General neighborhood sequences in $\mathbb{Z}^n$

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Abstract

Neighborhoods and neighborhood sequences play important roles in several branches of pattern analysis. In earlier papers in $\mathbb{Z}^n$ only certain special (e.g. periodic or octagonal) sequences were investigated. In this paper we study neighborhood sequences which are either ultimately periodic or allow at every neighborhood to do nothing at no cost. We give finite procedures and descriptive theoretical criteria for certain important (e.g. metrical) properties of the sequences. Our results are valid for several types of classical neighborhood sequences and for generated distance functions (e.g. octagonal and chamfer distances) which are widely applied in digital image processing. We conclude the paper by showing how our results contribute to the theory of distance transformations.

Key words: Combinatorial algorithms, Path and circuit problems, Geometric algorithms, languages and systems, Image processing and computer vision
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1 Introduction

In [30] Yamashita and Ibaraki introduced the concept of general periodic neighborhood sequences in $\mathbb{Z}^n$. They investigated when such sequences gen-

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erate metrics, and the relation of these metrics and the Euclidean one. Their main results are the exhibition of certain procedures, which decide about the metricity and related properties.

Das et al. [6] specialized this theory to the so-called octagonal sequences, based on the traditional neighboring relations of digital image processing. For various results in this direction we refer to [1,3,5,7–10,23,27], and the references given there. Recently, Fazekas et al. [12] dropped the periodicity requirement from the model of [6] by introducing general (not necessarily periodic) octagonal neighborhood sequences. This extension is important not only from a theoretical but also from a practical point of view, since e.g. the Euclidean metric can be approximated more precisely by general octagonal neighborhood sequences than by periodic ones, see [18]. Further results about general octagonal neighborhood sequences can be found e.g. in [16,17,24–26].

The purpose of this paper is twofold. On the one hand, we extend the theory of general (not necessarily octagonal) neighborhood sequences from the periodic case (investigated in [30]) to the case of neighborhood sequences which are defined over an arbitrary finite alphabet, and are either ultimately periodic or allow at every neighborhood to do nothing at no cost. The set of ultimately periodic neighborhood sequences has the advantage to be rather large, although such a sequence is determined by only finitely many data. We give procedures and provide theorems for certain important criteria (e.g. metricity). These theoretical results give a good insight into the behavior of such sequences. We have not computed the complexity of our procedures, it is left as an open issue (see Section 9).

The structure of the paper is as follows. In Section 2 we give the basic notation and definitions. Some preliminary results are presented in Section 3. In Section 4 we show how certain sequences can be simplified keeping their metrical properties. We characterize the neighborhood sequences for which $\mathbb{Z}^n$ is connected in Section 5. In Section 6 the metrical behaviour of neighborhood sequences is investigated. Section 7 contains our results on approximating the Euclidean metric with distance functions based on neighborhood sequences. In Section 8 we show how our model can be used e.g. in the theory of distance transformations. Finally, we discuss on current relating research results, and conclude in Section 9.

2 Basic concepts and notation

Let $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ denote the sets of real numbers, rational numbers, integers and positive integers, and write $\mathbb{R}_{\geq 0}$, $\mathbb{Q}_{\geq 0}$, $\mathbb{Z}_{\geq 0}$ for the subsets consisting of the non-negative elements of these sets, respectively. We fix a positive integer

n for the whole paper, and write $\mathbf{O}$ for the origin of $\mathbb{R}^n$.

Let $H = \{h_i \in \mathbb{Z}^n \mid i = 1, \ldots, m\}$ and $\mathbb{X} \subseteq \mathbb{R}$. Then the cone generated by H over $\mathbb{X}$ is defined as

$$H^{<}(\mathbb{X}) = \left\{ \sum_{i=1}^{m} \lambda_i h_i \,\middle|\, \lambda_i \in \mathbb{X},\ i = 1, \ldots, m \right\}.$$

A neighborhood is a pair (P, w), where the point set $P \subseteq \mathbb{Z}^n$ is finite, and $w : P \to \mathbb{R}_{\geq 0}$ is a so-called weight function on P. For $p \in P$, $w(p)$ is the weight of p with respect to w. The neighborhood (P, w) is called symmetric, if for any $p \in P$, $-p \in P$ and $w(p) = w(-p)$.

Let Λ be a finite set of neighborhoods. An n-dimensional (shortly nD) neighborhood sequence is defined as a sequence $N = (N_i)_{i=1}^{\infty}$ over Λ, that is, $N_i \in \Lambda$ for all $i \in \mathbb{N}$. We call Λ the alphabet used for N. Let S_n denote the set of all nD-neighborhood sequences. If for some $j \in \mathbb{N}$, $N_i = N_{i+j}$ for all $i \in \mathbb{N}$, then N is called periodic with period j. In this case we use the brief notation $N = \overline{N_1 N_2 \ldots N_j}$. If $j = 1$ then N is called a constant neighborhood sequence.

Let $N^{(k)}$ denote the neighborhood sequence obtained by omitting the first k elements of $N \in S_n$. The sequence $N = (N_i)_{i=1}^{\infty}$ is called ultimately periodic, if $N^{(k)}$ is periodic for some $k \in \mathbb{N}$. If $N^{(k)}$ has period length $l - k$, then we write $N = N_1 N_2 \ldots N_k \overline{N_{k+1} \ldots N_l}$. For technical reasons it is useful to avoid the degenerate case $k = 0$. Thus throughout the paper we assume that the periodic sequence $N = \overline{N_1 N_2 \ldots N_l}$ is given by $N = N_1 N_2 \ldots N_l \overline{N_1 N_2 \ldots N_l}$.

We use the following notation for some special subsets of the set of nD-neighborhood sequences S_n:

$S_n^p = \{N \in S_n \mid N \text{ is periodic}\}$,
$S_n^u = \{N \in S_n \mid N \text{ is ultimately periodic}\}$,
$S_n^{\mathbf{O}} = \{N = (N_i)_{i=1}^{\infty} \in S_n \mid N_i = (P_i, w_i),\ \mathbf{O} \in P_i,\ w_i(\mathbf{O}) = 0 \text{ for all } i \in \mathbb{N}\}$.

