over $K = \mathbb{Q}$.

Consider first the cyclic graph $C_4 = (v_1, v_2, v_3, v_4)$ (with points v_1, v_2, v_3, v_4 and edges $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}$). As in the above proof, we use S and A as 'variables', to be changed during the algorithm. Recall that the starting set $S = S_0$ should contain all primes less than n. Since in our case $n = 4$, we shall start with the set of primes $\{2, 3\}$.

Initialization. We put $S = \{2, 3\}$ and $A = \emptyset$.

Step 1. We can choose any integer a_1 to represent v_1; take $a_1 := 0$, and put $A := \{0\}$. S is left unchanged.

Step 2. Since v_2 is connected with v_1, here no restriction is needed; we can represent v_2 by $a_2 := 1$, and put $A := \{0, 1\}$. S is left unchanged once again.

Step 3. Since v_1 and v_2 **are** connected, by an edge, we let $D := \emptyset$. To choose a representative for v_3, firstly we need to be sure that the representatives of v_1 and v_3 **are not** connected. For this we take an arbitrary prime q_1 outside S (and also outside D; the latter condition is needed to have distinct moduli in the linear congruence system (3)). Here the choice $q_1 := 5$ is appropriate. Since D is empty, by (3) we may take $a_3 := 5$, whence $A := \{0, 1, 5\}$. Then we need to insert the prime divisors of $a_3 - a_2 = 4$ into S, to make sure that a_2 and a_3 (just as v_2 and v_3) are connected. In the present case this just means that S remains still unchanged.

Step 4. Now since v_1 and v_3 **are not** connected, we have $D := \{5\}$. Further, v_4 **is not** connected with v_2. So we need to find a prime q_1, similarly as in Step 3, i.e. such that $q_1 \notin D \cup S$. So we can take $q_1 := 7$. Now we need to find a_4 subject to (3). Since $a_2 = 1$ and we can take $x_5 := 2$, this now reads as

$$\begin{cases} a_4 \equiv 2 \pmod 5 \\ a_4 \equiv 1 \pmod 7. \end{cases} \tag{4}$$

Hence we can take $a_4 := 22$, and let $A := \{0, 1, 5, 22\}$. Finally, we include the prime divisors of $a_4 - a_1 = 22$ and $a_4 - a_3 = 17$ to S, to get $S := \{2, 3, 11, 17\}$.

Output. We output $A = \{0, 1, 5, 22\}$ and $S := \{2, 3, 11, 17\}$. One can easily check that with these choices, $\mathcal{G}_S(A)$ is isomorphic to C_4. $\square$

In what follows, we shall need some algorithmic results for the Chinese Remainder Theorem in number fields (see Algorithm 4.2.2, p. 188 in [2]), for finding an S-unit of bounded height (see Algorithm 7.4.8, p. 376 in [2]), for listing all prime ideals of bounded norm (see Algorithm 2.3.23, p. 100 in [2]) and for finding S-integers of bounded height (which can be reduced e.g. to listing all prime ideals of bounded norm).

Proof of Corollary 3.1. Suppose that K is effectively given in the sense defined in Section 3. By Theorem 3.1 there exist a finite set S of prime ideals of K and a set $A \subset O_S$ with $|A| = n$ such that G is isomorphic to $\mathcal{G}_S(A)$, and $N(S) \leq c_1(n, d, D_K)$, $h(A) \leq c_2(n, d, D_K)$, where c_1, c_2 are effectively computable. However, there are only finitely many such finite sets S of prime ideals in K, and for each given S, there are only finitely many finite subsets A of O_S with these properties, and all pairs

S, A can be effectively determined. Finally, we can select a pair S, A for which $\mathcal{G}_S(A)$ is isomorphic to G. In this way we find a representation of G with S. □

Proof of Theorem 3.4. The proof of the statement is similar to the proof of Theorem 4.2 in [13], we only need to use O_S in place of $\mathbb{Z}_S$. □

We shall use the following finiteness result, due to Evertse, [4] at several places.

Theorem A. (Evertse [4]) *The S-unit equation*

$$\alpha x + \beta y = 1 \tag{5}$$

in $x, y \in O_S^*$ where α and β are nonzero elements of K has at most

$$3 \cdot 7^{d+2|S|} \tag{6}$$

solutions where $d = [K : \mathbb{Q}]$.

