ON A DIOPHANTINE EQUATION CONCERNING THE NUMBER OF INTEGER POINTS IN SPECIAL DOMAINS

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1. Introduction

Many number theoretical problems have geometrical backgrounds. We just refer to two new results, [3] and [5], and the references given there. The first paper deals with the asymptotic behaviour of the number of lattice points inside a sphere, while the second one gives an upper bound for the number of lattice points on an arc with given length of an ellipse.

In this paper a similar problem is considered. We deal with the diophantine equation $\#\{(x_1, x_2) \in \mathbb{Z}^2 : \ |x_1| + |x_2| \leq r\} = \#\{(y_1, \dots, y_n) \in \mathbb{Z}^n : \ |y_1| + \dots + |y_n| \leq R\}$. This equation has some geometrical and combinatorial aspects. The problem, for example, can be interpreted in the following way. Consider $\mathbb{R}^n$ with the counting measure (with respect to the integer points). Now one can ask: for which values of r and R will the two dimensional "octahedron" with centre $\underline{O}$ and with diameter $2r$ (in $\mathbb{R}^2$) has the same "volume" as the n-dimensional "octahedron" with centre $\underline{O}$ and with diameter $2R$ (in $\mathbb{R}^n$)? For $n = 3$ this question leads to an elliptic diophantine equation. There are bounds for the solutions of elliptic equations. (For the best known bounds concerning rational integer solutions cf. [14], and for the general superelliptic case cf. [4].) Unfortunately, these bounds are too large to determine the solutions of a concrete equation. However, recently J. Gebel, A. Pethő and H. G. Zimmer [9], and independently R. J. Stroeker and N. Tzanakis [17] have developed an algorithm for finding all integer points on elliptic curves. Their methods are based on estimates of linear forms in elliptic logarithms. J. Gebel implemented the method of [9] in the computational number theoretical program package SIMATH (cf. [16]), and we will use this package to solve our elliptic equation.

The results in [9] and [17] was adapted by N. Tzanakis to the case of quartic equations as well (cf. [18]). However, this method is not implemented yet, so in case of $n = 4$ we will follow another argument. Using a method of Á. Pintér and B. M. M. de Weger (cf. [13] and [20], respectively), our equation can be reduced to the solution of some quartic Thue equations. These equations will be solved by the program package KANT (see [11]).

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2. Notation

First we introduce our notation. Let

$$f_n(r) = \#\{(x_1,\ldots,x_n) \in \mathbb{Z}^n : \ |x_1| + \ldots + |x_n| \le r\} \ n = 0,1,2,\ldots$$

For small values of n we have

$$f_0(r) = 1,\ f_1(r) = 2r+1,\ f_2(r) = 2r^2+2r+1,\ f_3(r) = \frac{4}{3}r^3 + 2r^2 + \frac{8}{3}r + 1$$

and

$$f_4(r) = \frac{2}{3}r^4 + \frac{4}{3}r^3 + \frac{10}{3}r^2 + \frac{8}{3}r + 1 \ .$$

It is easy to verify that f_n is of degree n, and for $n \ge 1$ the polynomials satisfy the following recursion:

$$f_n(r) = 2\sum_{k=0}^{r-1} f_{n-1}(k) \ + \ f_{n-1}(r) \ .$$

One can see that the polynomials f_n are closely related to the Bernoulli-polynomials. The equations $f_2(r) = f_n(R)$ in nonnegative integers r and R are trivial for $n = 0,1,2$. The aim of this paper is to solve the equations $f_2(r) = f_n(R)$ for $n = 3,4$.

3. Results

In this section we formulate our results concerning the equations

$$f_2(r) = f_3(R) \text{ in } \ r,\ R \in \mathbb{Z},\ r,\ R \ge 0 \tag{1}$$

and

$$f_2(r) = f_4(R) \text{ in } \ r,\ R \in \mathbb{Z},\ r,\ R \ge 0. \tag{2}$$

First we will prove that the only solutions of (1) are $(r,R) = (0,0),\ (3,2),\ (403,62)$ and $(6012,378)$. (The condition that r and R are nonnegative is natural, and the assumption that they are integers may be made without loss of generality.) This statement will follow from Theorem 1. Put $x = 6R+3$ and $y = 18r+9$. Then from equation (1) we get the elliptic equation

$$x^3 + 45x - 81 = y^2 \quad \text{in integers } x,\ y\ . \tag{3}$$

We have the following

Theorem 1. *The only integer solutions of equation (3) are* $(x,\pm y)$*=(3,9), (10,37), (15,63), (18,81), (375,7263) and (2271,108225).*

As a simple consequence of Theorem 1 we obtain our statement concerning the solutions of (1).