The set $S_n^{\mathbf{O}}$ will play a special and important role throughout the paper. Moreover, it is clear that $S_n^p \subsetneq S_n^u$.

We can measure distance by the help of neighborhood sequences in a natural way (see e.g. [6,12,30]). Let q and r be two points in $\mathbb{Z}^n$, and $N = (N_i)_{i=1}^{\infty} \in S_n$, with $N_i = (P_i, w_i)$. The point sequence $s = [q_0, q_1, \ldots, q_m]$, where $q = q_0$, $r = q_m$, and $q_i - q_{i-1} \in P_i$, is called an N-path between q and r. Define the relation $\sim$ on the set $A := \{(q_1 - q_0, 1), \ldots, (q_m - q_{m-1}, m)\}$ as $(q_i - q_{i-1}, i) \sim (q_j - q_{j-1}, j)$ if and only if $q_i - q_{i-1} = q_j - q_{j-1}$ and $w_i(q_i - q_{i-1}) = w_j(q_j - q_{j-1})$. Obviously, $\sim$ is an equivalence relation on A. Consider the partition $A = \bigcup_{i=1}^{t} A_i$ induced by $\sim$ on A with the appropriate $t \in \mathbb{N}$. Then, as a shorthand, we write

$s = r - q = \sum_{i=1}^{t} \lambda_i x_i$, where x_i is the common first entry of the pairs in A_i, and $\lambda_i = |A_i|$ denotes the cardinality of A_i ($i = 1, \dots, t$). The length of s is defined as $\ell(s; N) = \sum_{i=1}^{m} w_i(q_i - q_{i-1}) = \sum_{i=1}^{t} \lambda_i w^{(i)}(x_i)$, where $w^{(i)}$ denotes the common weight of the first entries of the pairs in A_i. If N is fixed, then we shortly write $\ell(s; N) = \ell(s)$.

The N-distance $W(q, r; N)$ between q and r is defined as the length of a shortest N-path between them, if such a path exists. Put $W(q, q; N) = 0$ for $q \in \mathbb{Z}^n$ (empty path). If there is no path between q and r, we set $W(q, r; N) = \infty$. We write $W(N)$ for the distance function itself defined by N on $\mathbb{Z}^n$, and also use the brief notation $W(q, r)$, if N is fixed. Moreover, we put $W(q; N) = W(q) = W(\mathbf{O}, q)$. If for all $q, r, s \in \mathbb{Z}^n$ we have

$$\begin{array}{ll} W(q,r;N) < \infty & (W \text{ is finite}) \\ W(q,r;N) \geq 0,\ W(q,r;N) = 0 \text{ iff } q = r & (W \text{ is positive definite}) \\ W(q,r;N) = W(r,q;N) & (W \text{ is symmetric}) \\ W(q,r;N) + W(r,s;N) \geq W(q,s;N) & (W \text{ satisfies triangle inequality}) \end{array}$$

then we call $W(N)$ a metric, and use the notation $d(q, r; N) = W(q, r; N)$, or shortly $d(q, r) = W(q, r)$ and $d(N) = W(N)$. Moreover, we write $d(q; N) = d(q)$ for $d(\mathbf{O}, q)$. For $x \in \mathbb{R}^n$, let $||x||_1 := \sum_{i=1}^{n} |x_i|$ and $||x||_2 := \sqrt{\sum_{i=1}^{n} x_i^2}$ denote the diamond norm and Euclidean norm, respectively.

We say that $\mathbb{Z}^n$ is N-connected, if for any two points of $\mathbb{Z}^n$ there exists an N-path between them. If N is fixed, we will shortly say that $\mathbb{Z}^n$ is connected. Note that $\mathbb{Z}^n$ is N-connected if and only if $W(q)$ is finite for all $q \in \mathbb{Z}^n$. A set $T \subseteq \mathbb{Z}^n$ is said to allow a finite covering of $\mathbb{Z}^n$, if $\mathbb{Z}^n$ can be covered by the union of finitely many translates of T. By a lattice we mean a subgroup of $\mathbb{Z}^n$. A lattice is called full if it has rank n.

We introduce a partial ordering relation on S_n, which will be important in our investigations. We note that for certain special neighborhood sequences such a relation was used by Das et al. [6], Fazekas [11] and by Fazekas et al. [12]. Let $N, N' \in S_n$. We define the relation $\sqsupseteq^*$ on S_n by

$$N \sqsupseteq^* N' \text{ if and only if } W(q, r; N) \leq W(q, r; N') \text{ for all } q, r \in \mathbb{Z}^n,$$

and we can also say that N is faster than N'.

3 Preliminary results

From the following proposition we can see that the neighborhood model is suitable for measuring distances, since it assures the existence of a shortest path, if $\mathbb{Z}^n$ is connected.

Proposition 1 *Let $N = (N_i)_{i=1}^{\infty} \in S_n$ with $N_i = (P_i, w_i)$. Then for all $q, r \in \mathbb{Z}^n$, $W(q, r; N)$ exists.*

PROOF. If there is no path between q and r then by definition $W(q, r; N) = \infty$. Otherwise, let s be an arbitrary but fixed path from q to r of length $\ell(s)$, having the short form $s = \sum_{i=1}^{t} \lambda_i x_i$. Suppose that $s' = \sum_{j=1}^{t'} \lambda'_j x'_j$ is a path from q to r of length $\ell(s')$ such that $\ell(s') < \ell(s)$. Then for any $j \in \{1, \ldots, t'\}$, for the coefficients λ'_j of those x'_j in s' for which $w^{(j)}(x'_j) > 0$, we have $\lambda'_j \leq \ell(s)/w^{(j)}(x'_j)$. Thus, as there are only finitely many neighborhoods, and every neighborhood contains only finitely many points, there are only finitely many possibilities for the lengths of such paths s' from q to r. Hence the statement follows. $\square$

The following examples explain why the restrictions $|P_i| < \infty$ for all $i \in \mathbb{N}$, and $|\Lambda| < \infty$ are necessary to have Proposition 1. In our first example we show why we avoid infinite neighborhoods.