Proof of Theorem 3.3. Write $A = \{a_1, \dots, a_n\}$.

To prove i) choose prime ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ in K, outside S. Note that here we may clearly assume that the norms of these ideals are bounded in terms of $N(S), d$. Consider the system of linear congruences

$$x \equiv a_i \pmod{\mathfrak{q}_i} \quad (i = 1, \dots, n)$$

in $x \in O_K$. By the Chinese Remainder Theorem, this system has infinitely many solutions. Let $a' \in O_K$ be a solution such that $a' \notin A$. Then obviously, a' is an isolated vertex of the graph $\mathcal{G}_S(A')$ where $A' = A \cup \{a'\}$. Further, it is also clear that one can effectively find such an a', with $h(a')$ bounded by a constant depending only on $n, N(S), d, D_K$.

To prove ii), take an arbitrary $a \in A$. Write

$$D := \{\pm(a_i - a_j) \;:\; 1 \leq i < j \leq n\},$$

and let $\varepsilon \in O_S^* \cap O_K$ such that $\varepsilon \notin D$, and for any $\eta \in O_S^* \cap O_K$ we also have $\varepsilon + \eta \notin D$. The existence of such an ε easily follows from Theorem A. Namely, for $d \in D$ the equation $x + y = d$ has only finitely many solutions in $x, y \in O_S^* \cap O_K$, and the number of solutions can be bounded by a constant $c_0(s)$ depending only on s, see [9]. Avoiding all such elements u, v, together with the at most $2\binom{n}{2}$ elements of D, in fact we can choose ε in infinitely many ways. We can bound $h(\varepsilon)$ in terms of $n, N(S), d, D_K$ as follows. Take an arbitrary S-unit ε_0 which is not a root of unity, with $h(\varepsilon_0)$ bounded in terms of $N(S), d, D_K$. Considering the powers ε_0^i with $i = 1, \dots, c_0(s) + 2\binom{n}{2}$, one of them will be an appropriate choice for ε.

Let $a' = a + \varepsilon$. Then $a' \notin A$, and obviously a' and a are connected by an edge in the graph $\mathcal{G}_S(A')$ where $A' = A \cup \{a'\}$. Assume that a' is also connected with some vertex $b \in A$ with $b \neq a$. Then $b-(a+\varepsilon) =:$ $\eta \in O_S^* \cap O_K$. However, this yields $\eta + \varepsilon = b - a$, whence $\eta + \varepsilon \in D$, contradicting the choice of ε. This shows that in the graph $\mathcal{G}_S(A')$ only the vertex a is connected by an edge with the vertex a'.

Finally, noting that $h(a') \leq h(a) + h(\varepsilon)$, our claim follows. □

Proof of Theorem 3.2. Let G be the disjoint union of the tree graphs $T_1, \dots, T_k$. Starting from one vertex $\alpha \in O_K$, using part ii) of Theorem 3.3, we can inductively build up a set $A_1 \subset O_K$ such that $\mathcal{G}_S(A_1)$ is isomorphic to T_1. Then by part i) of Theorem 3.3 we can adjoin an isolated vertex $a' \in O_K$ to this graph, and then build up a component $A_2 \subset O_K$ (with $a' \in A_2$) such that $\mathcal{G}_S(A_2)$ is isomorphic to T_2. Following this procedure, we can clearly construct a set $A = A_1 \cup A_2 \cup \dots \cup A_k$ with the property that $h(A)$ is bounded by a constant $c_3(n, N(S), d, D_K)$. □

Proof of Theorem 3.5. The proof is similar to the proof of Theorem 2.4 in [13]. We only need to work with K and O_S instead of $\mathbb{Q}$ and $\mathbb{Z}_S$, respectively. □

9. Proofs of the results stated in Section 4

We shall need the following two theorems. Theorem B will be used in the present section and in Section 12, and Theorem C in Section 10.

