POLYNOMIAL VALUES OF FIGURATE NUMBERS

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Abstract. There are a lot of effective, ineffective and explicit results concerning power values and polynomial values of binomial coefficients. Also, many papers deal with generalizations of these problems, involving polygonal numbers and pyramidal numbers. In this paper we prove effective and ineffective theorems concerning polynomial values of figurate numbers. Our results yield common extensions and generalizations of several previous theorems from the literature.

1. Introduction

Problems related to polynomial values of binomial coefficients have a long history and a vast literature. It was proved by Erdős [15] (cases $k \geq 4$) and Győry [19] (cases $k = 2, 3$) that the only non-trivial solution of the Diophantine equation

$$\binom{x}{k} = y^\ell$$

is given by $(x, k, y, \ell) = (50, 3, 140, 2)$. Yuan [39] proved that apart from certain explicitly given cases, the Diophantine equation

$$\binom{x}{k} = ay^\ell + b \ (a, b \in \mathbb{Q})$$

for any fixed $k \geq 3$ has only finitely many solutions in x, y, ℓ, and these solutions can be explicitly bounded. Stoll and Tichy [37] investigated the number of solutions of the Diophantine equation

$$\alpha\binom{x}{m} + \beta\binom{y}{n} = \gamma,$$

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where $\alpha, \beta, \gamma \in \mathbb{Q}$, $m, n \in \mathbb{N}, m \neq n$ and they obtained several results. The general equation

$$\binom{x}{k} = g(y) \tag{1}$$

where $g(y) \in \mathbb{Q}[y]$ is an arbitrary polynomial was taken up by Kulkarni and Sury [26]. They proved that this equation, up to certain completely described cases, has only finitely many solutions. Because of the ineffective nature of the methods they applied, this result is ineffective. (Note that in [26] the results are formulated for products of consecutive integers in place of binomial coefficients - however, the two statements are equivalent.) We mention that there are many further related results in the literature, concerning binomial coefficients and products of consecutive integers. Since their survey would be an enormous task, we only suggest the interested reader to consult the papers [16], [25], [8], [27], [32], [28], [33], [10], [11], [3], [21], [35], [4], [5], [30], [31], [20], [2] and in particular, the excellent survey papers [38], [17], [36], [18] and the references given there.

Let now k, m be integers with $k \geq 2$ and $m \geq 2$, and write

$$f_{k,m}(x) = \frac{x(x+1)\dots(x+k-2)((m-2)x+k+2-m)}{k!}$$

for the x^{th} figurate number with parameters k and m. For the introduction and basic properties of these numbers see e.g. the books [13] and [12]. Observe that as for $m = 2, 3$ we get back the binomial coefficients $\binom{x+k-2}{k-1}$, $\binom{x+k-1}{k}$, respectively, these numbers can be considered as generalizations of the binomial coefficients. Further, in the special cases when $k = 2$ and 3, we get the so-called x^{th} polygonal number and pyramidal number, respectively. For some open problems and theorems related to these families of combinatorial numbers, we refer to the books [13] and [12] again. The power values and equal values of $f_{k,m}(x)$ for certain special choices of the parameters have been studied intensively, see e.g. the papers [1], [9], [24], [26], [29], [14], [22] and the references therein. In particular, the present authors together with Pintér and Tengely [22] gave effective finiteness statements for the equation

$$f_{k,m}(x) = f_{2,n}(y) \tag{2}$$

in integers x and y and provided numerical results for small values of the parameters (k, m, n).

In the present paper we deal with the general equation

$$f_{k,m}(x) = g(y),$$

where g is a polynomial with rational coefficients. Observe that it is an essential generalization of the equations studied earlier, indeed: e.g., it includes (1), and contains an extra variable. We describe precisely and completely those polynomials g for which the above equation has infinitely many solutions in $x, y \in \mathbb{Z}$. In the cases where $\deg(g) = 2$ or g is of the form $g(y) = ay^\ell + b$ ($a, b \in \mathbb{Q},\ \ell \geq 2$) we give effective bounds for the solutions of the above equation, as well. These results are common extensions and generalizations of many theorems mentioned earlier; among others the ones from [39], [9], [3], [26], [31], [37], [22] and the related ones from [4].

Our proofs ultimately rely on Baker's method (via results of Brindza [7] and Schinzel and Tijdeman [34]) and a theorem of Bilu and Tichy [6]. For their use we need to understand well the root structures of the shifts $f_{k,m}(x) + t$ of our polynomials. So our tools are similar to the ones applied e.g. in the papers [39], [26]. However, because of the extra factor (parameter) in our case the situation requires a much more careful analysis than in case of the earlier similar statements. In particular, the polynomials $f_{k,m}(x)$ do not enjoy the so-called two interval monotonicity property (see e.g. [37]). So to get the required assertions, we need to apply a much more involved argument than say in [3] or [37] - see our Proposition 4.1.

2. Main results

Let $g(x) \in \mathbb{Q}[x]$ be an arbitrary polynomial, and consider the Diophantine equation

$$f_{k,m}(x) = g(y) \tag{3}$$

in $x, y \in \mathbb{Z}$.

Our first result gives a general (partly effective) finiteness theorem for equation (3). The second part of the statement involves Dickson polynomials; they will be discussed in the beginning of Section 4.

Theorem 2.1. *Let $k \geq 4$ and $m \geq 4$.*

(i) For $\deg g \in \{0, 2\}$, there exists an effectively computable constant $C_1(k, m, g)$ depending only on k, m and g such that $\max(|x|, |y|) < C_1(k, m, g)$ for each integer solution of equation (3), unless $k = 4$ and $m = 4, 6$.

(ii) For $\deg g \geq 3$, equation (3) has only finitely many integer solutions x, y, unless we are in one of the following cases:

- *$g(x) = f_{k,m}(S(x))$, where $S(x) \in \mathbb{Q}[x]$ with $deg(S) > 0$,*
- *$m = 4$, k is even and $g(x) = \varphi(T(x))$ with $\varphi(x) = \frac{2}{k!}\prod_{i=0}^{\frac{k-2}{2}}(x - i^2)$ and $T(x) \in \mathbb{Q}[x]$ has at most two roots of odd multiplicity,*

- *$k = m = 4$ and $g(x)$ is one of the polynomials*
$$\frac{1}{192u^{4t}}D_t(cx+d,u^8)-\frac{1}{96},\quad -\frac{1}{192u^t}D_{2t}(cx+d,u)-\frac{1}{96},$$
where u, c, d are arbitrary rationals with $uc \neq 0$ and $t \geq 3$ is an odd integer. Here $D_s(x, \alpha)$ is the Dickson polynomial with parameters s and α.