Example 2 *Let $N = \overline{N_1} \in S_1^p$ be a constant neighborhood sequence, with $N_1 = (\mathbb{Z}, w_1)$. For every $i \in \mathbb{Z}$, let $w_1(i) = |1/i|$ if $i \neq 0$, and let $w_1(0) = 0$. In this case $W(0, 1; N)$ does not exist, since there is no shortest path between 0 and 1. For example, the length of the path $0, -i, 1$ is $\frac{1}{i} + \frac{1}{i+1}$ for every $i \in \mathbb{N}$.*

Our next example shows why it would be inappropriate to define neighborhood sequences over an infinite alphabet.

Example 3 *Let $N = (N_i)_{i=1}^{\infty} \in S_1$, with $N_i = (P_i, w_i)$, where $P_i = \{\pm i, 0\}$, $w_i(\pm i) = |1/i|$ and $w_i(0) = 0$ for every $i \in \mathbb{N}$. Now the alphabet of neighborhoods Λ is not finite, and similarly to the previous example, $W(0, 1; N)$ does not exist, because there is no shortest path between 0 and 1.*

4 Equivalent neighborhood sequences

In this section we investigate under what circumstances the structure of a neighborhood sequence can be simplified without affecting its distance measurement.

Remark 4 *Note that our model allows positive weight for keeping place during the movement. Using sequences from $S_n^{\mathbf{O}}$ makes it possible to keep place and avoid involuntary movements to undesired places.*

In particular, we have the following statement:

Proposition 5 *For any $N \in S_n^{\mathbf{O}}$ there exists an $M \in S_n^u$ of the form $M = M_1 \dots M_k \overline{M_{k+1}}$, such that the functions $W(N)$ and $W(M)$ are identical on $\mathbb{Z}^n$.*

PROOF. The neighborhood $M_{k+1} = (P'_{k+1}, w'_{k+1})$ can be defined as follows. Let $I = \{i \mid N_i = (P_i, w_i) \text{ occurs infinitely often in } N\}$. Put $P'_{k+1} = \bigcup_{i \in I} P_i$, $w'_{k+1}(p) = \min_{i \in I}\{w_i(p) \mid p \in P_i\}$ for each $p \in P'_{k+1}$. Let the sequence of neighborhoods $M_1, \dots, M_k$ be the subsequence of N consisting of the elements of N which occur only finitely many times and write $M = M_1 \dots M_k \overline{M_{k+1}}$. Clearly, by these choices we have $W(N) = W(M)$. $\square$

Yamashita and Ibaraki [30] showed that if $N \in S_n^p$ and $W(N)$ is a metric, then there exists a constant neighborhood sequence $M \in S_n^p$ such that $d(N)$ is identical with $d(M)$. The following example shows that this result cannot be extended in this form to the ultimately periodic case.

Example 6 *Let $n = 1$, $N_1 = (P_1, w_1)$, $N_2 = (P_2, w_2)$, with $P_1 = \{0\}$, $P_2 = \{\pm 1\}$, $w_1(0) = 1$, $w_2(\pm 1) = 1$, and consider $N = N_1 \overline{N_2}$. It is obvious, that $W(N)$ is a metric, since $W(0; N) = 0$ (as always, by definition) and $W(x; N) = |x| + 1$ for any $x \in \mathbb{Z} \setminus \{0\}$. However, it can be easily seen that N cannot be replaced by a constant neighborhood sequence.*

It turns out that a variant of the above result is still valid for ultimately periodic sequences.

Proposition 7 *Let $N = N_1 \dots N_k \overline{N_{k+1} \dots N_l} \in S_n^u$ with $N_i = (P_i, w_i)$ for $i \in \{1, \dots, l\}$. Then there exists an $M \in S_n^u$ of the form $M = M_1 \dots M_k \overline{M_{k+1}}$ such that $W(N)$ and $W(M)$ are identical on $\mathbb{Z}^n$.*

PROOF. Let

$$P = \bigcup_{t=k}^{l-1} \left\{ \sum_{i=k}^{t} b_i \;\middle|\; b_i \in P_i,\ i = k, \dots, t \right\}, \text{ and}$$

$$P' = \left\{ \sum_{i=k+1}^{l} b_i \,\middle|\, b_i \in P_i,\ i = k+1, \dots, l \right\}.$$

Consider the neighborhood $T = (P, w_P)$, where for every $x \in P$

$$w_P(x) = \min \left\{ \sum_{i=k}^{t} w_i(b_i) \,\middle|\, x = \sum_{i=k}^{t} b_i,\ t = k, \dots, l-1,\ b_i \in P_i,\ i = k, \dots t \right\},$$

and $T' = (P', w_{P'})$, where for every $x \in P'$

$$w_{P'}(x) = \min \left\{ \sum_{i=k+1}^{l} w_i(b_i) \,\middle|\, x = \sum_{i=k+1}^{l} b_i,\ b_i \in P_i,\ i = k+1, \dots, l \right\}.$$

The neighborhood sequence $M = N_1 \dots N_{k-1} T \overline{T'}$ obviously has the desired property, and the proof of the proposition is complete. $\square$

From Example 6 we see that it is not true that for every neighborhood sequence we can find a constant one such that they induce the same metric. Now we show that, on the contrary, the elements of $S_n^{\mathbf{O}}$ have this property.

Theorem 8 *Suppose that $N \in S_n^{\mathbf{O}}$ induces a metric on $\mathbb{Z}^n$. Then there is a constant neighborhood sequence which induces the same metric on $\mathbb{Z}^n$.*

PROOF. By Proposition 5 we may assume that $N = N_1 \dots N_k \overline{N_{k+1}}$ with $N_i = (P_i, w_i)$ for $i \in \{1, \dots, k+1\}$. Let d be the metric induced by N on $\mathbb{Z}^n$, and let $P = \bigcup_{i=1}^{k+1} P_i$. Moreover, for every $x \in P$ set

$$w_P(x) = \min \{ w_i(x) \mid x \in P_i,\ i = 1, \dots, k+1 \},$$

and put $T = (P, w_P)$ and $M = \overline{T}$. We claim that $d = W(M)$.