We will also prove that the only nonnegative integer solutions of (2) are $(r,R) = (0,0)$ and $(4,2)$. (As we remarked in case of Theorem 1, the assumptions made on r and R in fact are not restrictions.) This statement will follow from Theorem 2. Put $x = R$ and $y = 2r+1$. Then from equation (2) we get the equation

$$(2x^2+2x+4)^2 - 13 = 3y^2 \ \text{ in integers } x,\ y\ . \tag{4}$$

We have the following

Theorem 2. *The only integer solutions of equation (4) are* $(x,\pm y)$*=(0,1), (-1,1), (2,9) and (-3,9).*

Our statement concerning the solutions of (2) now follows as a simple consequence.

4. Proofs of the Theorems

In the first half of this section we deal with Theorem 1. To the proof of this statement we need two lemmas and some (brief) new notation. For a more detailed study of elliptic curves we refer to [9].

Let E be an elliptic curve over $\mathbb{Q}$ defined by

$$E: \quad y^2 = x^3 + ax + b \quad (a, b \in \mathbb{Z}),$$

and suppose that the discriminant of E is nonzero. Denote by r the rank, by g the number of torsion points, and by j the j-invariant (or modular invariant) of E. For any point P of E let $\hat{h}(P)$ be the canonical height (or Néron-Tate height) of P. It is known, that $\hat{h}$ can be considered as a positive definite quadratic form; denote by λ_1 its smallest eigenvalue.

Let $P_1, \ldots, P_r$ be a basis of the Mordell-Weyl group of E. Then every point P of E has a unique representation of the form

$$P = \sum_{i=1}^{r} n_i P_i + P_{r+1} \ (n_i \in \mathbb{Z}) \ ,$$

where P_{r+1} is a torsion point. Let

$$N = \max_{1 \le i \le r} |n_i| \ .$$

Let μ_∞ denote the height of E, that is

$$\mu_\infty = \log \max\{|a|^{1/2}, |b|^{1/3}\} \ .$$

Let $\wp$ be Weierstrass' $\wp$ function corresponding to E, and let P be a point of E. Then we have

$$P = (\wp(u), \wp'(u))$$

for some complex number u with $|u| \le \frac{1}{2}$. Here u is the elliptic logarithm of P. Let ω_1 and ω_2 be the real and complex period of E, respectively, and let $\tau = \pm\omega_2/\omega_1$, such that $Im(\tau) > 0$. Denote by α the greatest real root of the polynomial $p(x) = x^3 + ax + b$, and let β and γ be the two other roots of $p(x)$. Put

$$M = \begin{cases} 0, & \text{if } \alpha \ge 0 \\ \frac{\exp(\mu_\infty)}{2^{\frac{1}{3}} - 1}, & \text{if } \alpha < 0 \end{cases}$$

and

$$\xi_0 = \begin{cases} 2\alpha + M & , \text{ if } \beta \in \mathbb{R} \\ 2\max\{\alpha, (\beta+\gamma)/2\} & , \text{ otherwise} \ . \end{cases}$$

Let $j = \frac{j_1}{j_2}$ with $j_1, j_2 \in \mathbb{Z}$, $(j_1, j_2) = 1$ and let $h = \log \max\{4|aj_2|, 4|bj_2|, |j_1|, |j_2|\}$. Choose real numbers $V_1, \ldots, V_r$ with

$$\log V_i \ge \max\left\{\hat{h}(P_i), h, \frac{3\pi|u_i|^2}{\omega_1^2 Im(\tau)}\right\} \text{ for } i = 1, \ldots, r \ ,$$

where u_i is the elliptic logarithm of P_i, $i = 1, \ldots, r$. Now, by the above notation, we have the following lemma.

Lemma 1. *Let $P=(\xi,\eta)=(\wp(u),\wp'(u))$ be an integer point on the elliptic curve E such that, by the above notation*

$$\xi > \max\{e^{\mu_\infty}, \xi_0\} \ .$$

Then the elliptic logarithm u of P satisfies the estimate

$$\exp\left\{-Ch^{r+1}\left(\log\left(\frac{r+1}{2}gN\right)+1\right)\left(\log\log\left(\frac{r+1}{2}gN\right)+1\right)^{r+1}\prod_{i=1}^{r}\log V_i\right\}$$

$$\leq |gu| < \exp\{-\lambda_1 N^2 + \log(g{c_1}')\} \ ,$$

where

$${c_1}' = \frac{2^{\frac{13}{6}}}{\omega_1}$$

and

$$C = 2.9\cdot 10^{6r+6}\cdot 4^{2r^2}\cdot (r+1)^{2r^2+9r+12.3} \ .$$

Proof. This statement is proved in [9] using the lower bound for linear forms in elliptic logarithms, due to S. David [7].