Remark 1. Obviously, in the exceptional cases $\deg g = 1$ and $g(x) = f_{k,m}(h(x))$, equation (3) can have infinitely many integer solutions x, y. From our proofs it will be clear that the same is true for the other exceptional choices of $g(x)$. We also note that our assumption $m \geq 4$ is not a real restriction. As we mentioned already, for $m = 2, 3$ we have $f_{k,m}(x) = \binom{x+k-2}{k-1}$, $\binom{x+k-1}{k}$, respectively. Hence in these cases parts (i) and (ii) of the above theorem are given by the results of Yuan [39] and Kulkarni and Sury [26], respectively. Further, the condition $k \geq 4$ is necessary. Indeed, for $k = 2$ we have
$$f_{2,m}(x) = \frac{1}{2}x((m-2)x + 4 - m),$$
and one could easily find counterexamples in both cases (i) and (ii). Also, in the case $k = 3$ the equation has infinitely many solutions which can be easily described. In fact we get that the equation $f_{3,m}(x) = ay^2 + b$ has infinitely many solutions if and only if $b = \frac{37m^2-274m+481}{36(m-2)^2}$. Finally, the cases $k = 4$, $m = 4, 6$ must also be excluded from part (i). This could be easily demonstrated by concrete examples, leading to Pellian equations.

We also mention that in the proof of part (ii) of our Theorem 2.1 we use the ineffective finiteness criterion of Bilu and Tichy [6] combined with Theorem 2.3 and other considerations. Thus, part (ii) is an ineffective statement.

Consider now the equation
$$f_{k,m}(x) = ay^\ell + b, \tag{4}$$
in integers x, y, ℓ with $\ell \geq 2$, where a, b are fixed rational numbers with $a \neq 0$.

The following theorem gives effective bounds for the solutions of (4).

Theorem 2.2. *Let $k \geq 3$ and $m \geq 4$. Then for all solutions x, y, ℓ of equation (4) with $|y| > 1$ we have $\ell < C_2(k, m, a, b)$. Further, for any fixed $\ell \geq 2$ for the solutions x, y of (4) we have $\max(|x|, |y|) < C_3(k, m, a, b)$, unless $(k, \ell) = (3, 2)$ or $(k, m, \ell) = (4, 4, 2)$, $(4, 6, 2)$, $(4, 4, 4)$. Here, $C_2(k, m, a, b)$ and $C_3(k, m, a, b)$ are effectively computable constants depending only on k, m, a and b.*

Remark 2. Similarly to the case of Theorem 2.1 also in the above theorem the conditions are natural and/or necessary. (See also Remark 1.)

Our last theorem concerns the decomposability of the polynomials $f_{k,m}(x)$. This plays an important role in the proof of part (ii) of Theorem 2.1. By a *decomposition* of a polynomial $F(x)$ over a field K we mean writing $F(x)$ as

$$F(x) = G(H(x)) \quad (G(x), H(x) \in K[x]),$$

which is *nontrivial* if $\deg G(x) > 1$ and $\deg H(x) > 1$. Two decompositions $F(x) = G_1(H_1(x))$ and $F(x) = G_2(H_2(x))$ are said to be *equivalent* if there exists a linear polynomial $\lambda(x) \in K[x]$ such that $G_1(x) = G_2(\lambda(x))$ and $H_1(x) = \lambda^{-1}(H_2(x))$. The polynomial $F(x)$ is called *decomposable* over K if it has at least one nontrivial decomposition over K; otherwise it is said to be *indecomposable*.

Theorem 2.3. *Let $k \geq 3, m \geq 4$. Then the polynomial $f_{k,m}(x)$ is indecomposable over $\mathbb{C}$, unless k is even and $m = 4$, when any nontrivial decomposition is equivalent to*

$$f_{k,m}(x) = \frac{2}{k!} \prod_{i=0}^{k-2} \left(\left(x + \frac{k-2}{2} \right)^2 - \frac{i^2}{4} \right).$$

Remark 3. Note that once again, the assumption $m \geq 4$ is not a real restriction. The cases $m = 2, 3$ have been settled by Theorem 1 of Kulkarni and Sury [26]. The case $k = 2$ is trivial.

3. Proof of the effective results

In this section we prove part (i) of Theorem 2.1 and Theorem 2.2. For this we need some lemmas and notations.

Let $f(x) = a_0x^N + \ldots + a_N = a_0 \prod_{i=1}^{n}(x - \gamma_i)^{r_i}$ be a polynomial with integer coefficients, with $a_0 \neq 0$ and $\gamma_i \neq \gamma_j$ for $i \neq j$, where γ_i $(i = 1, \ldots, n)$ are the distinct roots of $f(x)$ with multiplicities $r_1, \ldots, r_n$, respectively. Obviously for the degree N of the polynomial $f(x)$ we have $N = r_1 + \ldots + r_n$. Let $H(f)$ be the naive height of f, i.e. the maximum of the absolute values of its coefficients.

Consider the equation

$$f(x) = ay^\ell \tag{5}$$

in integers x, y, ℓ with $\ell \geq 2$, where a is a non-zero rational number.

Our first lemma is a well-known result of Schinzel and Tijdeman [34].

Lemma 3.1. *If $f(x)$ has at least two distinct zeros then for all solutions x, y, ℓ of equation* (5) *with $|y| > 1$ we have*

$$\ell < C_5(H(f), N, a),$$

where $C_5(H(f), N, a)$ is an effectively computable constant depending only on $H(f), N$ and a.

Proof. The statement is the main result of [34]. □

The following lemma is a theorem of Brindza [7]. To its formulation, we need furher notation. For any finite set S of primes, write $\mathbb{Z}_S$ for those rationals whose denominators are composed exclusively from the primes in S. By the height $h(s)$ of a rational number s we mean the maximum of the absolute values of the numerator and the denominator of s.