By the definition of M it is clear that $\mathbb{Z}^n$ is M-connected, and that for every $x \in \mathbb{Z}^n$ we have $d(x) \geq W(x; M)$. Take an arbitrary $x \in \mathbb{Z}^n$, and choose a shortest M-path $[\mathbf{O} = q_0, q_1, \dots, q_t = x]$ from $\mathbf{O}$ to x. Then using that $w_P(y) \geq d(y)$ for every $y \in P$ and that d satisfies the triangle inequality, we have

$$W(x; M) = \sum_{i=1}^{t} w_P(q_i - q_{i-1}) \geq \sum_{i=1}^{t} d(q_i - q_{i-1}) \geq d(x).$$

Hence $W(M) = d$ on $\mathbb{Z}^n$. $\square$

It is clear that if in Theorem 8 the neighborhoods in N are given, then the constant neighborhood sequence can be constructed by a simple procedure.

Proposition 9 *Suppose that $N = (N_i)_{i=1}^{\infty} \in S_n^{\mathbf{O}}$ induces a metric d on $\mathbb{Z}^n$, and $w_i(x) = 1$ for each $x \in P_i \setminus \{\mathbf{O}\}$, for all $i \in \mathbb{N}$. Then d is completely determined by the set $\{x \mid d(x) = 1\}$.*

PROOF. Clearly, $d(x) = 0$ if and only if $x = \mathbf{O}$, and $d(x) = 1$ if and only if $x \in P := \bigcup_{i=1}^{\infty} P_i \setminus \{\mathbf{O}\}$. Since $w_P \equiv 1$ and d is completely determined by (P, w_P), the statement follows. □

5 Connectedness of $\mathbb{Z}^n$

In this section we characterize the ultimately periodic neighborhood sequences for which $\mathbb{Z}^n$ is connected. For this purpose we need the following lemma.

Lemma 10 *Let $H = \{h_i \in \mathbb{Z}^n \mid i = 1, \ldots, m\}$. Then $H^<(\mathbb{Z}_{\geq 0})$ allows a finite covering of $\mathbb{Z}^n$ if and only if it is a full lattice.*

PROOF. Clearly, if $H^<(\mathbb{Z}_{\geq 0})$ is a full lattice, then it allows a finite covering of $\mathbb{Z}^n$. To prove the other direction, assume that $H^<(\mathbb{Z}_{\geq 0})$ allows a finite covering of $\mathbb{Z}^n$. We note that it is well-known that the set of integral points in the cone $H^<(\mathbb{Q}_{\geq 0})$ have a (so-called Hilbert's) basis; see e.g. [15]. As $H^<(\mathbb{Q}_{\geq 0})$ is a rational cone, either it is contained in a (rational) halfspace of $\mathbb{Q}^n$, or $H^<(\mathbb{Q}_{\geq 0}) = \mathbb{Q}^n$ holds. Since $H^<(\mathbb{Z}_{\geq 0})$ allows a finite covering of $\mathbb{Z}^n$, the former case can be excluded. In the latter case, let $l \in \{1, \ldots, m\}$ be arbitrary. Then there are non-negative integers r_i, s_i with $s_i \neq 0$ $(i = 1, \ldots, m)$, such that $-h_l = \sum_{i=1}^{m} \frac{r_i}{s_i} h_i$. Hence

$$-h_l = \left((r_l + s_l) \prod_{\substack{j=1 \\ j \neq l}}^{m} s_j - 1 \right) h_l + \sum_{\substack{i=1 \\ i \neq l}}^{m} \left(r_i \prod_{\substack{j=1 \\ j \neq i}}^{m} s_j \right) h_i,$$

implying $-h_l \in H^<(\mathbb{Z}_{\geq 0})$. Hence $H^<(\mathbb{Z}_{\geq 0})$ is a lattice. The fact that this lattice is full follows from $H^<(\mathbb{Q}_{\geq 0}) = \mathbb{Q}^n$, and the proof is complete. □

Theorem 11 *Let $N = N_1 \ldots N_k \overline{N_{k+1} \ldots N_l} \in S_n^u$ with $N_i = (P_i, w_i)$ for $i \in \{1, \ldots, l\}$, and put*

$$Q_f = \bigcup_{t=k}^{l-1} \left\{ \sum_{i=1}^{t} b_i \;\middle|\; b_i \in P_i,\ i = 1, \ldots, t \right\},$$

$$Q_\infty = \left\{ \sum_{i=k+1}^{l} b_i \;\middle|\; b_i \in P_i,\ i = k+1, \ldots, l \right\}.$$

Then $\mathbb{Z}^n$ is N-connected if and only if $Q_\infty^{<}(\mathbb{Z}_{\geq 0})$ is a full lattice in $\mathbb{R}^n$ and Q_f represents all the cosets of the lattice $Q_\infty^{<}(\mathbb{Z}_{\geq 0})$ in $\mathbb{Z}^n$.

Moreover, the connectedness of $\mathbb{Z}^n$ can be checked by a finite procedure.

PROOF. The sufficiency of the condition is clear. To prove the necessity, assume that $\mathbb{Z}^n$ is N-connected. Observe that then $Q_\infty^{<}(\mathbb{Z}_{\geq 0})$ allows a finite covering of $\mathbb{Z}^n$. By Lemma 10 this is equivalent to saying that $Q_\infty^{<}(\mathbb{Z}_{\geq 0})$ is a full lattice. Moreover, for the connectedness of $\mathbb{Z}^n$, we also need that all the cosets in $\mathbb{Z}^n$ of the lattice $Q_\infty^{<}(\mathbb{Z}_{\geq 0})$ are represented by Q_f.

Clearly, it is a finite procedure to check whether $Q_\infty^{<}(\mathbb{Z}_{\geq 0})$ is a full lattice or not. Actually, it is sufficient to check whether Q_∞ contains n linearly independent vectors over $\mathbb{Q}$, and that $-h \in Q_\infty^{<}(\mathbb{Z}_{\geq 0})$ for every $h \in Q_\infty$. The former problem is easy. The latter one leads to an integer programming problem of the form

$$Ax = b,\ x \geq 0 \qquad \text{in } x \in \mathbb{Z}^m, \tag{1}$$

where $b = -h \in \mathbb{Z}^n$, $m = |Q_\infty|$, and the column vectors of the $n \times m$ type matrix A are just the elements of Q_∞. The algorithmic solution of (1) is well-known, even if an objective function $\sum_{i=1}^{m} c_i x_i$, $x = (x_1, \ldots, x_m)$, $c_i \in \mathbb{R}$ $(i = 1, \ldots, m)$ should also be maximized (see e.g. [13]).