Consider first the equation

$$\alpha_1 x_1 + \dots + \alpha_n x_n = 1 \quad \text{in } x_1, \dots, x_n \in O_S^* \tag{7}$$

where $\alpha_1, \dots, \alpha_n$ are non-zero elements of K. A solution $(x_1, \dots, x_n)$ of (7) is called *non-degenerate* if

$$\sum_{i \in I} \alpha_i x_i \neq 0 \quad \text{for each non-empty subset } I \text{ of } \{1, \dots, n\}$$

and *degenerate* otherwise. Clearly, if (7) has a degenerate solution then it has infinitely many solutions. Evertse [5] gave the explicit upper bound below for the number, N_n, of non-degenerate solutions of (7). His bound was generalized, with a slightly weaker bound, by Evertse, Schlickewei and Schmidt [8] and Amoroso and Viada [1] for the case of finitely generated multiplicative subgroups of K^* where K is any field of characteristic 0.

Theorem B.

$$N_n \leq (2^{35} n^2)^{n^3(|S|+d)}. \tag{8}$$

This is Theorem 3 of Evertse [5].

Consider now the system of equations

$$\alpha_{i1}x_1 + \cdots + \alpha_{in}x_n = 0, \quad i = 1, \dots, m \tag{9}$$

in $x_1, \dots, x_n \in O_S^*$, where $\alpha_{ij} \in K$ for $i = 1, \dots, m; j = 1, \dots, n$. This is a generalization of the homogeneous version of equation (7). Two solutions $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ of (9) are called *S-equivalent* if $y_j = \varepsilon x_j$ $j = 1, \dots, n$ holds for some $\varepsilon \in O_S^*$. Further, a solution $(x_1, \dots, x_n)$ is called *degenerate*, if for some proper non-empty subset I of $\{1, \dots, n\}$

$$\sum_{j \in I} \alpha_{ij}x_j = 0 \quad \text{for } i = 1, \dots, m,$$

and *non-degenerate* otherwise. If (9) has a degenerate solution, then it has infinitely many S-equivalence classes of solutions.

We shall use the following consequence of Theorem 2 of Evertse and Győry [6]. It is a generalization of earlier work of Evertse, van der Poorten and Schlickewei on S-unit equations.

Theorem C. *The system of equations* (9) *has only finitely many non-degenerate equivalence classes of solutions.*

For more general versions see Laurent [15] and Győry [11].

Proof of Theorem C. Let $(x_1, \dots, x_n)$ be a non-degenerate solution of (9). Then at least one of the coefficients $\alpha_{1n}, \dots, \alpha_{mn}$ is different from zero. Putting $y_j = -x_j/x_n$ for $j = 1, \dots, n-1$, $(y_1, \dots, y_{n-1}, -1)$ is S-equivalent to the solution $(x_1, \dots, x_n)$. Further, it satisfies the system of equations

$$\alpha_{i1}y_1 + \cdots + \alpha_{i,n-1}y_{n-1} = \alpha_{in}, \quad i = 1, \dots, m,$$

such that there is no proper non-empty subset J of $\{1, \dots, n-1\}$ with $\sum_{j \in J} \alpha_{ij}y_j = 0$ for $i = 1, \dots, m$. But by Theorem 2 of Evertse and Győry [6], the number of such solutions $(y_1, \dots, y_{n-1})$ is finite, which completes the proof. $\square$

Remark. Using Theorem B we can derive an explicit upper bound for the number of non-degenerate equivalence classes of solutions of (9).

Proof of Theorem 4.1. i) Let n be a positive integer and $G = C_{2n}$. Then C_{2n} is infinitely representable according to the proof by induction of Theorem 3.1 of [16]. Since $G = K_{2,2}$ is isomorphic to C_4, the statement is also valid for this graph.

ii) Let $G = C_3$. Then every representation of G with S corresponds with a normalized equation $x + y = 1$ in $x, y \in O_S^*$. By Theorem A the number of solutions of this equation is finite. Therefore C_3 is finitely representable with S.

Let now $G = C_5$, and let $\mathcal{G}_S(A)$ be a representation of G, with $A = \{a_1, \dots, a_5\} \subset K$. Write $u_1, \dots, u_5$ for the S-units

$$a_2 - a_1,\ a_3 - a_2,\ a_4 - a_3,\ a_5 - a_4,\ a_1 - a_5.$$

Then we have

$$u_1 + \dots + u_5 = 0. \tag{10}$$

Suppose that the left hand side of (10) contains a vanishing subsum. Then there is such a subsum with two terms. Since these terms cannot be consecutive, we may assume that $u_1 + u_3 = 0$. However, then $a_4 - a_1 = a_3 - a_2$ is also an S-unit, which implies that a_1 and a_4 should also be connected by an edge in $\mathcal{G}_S(A)$. Since this is not the case, we conclude that the left hand side of (10) has no vanishing subsums. Now by Theorem B we get that the number of non-degenerate solutions of equation (10) is finite. Thus C_5 is finitely representable.