Lemma 3.2. *Let $\ell \geq 2$ be fixed and put $q_i = \frac{\ell}{\gcd(\ell, r_i)}$. Suppose that $(q_1, \ldots, q_n)$ is not a permutation of any of the tuples $(q, 1, \ldots, 1)$ $(q \geq 1)$ and $(2, 2, 1, \ldots, 1)$. Then all solutions of $(x, y) \in \mathbb{Z}_S$ of equation* (5) *satisfy*

$$\max\{h(x), h(y)\} < C_4(H(f), N, a, \ell, S)$$

where $C_4(H(f), N, a, \ell, S)$ is an effectively computable constant depending only on $H(f), N, a, \ell$ and S.

Proof. The statement is an immediate consequence of the main result of [7]. □

Let $k \geq 2$ and z be a rational number, and put

$$F_{k,z}(x) = x(x+1)\cdots(x+k-2)(x+z).$$

In the following part of this section we give some lemmas concerning the roots of the derivative of the polynomial $F_{k,z}(x)$, which are needed to prove our results. Obviously, all the roots of $F_{k,z}(x)$ are simple if $z \notin \{0, 1, \ldots, k-2\}$, and $F_{k,z}(x)$ has exactly one double root with all its other roots being simple if $z \in \{0, 1, \ldots, k-2\}$.

Lemma 3.3. *Let $k \geq 2$ and $z \in \mathbb{Q}$. Then all the roots of $F'_{k,z}(x)$ are real and simple. Further, writing $\alpha_1 > \cdots > \alpha_{k-1}$ for the roots of $F'_{k,z}(x)$, the polynomial $F_{k,z}(x)$ has local minimum at α_i if i is odd, and local maximum at α_i if i is even $(i = 1, \ldots, k-1)$. Finally, if $z \in \{0, 1, \ldots, k-2\}$, say $z = i$ then*

$$0 > \alpha_1 > -1 > \ldots > -i+2 > \alpha_{i-1} > -i+1 > \alpha_i = -i > \alpha_{i+1} >$$

$$> -i-1 > \ldots > -k+3 > \alpha_{k-1} > -k+2,$$

while if $z \notin \{0, 1, \dots, k-2\}$ *then writing* $c_0 > c_1 > \dots > c_i = z > \dots > c_{k-1}$ *for the roots of* $F_{k,z}$ *we have*

$$c_0 > \alpha_1 > c_1 > \alpha_2 > \dots > c_{i-1} > \alpha_i >$$
$$> c_i = z > \alpha_{i+1} > c_{i+1} > \dots > c_{k-2} > \alpha_{k-1} > c_{k-1}.$$

Proof. The statement trivially follows from Rolle's theorem and standard calculus. □

Corollary 3.1. *For any* $k \geq 2$ *and* $m \geq 4$ *the roots of* $f'_{k,m}(x)$ *are real and simple.*

Proof. Observe that $f_{k,m}(x) = wF_{k,z}(x)$ with $w = \frac{m-2}{k!}$ and $z = \frac{k}{m-2} - 1$. Thus the statement immediatley follows from Lemma 3.3. □

The next lemma deals with the sign of the function $|F_{k,z}(x)| - |F_{k,z}(x+1)|$. It plays a key role in the proof of part (i) of Theorem 2.1.

Lemma 3.4. *Let* $k \geq 12$ *and* $z \geq \frac{k-2}{2}$. *Then*

$$h(x) := |F_{k,z}(x)| - |F_{k,z}(x+1)| < 0$$

for all $-5 < x < 0$ *with* $x \notin \{-4, -3, -2, -1\}$.

Proof. We can write

$$h(x) = |(x+1)\cdots(x+k-2)|h^*(x),$$

where

$$h^*(x) = |x(x+z)| - |(x+k-1)(x+z+1)|.$$

As $-5 < x < 0$ and $z \geq \frac{k-2}{2}$ we have

$$h^*(x) = -x(x+z) - (x+k-1)(x+z+1) = -2x^2 - (2z+k)x - (k-1)(z+1).$$

The roots of $h^*(x)$ are given by

$$\frac{-(2z+k) \pm \sqrt{(2z+k)^2 - 8(k-1)(z+1)}}{4}.$$

Using $-5 < x < 0$ and $z \geq \frac{k-2}{2}$, a simple calculation shows that if these roots are real, then they are at most -5. Hence the statement immediately follows. □

Now we are ready to prove part (i) of Theorem 2.1 and Theorem 2.2.

Proof of part (i) of Theorem 2.1. The case $\deg g = 0$ is trivial, so assume that $\deg g = 2$. First we prove that for $k \geq 12$, $z \geq (k-2)/2$ and any rational t, the polynomial $F_{k,z}(x) + t$ is not of the form

$$F_{k,z}(x) + t = P(x)Q^2(x), \tag{6}$$

where $P, Q \in \mathbb{Q}[x]$, $\deg P \leq 2$ and $F_{k,z}(x)$ is defined earlier.

Assume to the contrary that with some $k \geq 12$ and $z \geq (k-2)/2$ (6) is valid. Then we have

$$F'_{k,z}(x) = Q(x)(P'(x)Q(x) + 2P(x)Q'(x)),$$

so the roots of $Q(x)$ are roots of $F'_{k,z}(x)$, too. (Recall that by Lemma 3.3, the roots of $F'_{k,z}(x)$ are real and simple.) On the other hand, (6) implies that for any root β of $Q(x)$ we have $F_{k,z}(\beta) = -t$. That is, for $\deg Q$ roots of $F'_{k,z}(x)$, the local maxima/minima of $F_{k,z}(x)$ taken at them should be equal. Observe that $\deg Q = k/2$ or $(k-2)/2$ if k is even, and $\deg Q = (k-1)/2$ if k is odd. Further, Lemma 3.3 shows that if k is even, then $F_{k,z}(x)$ has $k/2$ local minimum values and $(k-2)/2$ local maximum values, while if k is odd then the number of both such values is $(k-1)/2$. Putting these all together, we conclude that we have one of the following possibilities:

- k is even, and either all the local maximum values of $F_{k,z}(x)$ are equal, or all the local minimum values of $F_{k,z}(x)$ with at most one exception are equal,
- k is odd and either all the local maximum values of $F_{k,z}(x)$ are equal, or all the local minimum values of $F_{k,z}(x)$ are equal.