If $Q_\infty^{<}(\mathbb{Z}_{\geq 0})$ is a full lattice, then we need only a finite amount of computation to verify the second part of the condition. It suffices to enumerate Q_f and to check whether all the cosets in $\mathbb{Z}^n$ of the lattice $Q_\infty^{<}(\mathbb{Z}_{\geq 0})$ are represented or not. Thus we have a finite procedure to check the connectedness of $\mathbb{Z}^n$, and the theorem follows. □

Corollary 12 *If N is a constant neighborhood sequence then $\mathbb{Z}^n$ is N-connected if and only if $Q_\infty^{<}(\mathbb{Z}_{\geq 0}) = \mathbb{Z}^n$.*

Corollary 13 *Let $N \in S_n^{\mathbf{O}}$. Let $M = M_1 \ldots M_k \overline{M_{k+1}} \in S_n^u$ be the neighborhood sequence defined in the proof of Proposition 5. Put $Q'_f = \left\{ \sum_{i=1}^{k} b_i \;\middle|\; b_i \in P_i,\ i = 1, \ldots, k \right\}$. Then $\mathbb{Z}^n$ is N-connected if and only if $M_{k+1}^{<}(\mathbb{Z}_{\geq 0})$ is a full lattice in $\mathbb{R}^n$ and Q'_f represents all the cosets of $M_{k+1}^{<}(\mathbb{Z}_{\geq 0})$ in $\mathbb{Z}^n$.*

In Fig. 1 we consider the 2D-neighborhood sequence $N = N_1 N_2 \overline{N_3 N_4} \in S_2^u$, with $N_i = (P_i, w_i)$, where $P_1 = \{(-1, 0)\}$, $P_2 = \{(0, 2),\ (3, 2)\}$, $P_3 = \{(0, 1)\}$, $P_4 = \{(-2, -1),\ (1, 1),\ (2, -1),\ (-1, -3)\}$ and the weights are arbitrary.

From the figure we can see that $\mathbb{Z}^2$ is N-connected, since the vectors in $Q_f = \{(-1,2),\ (2,2),\ (-1,3),\ (2,3)\}$ represent all the cosets of the full lattice $Q_\infty^<(\mathbb{Z}_{\geq 0}) = z_1 \cdot (2,0) + z_2 \cdot (1,2)$, where $z_1, z_2 \in \mathbb{Z}$.

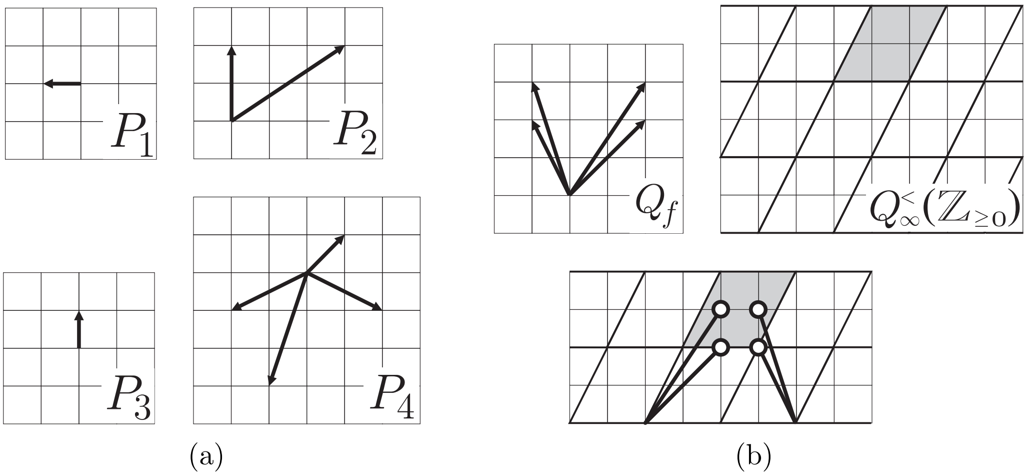

Fig. 1. Connectedness of $\mathbb{Z}^2$ for $N = N_1N_2\overline{N_3N_4} \in S_n^u$; (a) neighborhood vectors, (b) Q_f, the lattice $Q_\infty^<(\mathbb{Z}_{\geq 0})$, and the way how Q_f represents the cosets of $Q_\infty^<(\mathbb{Z}_{\geq 0})$.

6 Metrical properties of neighborhood sequences

In this section we study the metrical properties of distance functions defined by neighborhood sequences. We present the results about the triangle inequality, symmetry and metricity in separate subsections. We start with a statement which is interesting in itself, and plays a very important role later on.

Theorem 14 *Let $N = N_1 \ldots N_k\overline{N_{k+1} \ldots N_l} \in S_n^u$. Then for any points $q, r \in \mathbb{Z}^n$, $W(q, r; N)$ can be calculated by a finite procedure.*

PROOF. First we show that through finite steps we can find a path from q to r when such a path exists. Use the notation of Theorem 11, and observe that there is an N-path from q to r if and only if either

$$r - q \in \bigcup_{t=1}^{k-1} \left\{ \sum_{i=1}^{t} b_i \,\middle|\, b_i \in P_i,\ i = 1, \ldots, t \right\},$$

or

$$r - q = h + h' \ \text{ for some } \ h \in Q_f,\ h' \in Q_\infty^<(\mathbb{Z}_{\geq 0}).$$

Clearly, it is easy to check whether the first case holds or not, and if it holds, we get a path from q to r. As there are only finitely many such paths, we can choose a shortest among them, s_0 say. In the second case we determine whether the system

$$r - q - h = Ax, \;\; x \geq 0 \tag{2}$$

is solvable or not in $x \in \mathbb{Z}^m$ for some $h \in Q_f$, where $m = |Q_\infty|$ and the column vectors of the $n \times m$ type matrix A are the elements of Q_∞. As Q_f is finite, (2) leads to the solution of finitely many integer programming problems, without objective functions – just as in the proof of Theorem 11. Using an algorithm presented e.g. in [13], we can hence decide whether there is an N-path from q to r or not, and in the former case, we can actually determine such a path.