By a famous theorem of C. L. Siegel [15], the number of integer points on E is finite. Using the following Lemma, one can find, at least in principle, all the integer points on given elliptic curves.

Lemma 2. *Preserving the above notations, let $P=\sum_{i=1}^{r} n_iP_i + P_{r+1}$ be an integral point on the elliptic curve E, where $P_1,\dots,P_r$ is a basis of the Mordell-Weyl group of E, and P_{r+1} is a torsion point. Then the maximum*

$$N = \max_{1\leq i\leq r}\{|n_i|\}$$

satisfies the inequality

$$N \leq \max\left\{2^{r+2}\sqrt{c_1c_2}(\log(c_2(r+2)^{r+2}))^{(r+2)/2}, \frac{2\max\limits_{1\leq i\leq r}\{V_i\}}{r+1}\right\} \ ,$$

where

$$c_1 = \max\left\{\frac{\log(g{c_1}')}{\lambda_1},1\right\} \ \textit{and}\ c_2 = \max\left\{\frac{C}{\lambda_1},10^9\right\}\left(\frac{h}{2}\right)^{r+1}\prod_{i=1}^{r}\log V_i \ .$$

Proof. This Lemma is a consequence of Lemma 1. (See the Theorem in [9] on page 180.)

Proof of Theorem 1. We will follow the discussion in [10], and we preserve the above notations. Let

$$E = \{(x,y)|(x,y)\in\mathbb{Q}^2,\ x^3+45x-81=y^2\}\cup\{\mathcal{O}\} \ ,$$

where $\mathcal{O}$ denotes the point at infinity. In the sequel we determine some parameters of E using SIMATH.

The modular invariant of E is

$$j = \frac{j_1}{j_2} = \frac{864000}{743} ,$$

and the height of E is

$$\mu_\infty = 1.90333124... \quad .$$

To use Lemma 2, one has to know a basis as well as the torsion group of E. Using SIMATH, it turns out that the only torsion point of E is $\mathcal{O}$, hence $g = 1$. The rank of E is $r = 2$, and a basis of the Mordell-Weyl group of E is $\{P_1 = (3, 9),\ P_2 = (15, 63)\}$ with

$$\hat{h}(P_1) = 0.66267054...,\ \hat{h}(P_2) = 1.49218454... \ .$$

We obtain $\lambda_1 = 0.65368290...$.

The real and complex periods of E are

$$\omega_1 = 1.27864816...$$

and

$$\omega_2 = 0.63932408... + i \cdot 0.75319186... ,$$

respectively, whence

$$Im(\tau) = 0.58905325... \ .$$

We have $c_1{}' < 3.51140236$ and $c_1 < 1.92144460$. Moreover, we obtain $h < 13.66932804$, whence

$$\max\left\{\hat{h}(P_i), h, \frac{3\pi|u_i|^2}{\omega_1^2 Im(\tau)}\right\} = h \text{ for } i = 1, 2 \ .$$

Therefore we may choose

$$V_i = e^h = 864000 \text{ for } i = 1, 2 \ .$$

The constant $C < 3.57 \cdot 10^{41}$, hence

$$c_2 = \max\left\{\frac{C}{\lambda_1}, 10^9\right\}\left(\frac{h}{2}\right)^{r+1}\prod_{i=1}^{r}\log V_i < 3.26 \cdot 10^{46} \ .$$

Using the above parameters, Lemma 2 yields the estimate

$$N < 5.1 \cdot 10^{28} ,$$

that is, we have obtained an initial bound for the coefficients of P with respect to the basis P_1, P_2. Now using B. M. M. de Weger's method (see [19]), this initial bound can be reduced, and using SIMATH we obtain that the only integral points on E are the followings:

$$(x,\pm y) = (3,9),\ (10,37),\ (15,63),\ (18,81),\ (375,7263) \text{ and } (2271,108225)\ ,$$

and Theorem 1 follows.