By our assumptions Lemma 3.4 shows that none of the above cases may occur. Hence for $k \geq 12$ and $z \geq (k-2)/2$, the polynomial $F_{k,z}(x) + t$ is not of the form (6), indeed.

Now we claim that for any $k \geq 12$ and rational t, the polynomial $f_{k,m}(x) + t$ is also not of the form

$$f_{k,m}(x) + t = P(x)Q^2(x), \tag{7}$$

where $P, Q \in \mathbb{Q}[x]$, $\deg P \leq 2$. If $k/(m-2) - 1 \geq (k-2)/2$, then this immediately follows from that (6) does not hold for $F_{k,z}(x) + t$ with $k \geq 12$ and $z \geq (k-2)/2$. In case of $k/(m-2) - 1 < (k-2)/2$, on applying the substitution $x \to k-2-x$, the assertion also follows from (6).

Now the statement is an immediate consequence of Lemma 3.2, whenever $k \geq 12$.

So we are left with the cases $4 \leq k \leq 11$. We can write

$$f_{k,m}(x) + t = \frac{m-2}{k!}(F_{k,z}(x) + t^*),$$

where $z = \frac{k}{m-2} - 1$, $t^* = \frac{tk!}{m-2}$. By Lemma 3.2 it is sufficient to show that $F_{k,z}(x) + t^*$ is not of the form (6). As the argument is similar for the values $4 \leq k \leq 11$, we illustrate our method only for $k = 6$. In this case we should show that $F_{6,z}(x) + t^*$ is neither of the form $(x^2 + ax + b)(x^2 + cx + d)^2$, nor of the shape $(x^3 + ax^2 + bx + c)^2$, where

$a, b, c, d \in \mathbb{Q}$. We consider only the first possibility, the second one is similar (even simpler). Assume to the contrary that we have

$$F_{6,z}(x) + t^* = (x^2 + ax + b)(x^2 + cx + d)^2.$$

Comparing the coefficients in the above equation and eliminating the unknowns a, b, c, d, z successively (using Maple [23]), ultimately we get the equation $-27t^{*2} - 320t^* + 2304 = 0$. However, it has no rational solution in t^*. Hence our claim follows in this case.

Altogether we get that under our assumptions $k \geq 4$, $m \geq 4$ and $(k, m) \neq (4, 4), (4, 6)$ the polynomial $F_{k,z}(x) + t^*$ is not of the form (6). Hence the theorem follows. □

Proof of Theorem 2.2. We rewrite equation (4) as

$$f_{k,m}(x) - b = ay^\ell. \tag{8}$$

As $(f_{k,m}(x) - b)' = f_{k,m}(x)'$, by Corollary 3.1 we see that $f_{k,m}(x) - b$ has at most double roots. Thus for any $k \geq 3$, $f_{k,m}(x) - b$ has two distinct roots. Thus the bound for ℓ immediately follows by applying Lemma 3.1 to (8).

Now we give a bound for $\max\{|x|, |y|\}$. If $|y| \leq 1$ then this expression is trivially bounded by the required parameters. So we may assume that $|y| > 1$. Then, by what we have already proved, we have $\ell < C_2(k, m, a, b)$. Observe that by part (i) of Theorem 2.1, we may suppose that $\ell \geq 3$. Recall that all the roots of $f_{k,m}(x) - b$ are at most double. So the required bound immediately follows from Lemma 3.2, unless $k = 4$ and $\ell = 4$. To check this remaining case, we only need to find those values of m and b for which $f_{4,m}(x) - b$ has two double roots. A simple calculation with Maple [23] gives that the only possibility is given by $m = 4$ (and $b = -1/48$). Since the case $k = m = \ell = 4$ is excluded, the theorem follows. □

4. Proof of the ineffective results

In this section we prove part (ii) of Theorem 2.1 and Theorem 2.3. In the first lemma we reformulate a famous theorem of Bilu and Tichy from [6]. For this, we need some notation. Following [6] we define five kinds of standard pairs of polynomials.

Let α and β be nonzero rational numbers, q, s and t positive integers, r a nonnegative integer and $v(x) \in \mathbb{Q}[x]$ a nonzero polynomial, which may be constant. Denote by $D_s(x, \alpha)$ the sth Dickson polynomial, defined by, the formula

$$D_s(x,\alpha)=\sum_{i=0}^{\lfloor s/2\rfloor}\frac{s}{s-i}\binom{s-i}{i}(-\alpha)^i x^{s-2i}.$$

A standard pair of the *first kind* is $(x^q, \alpha x^r v(x)^q)$ (or switched), where $0\le r<q$, $(r,q)=1$ and $\deg v(x)+r>0$.
A standard pair of the *second kind* is $(x^2, (\alpha x^2+\beta)v(x)^2)$ (or switched).

A standard pair of the *third kind* is $(D_s(x,\alpha^t),\ D_t(x,\alpha^s))$, where $\gcd(s,\ t)=1$.
A standard pair of the *fourth kind* is $((\alpha^{-s/2}(D_s(x,\alpha)), -\beta^{-t/2}(D_t(x,\beta)))$ (or switched), where $\gcd(s,t)=2$.
A standard pair of the *fifth kind* is $((\alpha x^2-1)^3, 3x^4-4x^3)$ (or switched).