Finally, let $G = K_{m,n}$ with $m > n > 1$ or $m = n \geq 3$. Choose two vertices P, Q from the n-set of vertices of G. Then, after normalization, this yields $m \geq 3$ distinct solutions of the equation $\alpha x + \alpha y = 1$ in $x, y \in O_S^*$, where α is some non-zero element of K. By Theorem A this equation has only finitely many solutions. Thus there are only finitely many ways to represent G with S. □

Proof of Theorem 4.2. Theorem 4.2 is an immediate consequence of the following theorem, since $\overline{K_{m,n}}$ with $m \geq n \geq 2$ has two components each of size ≥ 2. □

Theorem D. (Győry [12]) *Let A be an ordered n-term subset in O_S. If*

$$n > 3 \cdot 2^{16(|S|+d)}$$

then $\overline{\mathcal{G}_S(A)}$ has at most two components, and one of them is of order at most 1. Here d is the degree of the underlying number field K.

Proof. This is a special case of Theorem 2.3 of [12]. □

Proof of Theorem 4.4. The proof is similar to the proof of Theorem 4.1 in [13], working with K and O_S in place of $\mathbb{Q}$ and $\mathbb{Z}_S$, respectively. □

Proof of Theorem 4.3. Assume first that G is representable with all K and S. Then, by Theorem 6.1 of [13], G is cubical.

Suppose now that G is cubical. Then, by Theorem 6.1 of [13], G is infinitely representable with $K = \mathbb{Q}$ and $S = \{p\}$, where p is as in Theorem 4.4. Hence the statement follows from Theorem 4.4. □

10. Proofs of the results stated in Section 5

Proof of Lemma 5.1. Let $\mathcal{G}$ be a representation of G. Fix a vertex v of G. Since G is connected, there is a path from v to any other vertex w. The vertex value of w is determined by the vertex value of v and the arrow values of a path from v to w (and apparently independent of the chosen path because of the existence of $\mathcal{G}$). Hence all the vertex values of any representation of G are fixed by the vertex value of v and the arrow values. Observe that all the generated representations of G are 'shifts' of $\mathcal{G}$ and therefore equivalent by definition. □

Proof of Lemma 5.2. Suppose we have a representation $\mathcal{G}$ of G. If the length of a path is 0 and its endpoints are not equal, then there are two vertices with the same value, a contradiction. By definition the length of some path in $\mathcal{G}$ is in O_S^* if and only if the endpoints of the path are connected by an edge in $\mathcal{G}$.

On the other hand, suppose the conditions of the lemma are fulfilled. Fix a vertex v of G and give it a value a. For any other vertex w consider some path from v to w and give w the induced value, b say. Since all paths from v to w have the same path value the value b is independent of the chosen path. Thus b is determined by a and the arrow values. Moreover $a \neq b$, and v and w are connected by an edge if and only if $b - a \in O_S^*$. Thus we have a representation $\mathcal{G}$ of G. The equivalence class to which $\mathcal{G}$ belongs is determined by the arrow values. □

Proof of Lemma 5.3. Suppose there are infinitely many pairs $(\mathcal{E}, \varepsilon)$ as in the statement of the lemma such that $G_{\mathcal{E},\varepsilon}$ is not isomorphic to G. Since there are only finitely many possibilities to choose $\mathcal{E}$, there exists an $\mathcal{E}$ for which this is true for infinitely many S-units ε. Fix this $\mathcal{E}$. Every arrow value of $\mathcal{G}$ is an S-unit. After multiplying it by an S-unit ε it becomes again an S-unit. Thus every edge of G leads to an edge of $G_{\mathcal{E},\varepsilon}$. The only reason that G and $G_{\mathcal{E},\varepsilon}$ are not isomorphic can be that $G_{\mathcal{E},\varepsilon}$ has an edge where G has no edge.