So suppose that we have an N-path s_1 from q to r of length l_1. For every $h \in Q_\infty$ with $\ell(h) > 0$, let $\lambda_h \in \mathbb{Z}_{\geq 0}$ be the largest integer, for which $\lambda_h \ell(h) < l_1$. Moreover, let

$$T'_f = \bigcup_{x \in Q_f} \left\{ x + \sum_{\substack{h_i \in Q_\infty \\ \ell(h_i) > 0}} \lambda_i h_i \;\middle|\; \lambda_i \leq \lambda_{h_i}, \lambda_i \in \mathbb{Z}_{\geq 0} \right\},$$

$$T_\infty = \{h \in Q_\infty \mid \ell(h) = 0\}\,.$$

Let T_f be the set of elements of T'_f whose lengths are smaller than l_1. Note that both T_f and T_∞ are finite sets. Observe that if an N-path s from q to r of length $< l_1$ contains at least k steps, then it can be decomposed as $s = y + y'$, $y \in T_f$, $y' \in T_\infty^{\leq}(\mathbb{Z}_{\geq 0})$. Thus for every $y \in T_f$, do the following. Solve the integer programming problem

$$r - q - y = Ax, \;\; x \geq 0, \tag{3}$$

which is just (2) with h replaced by y. If none of these problems is solvable, then s_0 or s_1 is a shortest path between q and r. Otherwise, let y_2 be a value of $y \in T_f$ for which (3) is solvable and $\ell(y)$ is minimal, and let s_2 be the corresponding path. Then either s_0 or s_2 is a shortest N-path between q and r. $\square$

Note that, by Proposition 5, Theorem 14 also holds for $N \in S_n^{\mathbf{O}}$.

6.1 The triangle inequality

It is well-known that not every neighborhood sequence generates a metric; see e.g. [30] for periodic neighborhood sequences, and [6] and [12] for periodic octagonal and for general octagonal ones, respectively. Using the relation $\sqsupseteq^*$ introduced in Section 2, we give a criterion for $W(N)$ to satisfy the triangle inequality on $\mathbb{Z}^n$. We define a special subset of S_n, namely

$S_n^* = \{N \in S_n \mid$ for all $j \in \mathbb{N},\ q \in \mathbb{Z}^n$ there exist a $y \in \mathbb{Z}^n$ and a shortest N-path $[\mathbf{O}, x_1, \ldots, x_j, x_{j+1}, \ldots, x_t = y]$ from $\mathbf{O}$ to y such that $q = x_t - x_j\}$.

Though at first sight the above defining property may seem to be strongly restrictive, S_n^* is a large set. It covers the classical octagonal neighborhood sequences investigated by several authors, e.g. in [6,12,25,30].

Theorem 15 *For any $N \in S_n$*

$$N^{(j)} \sqsupseteq^* N \text{ for every } j \in \mathbb{N} \tag{4}$$

implies the triangle inequality for $W(N)$. On the other hand, if $N \in S_n^*$ and $W(N)$ meets the triangle inequality then property (4) holds.

PROOF. We start with the first part of the statement. Let p, q and r be arbitrary points in $\mathbb{Z}^n$. Suppose that s_1 and s_2 are shortest N-paths between p and q, and between q and r, respectively. Let j denote the number of steps in the path s_1. By our assumption there exists an $N^{(j)}$-path s_3 from q to r, such that $\ell(s_3; N^{(j)}) \leq \ell(s_2; N)$. However, then for the concatenation s_0 of s_1 and s_3, we have

$$\begin{aligned} W(p, r; N) \leq \ell(s_0; N) = \ell(s_1; N) + \ell(s_3; N^{(j)}) \leq \\ \leq \ell(s_1; N) + \ell(s_2; N) = W(p, q; N) + W(q, r; N) \end{aligned}$$

and the proof of the first part is complete.

Now turn to the second part of the statement. Let $N \in S_n^*$ and let $j \in \mathbb{N}$ and $q \in \mathbb{Z}^n$ be arbitrary but fixed. By the defining property of S_n^*, there exist a $y \in \mathbb{Z}^n$ and a shortest N-path $s = [\mathbf{O}, x_1, \ldots, x_j, x_{j+1}, \ldots, x_t = y]$ from $\mathbf{O}$ to y such that $t \geq j$ and $q = x_t - x_j$. Now from the triangle property of $W(N)$, we obtain that

$$W(x_j + q; N) \leq W(x_j; N) + W(q; N). \tag{5}$$

On the other hand, as s is a minimal path,

$$W(x_j + q; N) = \sum_{i=1}^{j} w_i(x_i - x_{i-1}) + \sum_{i=j+1}^{t} w_i(x_i - x_{i-1}), \tag{6}$$

where $x_0 = \mathbf{O}$. Observe that

$$\sum_{i=1}^{j} w_i(x_i - x_{i-1}) \geq W(x_j; N), \text{ and } \sum_{i=j+1}^{t} w_i(x_i - x_{i-1}) \geq W(q; N^{(j)}). \tag{7}$$

Combining (5), (6) and (7), we obtain $W(q; N) \geq W(q; N^{(j)})$. Thus $N^{(j)} \sqsupseteq^* N$, which completes the proof. $\square$

In general, $W(N)$ may fail to satisfy the triangle inequality, even for $N \in S_n^{\mathbf{O}}$.

Example 16 *Let $n = 1$, $N_1 = (P_1, w_1)$, $N_2 = (P_2, w_2)$, with $P_1 = \{0, \pm 1\}$, $P_2 = \{0, \pm 2\}$, $w_1(0) = 0$, $w_1(\pm 1) = 1$, $w_2(0) = 0$, and $w_2(\pm 2) = 3$. Consider the neighborhood sequence $N = N_1 N_1 \overline{N_2}$. Then $W(0, 1; N) = 1$, $W(1, 3; N) = 2$ and $W(0, 3; N) = 4$. Thus $W(N)$ does not satisfy the triangle inequality.*

Theorem 17 *Let $N = N_1 \dots N_k \overline{N_{k+1} \dots N_l} \in S_n^u$ with $N_i = (P_i, w_i)$ for $i \in \{1, \dots, l\}$. Then there is a finite procedure to decide whether $W(N)$ satisfies the triangle inequality on $\mathbb{Z}^n$ or not.*

PROOF. Observe that W satisfies the triangle inequality if and only if the following condition holds:
for every $a_1, \dots, a_{t_1}$ and $a'_1, \dots, a'_{t_2}$ there exist $a''_1, \dots, a''_{t_3}$ ($a_i, a'_i, a''_i \in P_i$), such that

$$x := \sum_{i=1}^{t_1} a_i + \sum_{i=1}^{t_2} a'_i = \sum_{i=1}^{t_3} a''_i \tag{8}$$

and

$$\sum_{i=1}^{t_1} w_i(a_i) + \sum_{i=1}^{t_2} w_i(a'_i) \geq \sum_{i=1}^{t_3} w_i(a''_i). \tag{9}$$

Note that the number of nonzero terms on the right-hand side of (9) is bounded by the left-hand side divided by $\min\{w_i(x) \mid x \in P_i,\ w_i(x) > 0,\ i = 1, \dots, l\}$.