Lemma 4.1. *Let $f(x), g(x)\in\mathbb{Q}[x]$ be non-constant polynomials. Then the following two assertions are equivalent.*
(1) *The equation $f(x)=g(y)$ has infinitely many rational solutions with a bounded denominator.*
(2) *We have $f=\varphi\circ f_1\circ\lambda$ and $g=\varphi\circ g_1\circ\mu$ where λ and $\mu\in\mathbb{Q}[x]$ are linear polynomials, $\varphi(x)\in\mathbb{Q}[x]$ and $(f_1(x), g_1(x))$ is a standard pair over $\mathbb{Q}$ such that the equation $f_1(x)=g_1(y)$ has infinitely many rational solutions with a bounded denominator.*

Proof. This is the main result of Bilu and Tichy [6]. □

Theorem 2.3 is an immediate consequence of the following more general result, which can be of independent interest. Recall the notation

$$F_{k,z}(x)=x(x+1)\dots(x+k-2)(x+z).$$

Proposition 4.1. *Let $k\ge 3$ and $z\in\mathbb{Q}$. Then the polynomial $F_{k,z}(x)$ is decomposable over $\mathbb{C}$ if and only if k is even and $z=-1$, $(k-2)/2$, $k-1$. In the exceptional cases any non-trivial decomposition is equivalent to*

$$F_{k,z}(x)=\frac{(x+s)^2-i^2}{4}\cdot\frac{(x+s)^2-(i+2)^2}{4}\cdots\frac{(x+s)^2-(i+k-2)^2}{4}$$

with

$$(s,i)=\left(\frac{k-3}{2},\frac{1}{2}\right),\left(\frac{k-2}{2},0\right),\left(\frac{k-1}{2},\frac{1}{2}\right),$$

respectively.

Proof. Suppose that we have $F_{k,z}(x)=G(H(x))$ with some $G,H\in\mathbb{C}[x]$, such that $\deg(G)\ge 2$ and $\deg(H)\ge 2$. We may clearly assume that here G,H are monic polynomials. Write $\beta_1,\dots,\beta_t$ for the roots of $G(x)$. Observe that they must be simple. Indeed, otherwise $(H(x)-\beta_i)^u$

would divide $F_{k,z}(x)$ with some $i \in \{1, \dots, t\}$ and $u \geq 2$, implying $\deg(H(x)) = 1$. Thus we have

$$\prod_{i=1}^{t} (H(x) - \beta_i) = F_{k,z}(x).$$

Without loss of generality we may assume that $(x + z) \mid (H(x) - \beta_1)$. Write

$$H(x) - \beta_1 = (x + z)(x + a_1) \dots (x + a_s) \tag{9}$$

and

$$H(x) - \beta_2 = (x + b_1) \dots (x + b_{s+1})$$

with some $s \geq 1$ where a_i, b_j are pairwise distinct elements of $\{0, \dots, k-2\}$ $(1 \leq i \leq s, 1 \leq j \leq s + 1)$. Then we have

$$(x + z)(x + a_1) \dots (x + a_s) - (x + b_1) \dots (x + b_{s+1}) = \beta_2 - \beta_1,$$

whence $z \in \mathbb{Z}$.

By symmetry, we may clearly assume that $z \geq (k - 2)/2$. We prove that the degree of $\gcd(F_{k,z}(x)+t, F'_{k,z}(x))$ is at most two for any $t \in \mathbb{C}$. For this, as in Lemma 3.3 we write $\alpha_1, \dots, \alpha_{k-1}$ for the roots of $F'_{k,z}(x)$, and investigate the numbers

$$|F_{k,z}(\alpha_i)| \quad (i = 1, \dots, k - 1).$$

It is convenient to distinguish two subcases.

I) Suppose first that $z \geq k - 1$. Then Lemma 3.3 gives that

$$0 > \alpha_1 > -1 > \alpha_2 > -2 > \dots > -k+3 > \alpha_{k-2} > -k+2 > \alpha_{k-1} > -z.$$

We investigate the sign of the function

$$h(x) := |F_{k,z}(x)| - |F_{k,z}(x + 1)|.$$

Note that a similar analysis can be found in [3] for the polynomial $\frac{F_{k,z}(x)}{x+z}$. However, here - just because of the extra factor - the situation is much more complicated, and a more involved analysis is needed.

As in Lemma 3.4, we can write

$$h(x) = |(x + 1) \dots (x + k - 2)| h^*(x).$$

with

$$h^*(x) = |x(x + z)| - |(x + k - 1)(x + z + 1)|. \tag{10}$$

Observe that if h^* is positive for $-i < x < -i + 1$ then $|F_{k,z}(\alpha_i)| > |F_{k,z}(\alpha_{i-1})|$, while if h^* is negative for $-i < x < -i+1$ then $|F_{k,z}(\alpha_i)| < |F_{k,z}(\alpha_{i-1})|$ $(i \in \{2, \dots, k - 2\})$. Indeed, in the first case we have

$$|F_{k,z}(\alpha_i)| \geq |F_{k,z}(\alpha_{i-1} - 1)| > |F_{k,z}(\alpha_{i-1})|,$$

while in the second case

$$|F_{k,z}(\alpha_i)| < |F_{k,z}(\alpha_i+1)| \le |F_{k,z}(\alpha_{i-1})|$$

holds. Further, for any x with $-k+1 \le x \le 0$ we have

$$h^*(x) = -x(x+z)-(x+k-1)(x+z+1) = -2x^2-(k+2z)x-(k-1)(z+1).$$

The roots of $h^*(x)$ are given by

$$\frac{-(2z+k) \pm \sqrt{(2z+k)^2 - 8(k-1)(z+1)}}{4}. \tag{11}$$

One can easily check that now these roots are real. Further, one of them is at most $-z$, while the other root belongs to the interval $[-k/2, (1-k)/2]$. Thus, writing

$$i_0 = \begin{cases} (k-2)/2 & \text{if } k \text{ is even}, \\ (k-1)/2 & \text{if } k \text{ is odd} \end{cases}$$

we have

$$|F_{k,z}(\alpha_{k-1})| > \cdots > |F_{k,z}(\alpha_{i_0})| \quad \text{and} \quad |F_{k,z}(\alpha_{i_0-1})| < \cdots < |F_{k,z}(\alpha_1)|.$$

This shows that for any $t \in \mathbb{C}$ there are no three distinct indices $i_1, i_2, i_3 \in \{1, \dots, k-1\}$ with

$$F_{k,z}(\alpha_{i_1}) = F_{k,z}(\alpha_{i_2}) = F_{k,z}(\alpha_{i_3}).$$

That is, we have $\deg(\gcd(F_{k,z}(x)+t, F'_{k,z}(x))) \le 2$ for any $t \in \mathbb{C}$ in this case. Thus as it is well-known (see e.g. the proof of Theorem 4.3 of [4]) we obtain $\deg(H) \le 2$. Write

$$H(x) = x^2 + Ax + B \quad (A, B \in \mathbb{C}).$$

Now k must be even, and there must exist a partition

$$\{a_1, a_2\}, \dots, \{a_{k/2-1}, a_{k/2}\}$$

of $\{0, 1, \dots, k-2, z\}$ such that

$$a_{2j-1} + a_{2j} = A \quad (j = 1, \dots, k/2).$$

This by a simple calculation yields that $z = k-1$, and the above partition is given by

$$\{0, k-1\}, \dots, \{(k-2)/2, k/2\}.$$

Now the statement immediately follows from Theorem 4.3 of [4].