There are only finitely many edges which can be added to G. Therefore we can fix two vertices v and w in G for which there are infinitely many S-units ε such that multiplying the arrow values of $\mathcal{E}$ in $\mathcal{G}$ by ε and leaving the others unchanged causes an edge in $G_{\mathcal{E},\varepsilon}$ between v and w. Let r and r_ε denote the path values from v to w in $\mathcal{G}$ and $\mathcal{G}_{\mathcal{E},\varepsilon}$, respectively. Both r and r_ε are independent of the chosen path. Write $r = P + Q$ where P is the contribution of the arrows from $\mathcal{E}$ to a path from v to w in $\mathcal{G}$ and Q is the contribution of the other arrows along that path. Then $r_\varepsilon = \varepsilon \cdot P + Q$. Furthermore P and Q are constants (that is, independent of ε) and r_ε is an S-unit, η say. This yields the

S-unit equation $\eta = \varepsilon \cdot P + Q$ with infinitely many solutions in S-units ε, η. By Theorem A this equation has only finitely many solutions ε, η, unless $PQ = 0$. We conclude $PQ = 0$. If $P = 0$, then $r = Q = r_\varepsilon$ and therefore there was already an edge in $\mathcal{G}$ between v and w. If $Q = 0$, than $r = P = \varepsilon^{-1} r_\varepsilon$ is also an S-unit, and we have the same conclusion. □

Proof of Theorem 5.1. Obviously c) implies both a) and b).
a) $\Rightarrow$ c). Let G be a doubly connected graph with n edges. Suppose there are infinitely many equivalence classes of representations of G. Let $\mathcal{G}$ be any representation of G. Let $\mathcal{G}$ have arrow values $\pm x_1, \ldots, \pm x_n$ where the value of each arrow is fixed and nonzero. Since G is doubly connected, every edge of G is part of a cycle and the edges of G are determined by the cycles. Every cycle of $\mathcal{G}$ corresponds to an equation (9) with $\alpha_{ij} \in \{-1, 0, 1\}$ for all j, x_j the corresponding arrow value and i numbering the (finitely many) cycles. Note that, by Lemmas 5.1 and 5.2, there is a bijection between the solutions of (9) and the arrow value sets which generate representations of graphs G' which have the same vertices and edges as G, but possibly other edges too. By Theorem C there are only finitely many non-degenerate equivalence classes of solutions. Since we have infinitely many equivalence classes of solutions, there is a degenerate equivalence class of solutions to system (9). This corresponds to a representation $\mathcal{G}$ of G, a proper subset $\mathcal{E}$ of the edges of $\mathcal{G}$ and infinitely many S-units ε such that multiplying the arrow values of $\mathcal{G}$ which belong to $\mathcal{E}$ by an S-unit ε and leaving the other arrow values unchanged leaves the sum of the arrow values of every cycle in $\mathcal{G}$ equal to 0. According to Lemma 5.3 there are only finitely many S-units ε such that the resulting graph $G_{\mathcal{E},\varepsilon}$ is not isomorphic to G. Hence there are infinitely many S-units ε such that multiplying the arrow values of $\mathcal{E}$ by ε and leaving the other arrow values unchanged yields another representation of G.
b) $\Rightarrow$ c). Suppose there are a representation of $\mathcal{G}$ of G, a non-empty proper subset $\mathcal{E}$ of the edges of $\mathcal{G}$ and an S-unit $\varepsilon_0 \neq 1$ such that multiplying the arrow values of $\mathcal{E}$ by ε_0 and leaving the other arrow values invariant yields another representation of G. Consider any closed path in $\mathcal{G}$. Let P be the total contribution of the edges from $\mathcal{E}$ to this closed path and Q the total contribution of the edges not in $\mathcal{E}$. Then both $P + Q = 0$ and $\varepsilon_0 \cdot P + Q = 0$ in view of Lemma 5.2. Since $\varepsilon_0 \neq 1$ we obtain $P = 0$. Thus the contribution of the arrow values from $\mathcal{E}$ to any closed path of $\mathcal{G}$ is 0.

Consider the graph $\mathcal{G}_{\mathcal{E},\varepsilon}$ which arises by multiplying the arrow values of $\mathcal{E}$ in $\mathcal{G}$ by the S-unit ε and leaving the other arrow values unchanged.