Remark. We would like to mention that there is another way to solve our elliptic equation. Using an argument of L. J. Mordell [12] and B. M. M. de Weger [19], the equation can be reduced to several quartic Thue equations. These equations can be solved by the program package KANT. (The first reduction algorithm for solving Thue equations is due to W. J. Ellison (cf. [8]), who applied a method of A. Baker and H. Davenport, see [1]. The algorithm implemented in KANT is based on a result of Y. Bilu and G. Hanrot (cf. [2]). We have solved our elliptic equation in this way too, and of course, we obtained the same result. We omit the details.

Now we turn to the proof of Theorem 2.

Proof of Theorem 2. Let $K = \mathbb{Q}(\sqrt{13})$. The main arithmetical invariants of K are well known. The discriminant of K is 13, $\{1, \frac{-1+\sqrt{13}}{2}\}$ is an integral basis and $\eta = \frac{3}{2} - \frac{\sqrt{13}}{2}$ is a fundamental unit (with norm -1). The class number of K is one, so the ring of integers of K is a ring with unique factorization.

Equation (4) can be rewritten in the following form

$$(2x^2+2x+4+\sqrt{13})(2x^2+2x+4-\sqrt{13}) = \left(\frac{-1}{2}+\frac{\sqrt{13}}{2}\right)\left(\frac{1}{2}+\frac{\sqrt{13}}{2}\right)y^2\ , \quad (5)$$

and the greatest common divisor D of the factors on the left hand side divides $2\sqrt{13}$. (Both 2 and $\sqrt{13}$ are primes in K.) However, one can simply verify that D divides neither 2, nor $\sqrt{13}$, hence $D = 1$. Therefore, by (5) we have that

$$2x^2+2x+4+\sqrt{13} = (-1)^{\alpha}\eta^{\beta}\left(\frac{-1}{2}+\frac{\sqrt{13}}{2}\right)^{\gamma}\left(\frac{1}{2}+\frac{\sqrt{13}}{2}\right)^{\delta}{y_1}^2\ , \qquad (6)$$

with $\alpha, \beta, \gamma, \delta \in \{0,\ 1\}$ and $y_1 = a + b\frac{-1+\sqrt{13}}{2}$, where a and b are rational integers. By taking the norms of both sides of (6) we get $\beta = 1$ and $\gamma + \delta = 1$. It means that we have to study four cases, listed in Table 1. In all the four cases we represent the right hand side of (6) in the integral basis $\{1, \frac{-1+\sqrt{13}}{2}\}$, and we get a system of two equations by comparing the coefficients of 1 and $\sqrt{13}$ on the left and right hand sides of (6):

No.	(α,γ,δ)	system of equations
I.	$(0,1,0)$	$2x^2+2x+4 = -4a^2+17ab-\frac{41}{2}b^2$ $1 = a^2-5ab+\frac{11}{2}b^2$
II.	$(1,1,0)$	$2x^2+2x+4 = 4a^2-17ab+\frac{41}{2}b^2$ $1 = -a^2+5ab-\frac{11}{2}b^2$
III.	$(0,0,1)$	$2x^2+2x+4 = -\frac{5}{2}a^2+9ab-12b^2$ $1 = \frac{1}{2}a^2-3ab+3b^2$
IV.	$(1,0,1)$	$2x^2+2x+4 = \frac{5}{2}a^2-9ab+12b^2$ $1 = -\frac{1}{2}a^2+3ab-3b^2$

Table 1

The cases (I) and (II)

In the cases (I) and (II) the corresponding systems of equations are unsolvable. The proof of this statement is the following. (The signs correspond to the cases (I) and (II), respectively.)

From the corresponding first equation we get

$$\pm(2x+1)^2 \pm 7 = -8a^2 + 34ab - 41b^2 \ .$$

Into this equation we substitute the corresponding second equation to obtain

$$\mp 15(4x+2)^2 = (30a - 69b)^2 + (3b)^2 \ .$$

Using congruence conditions modulo 3, this can be shown to hold only if

$$4x + 2 = 0 \ , \ 30a - 69b = 0 \ , \ 3b = 0 \ ,$$

but this is impossible.

The case (III)

In the case (III) of Table 1 the system of equations obtained from (6) is also unsolvable. Really, from the second equation we obtain

$$2 \equiv a^2 \ (mod \ 3) \ ,$$

which is impossible.