II) Assume next that $z < k-1$. Recall that $z \in \mathbb{Z}$ and $z \ge (k-2)/2$. We follow a similar strategy as in case I). Now Lemma 3.3 gives that for the roots $\alpha_1, \dots, \alpha_{k-1}$ of $F'_{k,z}(x)$ we have

$$0 > \alpha_1 > -1 > \cdots > -z-1 > \alpha_z > -z = \alpha_{z+1} =$$

$$= -z > \alpha_{z+2} > -z-1 > \cdots > -k+3 > \alpha_{k-1} > -k+2.$$

We define $h^*(x)$ with (10), as in case I). Consider first $h^*(x)$ for $-z < x < 0$. Then we have

$$h^*(x) = -x(x+z)-(x+k-1)(x+z+1) = -2x^2-(k+2z)x-(k-1)(z+1).$$

The roots of $h^*(x)$ are given by (11) again. It is easy to check that now they are either non-real complex numbers, or they belong to the interval $(-z,-k/2)$. If $h^*(x)$ has no real roots then we have $h^*(x) < 0$ for all $x \in \mathbb{R}$. Hence in this case

$$|F_{k,z}(\alpha_z)| < \cdots < |F_{k,z}(\alpha_1)|.$$

If the roots x_1, x_2 of $h^*(x)$ with $x_1 \leq x_2$ are in $(-z,-k/2)$, then find the uniqe $i_1, i_2 \in \{0,1,\ldots,k-2\}$ with

$$-z \leq -i_1 \leq x_1 < -i_1+1 \leq -k/2$$

and

$$-z \leq -i_2 \leq x_2 < -i_2+1 \leq -k/2.$$

Now we have

$$|F_{k,z}(\alpha_z)| < \cdots < |F_{k,z}(\alpha_{i_1})|, \quad |F_{k,z}(\alpha_{i_1-1})| < \cdots < |F_{k,z}(\alpha_{i_2})|,$$
$$|F_{k,z}(\alpha_{i_2-1})| < \cdots < |F_{k,z}(\alpha_1)|.$$

Note that in both cases, as $F_{k,z}(\alpha_{z+1}) = F_{k,z}(-z) = 0$, we have

$$|F_{k,z}(\alpha_{z+2})| > |F_{k,z}(\alpha_{z+1})| < |F_{k,z}(\alpha_z)|.$$

Consider now $h^*(x)$ for $-k+2 < x < -z-1$. Then we have

$$h^*(x) = x(x+z)+(x+k-1)(x+z+1) = 2x^2+(k+2z)x+(k-1)(z+1).$$

So the roots of $h^*(x)$ are again given by (11). Now by what we have proved above, we have that

$$|F_{k,z}(\alpha_{k-1})| > \cdots > |F_{k,z}(\alpha_{z+2})|.$$

We conclude that for any $t \in \mathbb{C}$ there are no five distinct indices $i_1, i_2, i_3, i_4, i_5 \in \{1,\ldots,k-1\}$ with

$$F_{k,z}(\alpha_{i_1}) = F_{k,z}(\alpha_{i_2}) = F_{k,z}(\alpha_{i_3}) = F_{k,z}(\alpha_{i_4}) = F_{k,z}(\alpha_{i_5}).$$

Hence $\deg(\gcd(F_{k,z}(x)+t, F'_{k,z}(x))) \leq 4$ for any $t \in \mathbb{C}$ in this case. Thus now we get $\deg(H) \leq 4$. We show that $\deg(H) = 3, 4$ is not possible. Assume first that $\deg(H) = 3$, and write

$$H(x) = x^3 + Ax^2 + Bx + C \quad (A,B,C \in \mathbb{C}).$$

In this case k must be divisible by 3, and there must exist a partition

$$\{a_1,a_2,a_3\},\ldots,\{a_{k/3-2},a_{k/3-1},a_{k/3}\}$$

of the set $\{0, 1, \ldots, k-2, z\}$ such that

$$a_{3j-2} + a_{3j-1} + a_{3j} = A$$

and

$$a_{3j-2}a_{3j-1} + a_{3j-1}a_{3j} + a_{3j-2}a_{3j} = B$$

for $j = 1, \ldots, k/3$. Here we used the convention that z occurs in the set $\{0, 1, \ldots, k-2, z\}$ twice. Further, observe that $(x+z)^2$ must divide $H(x) - \beta_1$ in (9). Indeed, otherwise $(x + z) \mid H(x) - \beta_i$ would hold with some $i \neq 1$, which by $\beta_1 = H(z) = \beta_i$ would contradict $\beta_i \neq \beta_1$. So we may assume that one of the classes in the above partition is of the form $\{z, z, u\}$. Since the sums of the elements and the sums of the squares of the elements in each set of the above partition must be the same, we get

$$\frac{(k-2)(k-1)}{2} + z - \frac{k(2z+u)}{3} = 0$$

and

$$\frac{(k-2)(k-1)(2k-3)}{6} + z^2 - \frac{k(2z^2+u^2)}{3} = 0.$$

Solving this system, we obtain that

$$z = \frac{6k^2 - 21k + 18 \pm k\sqrt{6k^2 - 21k + 18}}{12k - 18}.$$

However, it is easy to check that then z is an integer only for $k = 3$, which by $\deg(G) > 1$ cannot be the case. Hence $\deg(H) = 3$ is not possible. Suppose next that $\deg(H) = 4$. In this case k must be divisible by four, and we must have a partition of $\{0, 1, \ldots, k-2, z\}$ into sets having four elements, with equal sums, square sums and cube sums. (It follows from the fact that the values of the first three symmetric polynomials of these quadruples must coincide.) Again, it is easy to check that one of the subsets should be of the form $\{z, z, u, v\}$. Now a simple calculation yields that