Since the contribution of the arrow values from $\mathcal{E}$ to the path values of any closed path is 0, the graph $\mathcal{G}_{\mathcal{E},\varepsilon}$ is a representation of some graph $G_{\mathcal{E},\varepsilon}$ in view of Lemma 5.2. By Lemma 5.3 there exist only finitely many S-units ε for which the graph $G_{\mathcal{E},\varepsilon}$ is not isomorphic with G. Thus there are infinitely many S-units ε such that multiplying the arrow values of $\mathcal{E}$ by ε and leaving the other arrow values invariant yields another representation of G. Each such a representation belongs to a different equivalence class of representations of G. Thus G is infinitely representable with S. □

11. Proofs of the results stated in Section 6

In the proofs we use the following observations. By the definition of equivalence of representations the values of all the vertices of a representation $\mathcal{G}$ with S of some connected graph G are uniquely determined within an equivalence class by the value of one vertex and the value of one arrow. (Conversely, the value of the vertex may be any element of K and the value of the arrow any S-unit.) Since vertices have distinct values, within an equivalence class a representation of G is also uniquely determined by the values of two vertices.

Proof of Theorem 6.1. Consider the set of representations of $G = G_1 \cup G_2$ for which two vertices v_1 and v_2 in $G_1 \cap G_2$ have fixed distinct values. Then, by the finite representability of G_1, there are only finitely many representations of G_1. But for the same reason there exist only finitely many representations of G_2. Hence there are only finitely many possibilities to give values to the other vertices of G. Thus G is finitely representable. □

Proof of Theorem 6.2. If G_1 and G_2 have no vertices in common, then the statement is trivial. If G_1 and G_2 have precisely one vertex in common, then the situation is still simple. Indeed, consider any representation of G_1 and any degenerate representation of G_2. Suppose that in these representations of G_1 and G_2 the values a and a' are attached to the common vertex of G_1 and G_2, respectively. Now adding $a' - a$ to the values attached to the vertices of G_1, we get a representation of G of the required form, since G_1 and G_2 have no common edge. Thus the infinitely many pairwise non-equivalent representations of G_2 yield infinitely many non-equivalent representations of G which leave G_1 unchanged.

So from this point on we may assume that G_1 and G_2 have at least two vertices in common. Every representation of G induces a representation of G_1 and a representation of G_2. Fix the values of two vertices

$v, w \in G_1 \cap G_2$ and consider representations of G with these fixed vertex values. Then, by the finite representability of G_1, there are only finitely many different induced representations of G_1.

Suppose G is infinitely representable. Then, by Theorem 5.1, there are a representation $\mathcal{G}$ of G, a non-empty proper subset $\mathcal{E}$ of the edges of $\mathcal{G}$ and infinitely many S-units ε such that multiplying the values of the arrows of $\mathcal{E}$ by ε and leaving the other arrow values of $\mathcal{G}$ unchanged yields another representation of G. Since G_1 has only finitely many representations, there exists a set E of infinitely many S-units ε such that multiplication of the arrow values of $\mathcal{E}$ by ε and leaving the other arrow values unchanged yields a representation of G such that its restriction to G_1 is in the equivalence class of some representation $\mathcal{G}_1$ of G_1.

Let c be the difference of the values of v and w in $\mathcal{G}_1$ and c_ε the difference after the multiplication of the arrow values of $\mathcal{E}$ by $\varepsilon \in E$ leaving the other arrow values invariant. Then $c \neq 0$ and, since both representations belong to the same equivalence class of G_1, $c_\varepsilon = \beta_\varepsilon c$ where β_ε is an S-unit. But for these ε's we also have along any path in G_1 from v to w that $c_\varepsilon = \varepsilon P + Q$ where P is the sum of the contributions of the arrows in $\mathcal{E}$, Q the contribution of the arrows not in $\mathcal{E}$, and $P + Q = c$. Thus the equation $\beta_\varepsilon c = \varepsilon P + Q$ with constants c, P, Q has infinitely many solutions in S-units $\varepsilon, \beta_\varepsilon$. By Theorem A this implies $P = 0$ or $Q = 0$ as $c \neq 0$. If $P = 0$, then $c_\varepsilon = Q = c$ for infinitely many ε's. Then $c_\varepsilon = c$ for infinitely many ε's. If $Q = 0$, then we repeat the above procedure with $\mathcal{E}$ replaced by its complement in G and conclude that $c_\varepsilon = c$ for infinitely many ε's too. For such ε's the vertex values of v and w remain unchanged. But this means that the restriction to G_1 is the representation $\mathcal{G}_1$ itself. Thus $\mathcal{E}$ belongs to $G \setminus G_1$. □

Remark. In the proof of Theorem 6.2 $P = 0$ corresponds to the case that no edge of G_1 belongs to $\mathcal{E}$ and $Q = 0$ to the case that all edges of G_1 belong to $\mathcal{E}$.