In the finitely many cases when $\max\{t_1, t_2\} < l$, we can use Theorem 14 to check criterion (8)–(9). If it fails for some x then W is not a metric and we are done. So in the remaining part of the proof we assume that $\max\{t_1, t_2\} \geq l$.

Fix an x (i.e. the sum $\sum_{i=1}^{t_1} a_i + \sum_{i=1}^{t_2} a'_i$) in (8), and define the (finite) set

$$E = \bigcup_{t=1}^{k-1} \left\{ \sum_{i=1}^{t} a_i + \sum_{i=k+1}^{l} a_i \,\middle|\, a_i \in P_i,\ i = 1, \dots, t, k+1, \dots, l \right\}.$$

Suppose first that $x \in E$. By Theorem 14 we can calculate $W(x)$. Clearly, we may fix $a''_1, \dots, a''_{t_3}$ as the vectors yielding a shortest path to x from $\mathbf{O}$.

Let Q_f and Q_∞ be the sets defined in Theorem 11. As in the proof of Theorem 14, for every $h \in Q_\infty$ with $\ell(h) > 0$, let $\lambda_h \in \mathbb{Z}_{\geq 0}$ be the largest integer for which $\lambda_h \ell(h) < W(x)$. Moreover, let T'_f and T_∞ be the sets defined in the proof of Theorem 14. Write $T_\infty = \{h_1, \dots, h_m\}$ and let T_f denote the subset of T'_f consisting of those elements whose lengths are smaller than $W(x)$.

Assume that for some $a_1, \dots, a_{t_1}$, $a'_1, \dots, a'_{t_2}$ (8) holds but (9) is not valid with the above choice of $a''_1, \ldots, a''_{t_3}$. If $\min\{t_1, t_2\} \geq k$ then we have $\sum_{i=1}^{t_1} a_i = a + \sum_{i=1}^{m} m_i h_i$ and $\sum_{i=1}^{t_2} a'_i = a' + \sum_{i=1}^{m} m'_i h_i$, with $a, a' \in T_f$, $\ell(a) + \ell(a') < W(x)$ and $m_i, m'_i \in \mathbb{Z}_{\geq 0}$. Thus

$$x = (a + a') + \sum_{i=1}^{m} (m_i + m'_i) h_i. \tag{10}$$

In case of $\min\{t_1, t_2\} < k$ we may assume that $t_2 < k < l \leq t_1$. Then we have $\sum_{i=1}^{t_1} a_i = a + \sum_{i=1}^{m} m_i h_i$ and $\sum_{i=1}^{t_2} a'_i = z$ with $a \in T_f$, $m_i \in \mathbb{Z}_{\geq 0}$ $(i = 1, \ldots, m)$, $z \in \bigcup_{t=1}^{k-1} \left\{ \sum_{i=1}^{t} a_i \,\middle|\, a_i \in P_i,\ i = 1, \ldots, t \right\}$ such that $\ell(a) + \ell(z) < W(x)$. Hence

$$x = z + a + \sum_{i=1}^{m} m_i h_i. \tag{11}$$

So we are led again to integer programming problems (in $m_i + m'_i$ or m_i, respectively) which are similar to those in the proof of Theorem 14. Solving (10) and (11) for each $a \in T_f$ and $a' \in T_f$ we can check criterion (8)–(9) in case $x \in E$, and we are done if it fails.

Assume now that $x \notin E$, and fix any $a_1, \ldots, a_{t_1}, a'_1, \ldots, a'_{t_2}$ in (8). Suppose that there do not exist $a''_1, \ldots, a''_{t_3}$ such that criterion (8)–(9) is satisfied. Without loss of generality we may assume that $t_1 + t_2$ is minimal with this property and $t_1 \geq t_2$ (whence $t_1 \geq l$). Consider the system $a_1, \ldots, a_{t_1-l+k}$, $a'_1, \ldots, a'_{t_2}$. By the minimality of $t_1 + t_2$, we can find a t_3^* and vectors $a''_i \in P_i$ $(i = 1, \ldots, t_3^*)$ such that $\sum_{i=1}^{t_1-l+k} a_i + \sum_{i=1}^{t_2} a'_i = \sum_{i=1}^{t_3^*} a''_i$ and $\sum_{i=1}^{t_1-l+k} w_i(a_i) + \sum_{i=1}^{t_2} w_i(a'_i) \geq \sum_{i=1}^{t_3^*} w_i(a''_i)$. Note that by $x = \sum_{i=1}^{t_3^*} a''_i + \sum_{i=t_1-l+k+1}^{t_1} a_i$ and $x \notin E$ we have $t_3^* \geq k$. Put $t_3 = t_3^* + l - k$ and define $a''_{t_3^*+i+1} = a_{t_1-j}$ for $i, j = 0, 1, \ldots, l-k-1$ with $t_3^* + i + 1 \equiv t_1 - j \pmod{l-k}$. It follows that criterion (8)–(9) is satisfied for the original system $a_1, \ldots, a_{t_1}$, $a'_1, \ldots, a'_{t_2}$. This contradiction shows that the triangularity of W can be decided by a finite procedure. $\square$

6.2 Symmetry

We start with some statements that will be useful to formalize the main result about symmetry.