$$\frac{(k-2)(k-1)}{2} + z - \frac{k(2z+u+v)}{4} = 0,$$

$$\frac{(k-2)(k-1)(2k-3)}{6} + z^2 - \frac{k(2z^2+u^2+v^2)}{4} = 0,$$

$$\left(\frac{(k-2)(k-1)}{2}\right)^2 + z^3 - \frac{k(2z^3+u^3+v^3)}{4} = 0.$$

Solving this system for z, we obtain that either $z = (k-2)/2$, or

$$z = \frac{3k^2 - 10k + 8 \pm k\sqrt{3k^2 - 10k + 8}}{6k - 8}.$$

However, the second expression is an integer only if

$$\frac{k^2(k-2)}{12k-16}$$

is an integer square, whence $k = 4$. But this by $\deg(G) > 1$ is impossible. So $z = (k-2)/2$. Consider now the set in the above partition containing 0; let it be given by $\{0, a, b, c\}$. Then we have

$$\frac{k(k-2)}{2} - \frac{k(a+b+c)}{4} = 0,$$

$$\frac{k(k-2)(4k-7)}{12} - \frac{k(a^2+b^2+c^2)}{4} = 0,$$

$$\frac{k(k-2)^2(2k-3)}{8} - \frac{k(a^3+b^3+c^3)}{4} = 0.$$

However, a simple calculation shows that this system does not allow a solution in positive integers a, b, c for $k > 4$. Namely, from the first equation we get

$$k = \frac{a+b+c+4}{2}.$$

Substituting it into the second and third equations, and then taking resultant in a, we obtain that

$$(6b^2 - 6bc + 2c^2 - c)(2b^2 - 2bc + 2c^2 - b - c)(2b^2 - 6bc + 6c^2 - b) = 0.$$

One can easily check that this implies $(b, c) = (1, 1), (1, 2), (2, 1)$, whence we get that (a, b, c) is a permutation of $(1, 1, 2)$, and $k = 4$. Thus $\deg(H) = 4$ is also impossible. So we are left with the case $\deg(H) = 2$. Then k must be even, and we must have a partition of $\{0, 1, \ldots, k-2, z\}$ into pairs having equal sums. It is easy to check that it is possible only if $z = (k-2)/2$, when the partition is given by

$$\{0, k-2\}, \ldots, \{(k-2)/2, (k-2)/2\}.$$

In this case we have

$$F_{k,z}(x) = \prod_{i=0}^{(k-2)/2} \left(\left(x + \frac{k-2}{2} \right)^2 - i^2 \right).$$

As we showed, in any other decomposition of $F_{k,z}(x)$ with $z = (k-2)/2$ of the form $G_0(H_0(x))$, we must have $\deg(H_0) = 2$. Writing $H_0(x) = \alpha(x-\beta)^2 + \gamma$, this decomposition is equivalent to $F_{k,z}(x) = P((x-\beta)^2)$ with some polynomial $P(x) \in \mathbb{C}[x]$. As the roots of $F_{k,z}(x)$ are symmetric to β, we get $\beta = (2-k)/2$ and the theorem follows. $\square$

Proof of Theorem 2.3. Using the method of Proposition 4.1, we have $f_{k,m}(x) = wF_{k,z}(x)$ with $w = \frac{m-2}{k!}$ and $z = \frac{k}{m-2} - 1$. Thus observing that $z \neq -1, k-1$, the statement immediately follows form Proposition 4.1. $\square$

Proof of Theorem 2.1 part (ii). Assume that (3) has infinitely many solutions. Then $f_{k,m}(x) = \varphi \circ F \circ \lambda$, where φ is arbitrary, λ is a linear polynomial, and F belongs to one of the five standard pairs of polynomials.

In view of Theorem 2.3, we need to distinguish only three cases, namely $\deg \varphi = k, k/2$ and 1.

If $\deg \varphi = k$, the polynomials $\varphi(x)$ and $f_{k,m}(x)$ are equivalent (i.e. $f_{k,m}(x)$ is of the form $\varphi(ax+b)$). So in this case the statement immediately follows.

When $\deg \varphi = k/2$, the k is necessarily even. By Theorem 2.3 we get that $m = 4$, and $\varphi(x)$ is a linear transformation of $\frac{2}{k!}\prod_{i=0}^{(k-2)/2}(x-i)^2$. Hence g is of the form $g(x) = \varphi(T(x))$ with some $T(x) \in \mathbb{Q}[x]$. As $f_{k,m}(x) = \varphi((x + \frac{k-2}{2})^2)$, Lemma 4.1 and Lemma 3.2 show that the equation $f_{k,m}(x) = g(y)$ can have infinitely many solutions only if $T(x)$ has at most two roots of odd multiplicity. Hence the theorem follows also in this case.

Finally, let $\deg \varphi = 1$.

First, consider the case when F belongs to a standard pair of the first kind over $\mathbb{Q}$. If F is the first entry of this pair, then with some $a, b, A, B \in \mathbb{Q}$ we have

$$f_{k,m}(x) = A(ax+b)^q + B.$$

However, as by Corollary 3.1 all the roots of $f'_{k,m}(x)$ are simple, here q is at most 2. But as $k \geq 4$ this is not possible.

If F is the second entry of a standard pair of the first kind, then

$$f_{k,m}(x) = A(ax+b)^r(v(ax+b))^q + B.$$

Here $a, b, A, B \in \mathbb{Q}$, $0 \leq r < q$ with $\gcd(r,q) = 1$, $v(x) \in \mathbb{Q}[x]$ and $r + \deg v > 0$. The derivative of the polynomial on the right hand side is

$$Aa(ax+b)^{r-1}(v(ax+b))^{q-1}\left(rv(ax+b) + q(ax+b)v'(ax+b)\right).$$

As by Corollary 3.1 we know that $f'_{k,m}(x)$ has only simple roots, we obtain that $r \leq 2$, and either $\deg v = 0$, or $q \leq 2$. Since the case $\deg v = 0$ reduces to the previous one, we may assume that $\deg v > 0$. Also, the case $r = 2$ can be excluded as $0 \leq r < q \leq 2$. Thus either $r = 0, q = 1$ or $r = 1, q = 2$. If $r = 0, q = 1$ we get that $f_{k,m}(x)$ and $v(x)$ are equivalent, and $g(x)$ is linear. However, this possibility is excluded.