Proof of Corollary 6.1. It suffices to notice that by Theorem 4.1 a triangle is finitely representable for any S and that two triangles with a common edge have two vertices in common. Thus Theorem 6.1 can be used inductively. □

Proof of Corollary 6.2. We show that G^∇ is connected if and only if $G^\triangle$ is connected. Then the statement follows from Corollary 6.1.

For given $G^\triangle$ we can construct G^∇ in the following way: replace every triangle of edges by a vertex and connect two vertices by an edge if and only if the corresponding triangles in $G^\triangle$ have a vertex in common. Conversely, for given G^{∇} we can construct $G^{\triangle}$ by replacing every vertex by a triangle such that if the vertices are connected by an edge in G^{∇} the corresponding triangles in $G^{\triangle}$ have a common vertex. It is obvious that there is a path in $G^{\triangle}$ between two triangles if and only if there is a path in G^{∇} between the corresponding vertices. This proves our claim. □

Proof of Corollary 6.3. Apply Theorem 6.1 to the components of $G^{\triangle}$. □

12. Proofs of the results stated in Section 7

We shall need some further preliminary results. The following theorem was established in terms of the complements of the graphs $\mathcal{G}_S(A)$ which formulation is more useful for certain applications.

Theorem E. (Győry [12]) *Let $n \geq 3$ be an integer, and fix S. Then for all but at most*

$$\big((n+1)^4 2^{16(|S|+d)}\big)^{n-2}$$

S-equivalence classes of ordered n-term subsets A from O_S, one of the following cases holds:

i) $\overline{\mathcal{G}_S(A)}$ is connected and at least one of $\mathcal{G}_S(A)$ and $\mathcal{G}_S(A)^{\triangle}$ is not connected;

ii) $\overline{\mathcal{G}_S(A)}$ has exactly two components, $\overline{\mathcal{G}_1}$, and $\overline{\mathcal{G}_2}$, say, such that $|\overline{\mathcal{G}_1}| = 1$, and $\mathcal{G}_2$ is not connected;

iii) $\overline{\mathcal{G}_S(A)}$ has exactly two components of orders ≥ 2.

Proof. This is a consequence of a special case of Theorem 2.2 of [12]. □

For $n \geq 5$, the following lemma provides an upper bound for the number of cases in Theorem E iii).

Lemma 12.1. *Let $n \geq 5$ be an integer, and let S be fixed. There are at most*

$$\big(n \cdot 5^{453(|S|+d)}\big)^{n-1}$$

S-equivalence classes of ordered n-term subsets A in O_S for which $\overline{\mathcal{G}_S(A)}$ consists of two components, of which one has order ≥ 3 and the other has order ≥ 2.

In the proof of Lemma 12.1 we use the following result.

Lemma 12.2. *Apart from an S-unit factor, there are at most*

$$2^{1053(|S|+d)}$$

elements $\gamma \in K^*$ such that

$$x + y = \gamma \quad \text{in } x, y \in O_S^*$$

has more than two solutions.

For a qualitative version of Lemma 12.2, see Evertse, Győry, Stewart and Tijdeman [7]. For the special case $K = \mathbb{Q}$, see Lemma 10.2 in Part I.