The case (IV)

Consider now the system of two equations in case (IV) of Table 1. The first equation in the system can be rewritten in the following form:

$$(2x+1)^2 + 7 = 5a^2 - 18ab + 24b^2 \ .$$

After substituting the corresponding second equation we get

$$5(4x+2)^2 = (30b - 13a)^2 + a^2 \ . \tag{7}$$

To determine the solutions of (7) we shall need the following lemma (which is a consequence of the general results obtained in [6]).

Lemma 3. *All nontrivial solutions of the equation* $5u^2 = v^2 + w^2$ *in* $u, v, w \in \mathbb{Z}$ *are of the form*

$$\pm u = \frac{d}{h}(m^2 + 4mn + 5n^2), \ v = \frac{d}{h}(2m^2 + 10mn + 10n^2), \ w = \frac{d}{h}(-m^2 + 5n^2),$$

with integers d, h, m, n satisfying

$$(m, n) = 1, \ n \geq 0, \ h \in \{1, 2, 5, 10\} \ .$$

If we parametrize the solutions of equation (7) using Lemma 3, and we substitute into the second equation of the system, then we obtain quartic Thue equations in the parameters m, n. In the followings we give all the occuring Thue equations (with their right hand sides). We give the parametrized values of a, b, x, too.

Let

$$F(s,t) = 59s^4 - 80s^3t - 850s^2t^2 + 1875t^4 \ .$$

We get the following equation ($d \in \mathbb{Z}$, $h \in \{1, 2, 5, 10\}$):

$$F(m,n) = \frac{300h^2}{d^2} \tag{8}$$

with

$$a = \frac{d(-m^2+5n^2)}{h}, \ b = \frac{d(-11m^2+10mn+75n^2)}{30h}$$

and

$$x = \pm\frac{d(m^2+4mn+5n^2)}{4h} - \frac{1}{2} \ .$$

One can easily see that some of the Thue equations of (8) turn out to be unsolvable (we omit the details). In Table 2 we give all the remaining Thue equations (in simpler form), indicating the values of x (in parametrized form), d and h as well.

Let

$$G(s,t) = 1593s^4 - 144s^3t - 102s^2t^2 + t^4 \ .$$

We get the following six equations:

No.	(d,h)	x	Thue equations
1.	$(\pm 2, 5)$	$\frac{\pm(m^2+4mn+5n^2)}{10} - \frac{1}{2}$	$G(m/5, 3n) = 81$
2.	$(\pm 1, 5)$	$\frac{\pm(m^2+4mn+5n^2)}{20} - \frac{1}{2}$	$G(m/5, 3n) = 324$
3.	$(\pm 1, 10)$	$\frac{\pm(m^2+4mn+5n^2)}{40} - \frac{1}{2}$	$G(m/5, 3n) = 1296$
4.	$(\pm 10, 1)$	$\frac{\pm 5(m^2+4mn+5n^2)}{2} - \frac{1}{2}$	$G(m, 15n) = 81$
5.	$(\pm 5, 1)$	$\frac{\pm 5(m^2+4mn+5n^2)}{4} - \frac{1}{2}$	$G(m, 15n) = 324$
6.	$(\pm 5, 2)$	$\frac{\pm 5(m^2+4mn+5n^2)}{8} - \frac{1}{2}$	$G(m, 15n) = 1296$

Table 2

We use the program package KANT to solve these Thue equations. We summarize the solutions in the following Table 3.

Thue equation	(s,t)
$G(s,t) = 81$	$(0, \pm 3), (\pm 1, \mp 6)$
$G(s,t) = 324$	$(\pm 1, \pm 3), (\pm 3, \mp 21)$
$G(s,t) = 1296$	$(0, \pm 6), (\pm 2, \mp 12)$

Table 3

As one can see, we get no solution for x in cases (4), (5) and (6) of Table 2. In the remaining cases (1), (2) and (3) we obtain the following values for x:

$$x = 0, -1\ ,\ x = 2, -3 \text{ and } x = 0, -1\ ,$$

respectively. Hence we get the solutions $(x, \pm y) = (0, 1)$, $(-1, 1)$, $(2, 9)$ and $(-3, 9)$ of equation (4), and Theorem 2 follows.

Remark. We mentioned in Section 2 that the polynomials f_n are closely related to the Bernoulli polynomials. Describing this connection one could hopefully prove some kind of finiteness criterion for the number of solutions in the general case

$$f_2(r) = f_n(R),\ n \geq 0\ . \tag{9}$$

Conjecture. For given $n \geq 0$, $n \neq 2$, equation (9) has only a finite number of solutions in nonnegative integers r and R.

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