So we are left with the case $(r, q) = (1, 2)$. But, then $\deg g = 2$. This case is discussed in part (i) of the theorem and is excluded from part (ii).

Now let $F(x)$ belong to a standard pair of the second kind. In view of $k \geq 4$, $F(x)$ must be of the form

$$(\alpha x^2 + \beta)(v(x))^2.$$

But then $\deg g = 2$, which is discussed in part (i) of the statement, and is excluded from part (ii). Assume next that $F(x)$ belongs to a standard pair of the fifth kind. Observe that then

$$f_{k,m}(x) = AF(ax + b) + B$$

with $a, b, A, B \in \mathbb{Q}$ implies that $f'_{k,m}(x)$ has a double root. However, this contradicts Corollary 3.1.

Finally, consider the cases where $F(x)$ belongs to a standard pair of the third or fourth type. Observe that then $F(x)$ is a constant multiple of a Dickson-polynomial of degree k, and we can write

$$f_{k,m}(ax + b) = AD_k(x, \delta) + B \tag{12}$$

with some $a, b, A, B, \delta \in \mathbb{Q}$, where $aA\delta \neq 0$. As the coefficients of x^{k-1} and x^{k-3} of $D_k(x, \delta)$ are both zero if $k \geq 4$, we get

$$wa^{k-1}\left(z + kb + \frac{(k-2)(k-1)}{2}\right) = 0 \tag{13}$$

and

$$wa^{k-3}\left((b+z)\sum_{i_1=1}^{k-2}\left((b+i_1)\sum_{i_2=0}^{i_1-1}(b+i_2)\right) + \right. \tag{14}$$

$$\left. + \sum_{i_1=2}^{k-2}\left((b+i_1)\sum_{i_2=1}^{i_1-1}\left((b+i_2)\sum_{i_3=0}^{i_2-1}(b+i_3)\right)\right)\right) = 0,$$

respectively, where $w = \frac{m-2}{k!}$ and $z = \frac{k}{m-2} - 1$. After simplification, expressing z from (13) and substituting it to (14) we get

$$\frac{-1}{24}k(k-1)(k-2)(2b+k-1)(2b+k-2)(2b+k-3) = 0$$

which yields

$$b \in \left\{-\frac{1}{2}k + \frac{1}{2}, -\frac{1}{2}k + 1, -\frac{1}{2}k + \frac{3}{2}\right\}$$

and then

$$z = k - 1, \frac{1}{2}k - 1, -1,$$

respectively. The cases $z=-1$ and $z=k-1$ have been considered by Theorem 4.3. in [4] and are excluded from Theorem 2.2.

So we are left with the case $z=\frac{1}{2}k-1$ and $b=-\frac{1}{2}k+1$. Then $m=4$.

If k is even we have

$$f_{k,4}(ax+\frac{1}{2}k-1)=\frac{2}{k!}(ax)^2\prod_{i=1}^{\frac{1}{2}k-1}\left(ax-\frac{1}{2}k+i\right)\left(ax+\frac{1}{2}k-i\right).$$

Writing out the first few terms of the Dickson-polynomial we get

$$D_k(x,\delta)=x^k-\delta kx^{k-2}+\frac{1}{2}k(k-3)\delta^2x^{k-4}-\frac{1}{6}k(k-4)(k-5)\delta^3x^{k-6}+$$
$$+\frac{1}{24}k(k-5)(k-6)(k-7)\delta^4x^{k-8}-\dots$$

Now, we deal with some nonzero coefficients of equation (12). Comparing the coefficients of x^k, x^{k-2} and x^{k-4} in (12) for $k\geq 5$ we get the following three equations:

$$wa^k=A,$$
$$\frac{-1}{24}wa^{k-2}k(k-1)(k-2)=A(-\delta)k,$$
$$\frac{1}{5760}wa^{k-4}k(k-1)(k-2)(k-3)(k-4)(5k+2)=\frac{1}{2}Ak(k-3)\delta^2.$$

They give $k=-6,0,1,2,3$, which cannot hold.

If k is odd, we have

$$f_{k,4}(ax+\frac{1}{2}k-1)=\frac{2}{k!}(ax)\prod_{i=1}^{\frac{1}{2}k-\frac{1}{2}}\left(ax-\frac{1}{2}k+i\right)\left(ax+\frac{1}{2}k-i\right),$$

and the statement follows by a similar argument.

So we are left with the case $k=4$. We already know that now in equation (12) we have $k=m=4$, $b=-1$. Comparing the coefficients of x^4 and x^2 and the constant terms in (12) we obtain the following equations:

$$\frac{a^4}{12}=A,\qquad -\frac{a^2}{12}=-4A\delta,\qquad 0=2A\delta^2+B.$$

By a simple calculation these yield that

$$A=\frac{a^4}{12},\qquad B=-\frac{1}{96},\qquad \delta=\frac{1}{4a^2}. \tag{15}$$

If F belongs to a standard pair of the third kind, then (15) gives

$$\varphi(x)=\frac{1}{192u^{4t}}x-\frac{1}{96},\quad G(x)=D_t(x,u^8),\quad \mu(x)=cx+d,$$

where u, c, d are arbitrary rationals with $uc \neq 0$ and $t \geq 3$ is an odd integer. This implies

$$g(x) = \frac{1}{192u^{4t}} D_t(cx + d, u^8) - \frac{1}{96}.$$

Finally, suppose that F belongs to a standard pair of the fourth kind. Then (15) yields

$$\varphi(x) = \pm\frac{1}{192}x - \frac{1}{96}, \quad G(x) = \mp u^{-t} D_{2t}(x, u), \quad \mu(x) = cx + d,$$

where u, c, d are arbitrary rationals with $uc \neq 0$ and $t \geq 3$ is an odd integer. This gives

$$g(x) = -\frac{1}{192u^t} D_{2t}(cx + d, u) - \frac{1}{96}.$$

Hence the theorem follows. $\square$

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