Proof of Lemma 12.2. A combination of the proof of Lemma 10.2 of Part I with O_S instead of $\mathbb{Z}_S$ and the inequality $N_3 \leq 5^{444(|S|+d)}$ from Theorem B proves the assertion. □

Proof of Lemma 12.1. Following the proof of Lemma 10.1 of Part I with the choice

$$C_1 = 5^{444(|S|+d)},\ C_2 = 3 \cdot 7^{3d+2|S|}$$

and working over O_S in place of $\mathbb{Z}_S$, the assertion follows. □

Proof of Theorem 7.3. Combine Theorem E and Lemma 12.1. □

Proof of Corollary 7.1. Let G be a graph of order $n \geq 3$ and suppose that G is more than

$$\left(n \cdot 5^{114(|S|+d)}\right)^{4(n-1)}$$

times representable for some S. This means that G is isomorphic to $\mathcal{G}_S(A)$ for as many S-equivalence classes of ordered subsets A from O_S. The assertion immediately follows from Theorem 7.3. □

Proof of Theorem 7.4. The theorem immediately follows from Theorem D. □

To prove Theorem 7.1, we shall need the following

Theorem F. (Győry [10]) *Let $n \geq 3$ be an integer. For given S, there are only finitely many S-equivalence classes of n-term subsets A in O_S such that both $\mathcal{G}_S(A)$ and $\mathcal{G}_S(A)^{\triangle}$ are connected. These classes of n-term subsets are effectively determinable.*

Proof. This is in fact an immediate consequence of Theorem 1 of Győry [10]. Indeed, if A is a subset of O_S^n for which $\mathcal{G}_S(A)$ and $\mathcal{G}_S(A)^{\triangle}$ are connected, then A is S-equivalent to a subset A' of O_S^n such that A' is of the form $A' = \{0, \alpha'_2, \ldots, \alpha'_n\}$ and $\mathcal{G}_S(A')$ and $\mathcal{G}_S(A')^{\triangle}$ are connected. We can now apply Theorem 1 of [10] to the complement of the graph $\mathcal{G}_S(A')$ and the assertion follows. □

Remark. We note that the proof of Theorem 1 of [10] is based on Győry's [9] effective finiteness results on the equation (5). This result gives also an explicit upper bound for the heights of the solutions. This bound has been improved by several people. The best known bound is due to Győry and Yu [14]. These bounds could be used to obtain quantitative versions of Theorem F.

Proof of Theorem 7.1. Let G be a graph of order $n \geq 3$ such that both G and $G^{\triangle}$ are connected. If G is representable with S and is isomorphic to $\mathcal{G}_S(A)$ for some n-term $A \subset O_S$, then $\mathcal{G}_S(A)$ and $\mathcal{G}_S(A)^{\triangle}$ must be connected. Now Theorem F applies and the assertion follows. □

Proof of Theorem 7.2. Let G be a graph of order ≥ 3. Suppose that G is representable with some S and that G and $\mathcal{H}(G)$ are connected. If $G^{\triangle}$ is connected then the assertion follows from Theorem 7.1. Consider the case when $G^{\triangle}$ is not connected. By Theorem 7.1 each $\triangle$-connected component of $G^{\triangle}$ is finitely representable and, for given S, each of these representations is effectively determinable. We claim that if two such components are connected in $\mathcal{H}(G)$ then the subgraph of G spanned by these components is also finitely representable, and all representations of this subgraph can be effectively determined.

Indeed, let $\mathcal{G}_S(A)$ be a graph isomorphic to G for some subset A of O_S, and let $\mathcal{G}_S(B)$, $\mathcal{G}_S(B')$ be the induced subgraphs of $\mathcal{G}_S(A)$, isomorphic to the respective subgraphs of G spanned by the two components under consideration. Then it follows that

$$b_1 - b_2 = \varepsilon\kappa_{b_1,b_2} \quad \text{and} \quad b_1' - b_2' = \eta\kappa'_{b_1',b_2'}$$

for each distinct b_1, b_2 from B and b_1', b_2' from B', where ε, η are S-units and κ_{b_1,b_2}, $\kappa'_{b_1',b_2'}$ can take only finitely many values from O_S, and these are effectively determinable. But by assumption B and B' have two common vertices, which implies that $\eta = \varepsilon\tau$ for some $\tau \in O_S$ which may take only finitely many and effectively determinable values. For each $b_1 \in B$ and $b_1' \in B'$ we have

$$b_1 - b_1' = (b_1 - b_2) + (b_2 - b_1')$$

where b_2 is a common vertex of B and B'. This means that up to the factor ε, $b_1 - b_1'$ may take only finitely many and effectively determinable values from O_S, whence our claim follows.

Finally, we can treat the remaining components by induction, and the assertion follows. □

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