Lemma 18 *Suppose that $b_1, \ldots, b_t$ are distinct nonzero vectors in $\mathbb{R}^n$ and*

$m_1, \ldots, m_t$ are positive integers such that

$$m_1 b_1 + \cdots + m_t b_t = \mathbf{O}. \tag{12}$$

Put $M = m_1 + \cdots + m_t$. Then there exists a sequence $i_1, \ldots, i_M$ with $i_l \in \{1, \ldots, t\}$ for $l = 1, \ldots, M$ such that each i occurs exactly m_i times in the sequence ($i \in \{1, \ldots, t\}$), and $\sum_{l=1}^{h} b_{i_l} \in T$ for every $h = 1, \ldots, M$, where

$$T = \left\{ \sum_{l=1}^{t} \lambda_l b_l \,\middle|\, -1 \le \lambda_l \le 0 \text{ for } l = 1, \ldots, t \right\}.$$

In particular, $\left| \sum_{l=1}^{h} b_{i_l} \right| \le \sum_{l=1}^{t} |b_l|$ ($h = 1, \ldots, M$).

PROOF. We define the numbers i_h inductively. Let i_1 be an index l ($1 \le l \le t$) for which m_l is maximal. Then by (12) we have $b_{i_1} = \sum_{l=1}^{t} \lambda_l^{(1)} b_l$ with $\lambda_l^{(1)} = -m_l/m_{i_1}$ for $l \ne i_1$ and $\lambda_{i_1}^{(1)} = 0$. Thus $b_{i_1} \in T$. Suppose that i_h and $\lambda_1^{(h)}, \ldots, \lambda_t^{(h)}$ for some h with $1 \le h < t$ are already defined such that $\sum_{l=1}^{h} b_{i_l} = \sum_{l=1}^{t} \lambda_l^{(h)} b_l \in T$, with $-1 \le \lambda_l^{(h)} \le 0$ for $l = 1, \ldots, t$. Write $\sum_{l=1}^{h} b_{i_l} = \sum_{l=1}^{t} n_l b_l$ with integers $n_1, \ldots, n_t$ such that $0 \le n_l \le m_l$ ($l = 1, \ldots, t$) and $n_1 + \cdots + n_t = h$. If for some l we have $n_l - m_l = \lambda_l^{(h)} = -1$, then set $i_{h+1} = l$, and $\lambda_{i_{h+1}}^{(h+1)} = 0$, $\lambda_i^{(h+1)} = \lambda_i^{(h)}$ ($i = 1, \ldots, t,\ i \ne i_{h+1}$). Otherwise, put $J = \{1 \le l \le t \,|\, n_l \ne m_l\}$ and define i_{h+1} as the index $l \in J$ for which $(-1 - \lambda_l^{(h)})/(m_l - n_l + \lambda_l^{(h)})$ is minimal, and write $\lambda = (-1 - \lambda_{i_{h+1}}^{(h)})/(m_{i_{h+1}} - n_{i_{h+1}} + \lambda_{i_{h+1}}^{(h)})$. Then (12) and $\sum_{l=1}^{t} n_l b_l = \sum_{l=1}^{t} \lambda_l^{(h)} b_l$ give that

$$\sum_{l=1}^{t} n_l b_l = \sum_{l=1}^{t} (\lambda_l^{(h)} + \lambda(m_l - n_l + \lambda_l^{(h)})) b_l.$$

Put $\lambda_l^{(h+1)} = \lambda_l^{(h)} + \lambda(m_l - n_l + \lambda_l^{(h)})$ for $l = 1, \ldots, t$. Observe that by the choice of λ, $\lambda_{i_{h+1}}^{(h+1)} = -1$ holds. Moreover, we have $\lambda_l^{(h+1)} \in [-1, 0]$, for $l = 1, \ldots, t$. Thus, by the choice of i_{h+1}, we get that $\sum_{l=1}^{h+1} b_{i_l} \in T$. Hence the induction hypothesis holds true for $h+1$. After M steps we obtain $n_l = m_l$ for $l = 1, \ldots, t$, and the lemma follows. $\square$

Corollary 19 *Let $b_0, b_1, \ldots, b_t$ be nonzero vectors in $\mathbb{R}^n$ such that $b_0 + b_1 + \cdots + b_t = \mathbf{O}$. Then there exists a permutation $(i_1, \ldots, i_t)$ of $(1, \ldots, t)$ such that* *for every h with $h \in \{1, \dots, t\}$ we have*

$$\left| b_0 + \sum_{l=1}^{h} b_{i_l} \right| \leq 2|b_0| + \sum_{b \in B} |b|,$$

where B is the set $\{b_j \mid j = 1, \dots, t,\ b_j \neq b_0\}$.

PROOF. By Lemma 18 there is a permutation $(j_0, j_1, \dots, j_t)$ of $(0, 1, \dots, t)$ such that for every h with $h \in \{0, \dots, t\}$ we have $\left| \sum_{l=0}^{h} b_{j_l} \right| \leq |b_0| + \sum_{b \in B} |b|$. Let r be the index for which $j_r = 0$ and put

$$i_l = \begin{cases} j_{l-1}, & \text{if } 1 \leq l < r \\ j_l, & \text{if } l \geq r. \end{cases}$$

Let $h \in \{1, \dots, t\}$. If $h \geq r$ then

$$\left| b_0 + \sum_{l=1}^{h} b_{i_l} \right| = \left| \sum_{l=0}^{h} b_{j_l} \right| \leq |b_0| + \sum_{b \in B} |b|.$$

On the other hand, if $h < r$ we have

$$\left| b_0 + \sum_{l=1}^{h} b_{i_l} \right| \leq |b_0| + \left| \sum_{l=1}^{h} b_{i_l} \right| = |b_0| + \left| \sum_{l=0}^{h-1} b_{j_l} \right| \leq 2|b_0| + \sum_{b \in B} |b|.$$

This implies the statement. $\square$

Theorem 20 *Let $N = N_1 \dots N_k \overline{N_{k+1} \dots N_l} \in S_n^u$ with $N_i = (P_i, w_i)$ for $i \in \{1, \dots, l\}$. Then there is a finite procedure to decide whether $W(N)$ is symmetric on $\mathbb{Z}^n$ or not.*

PROOF. First we introduce some notation. Put

$$A_r = \{a_1 + \dots + a_r \mid a_i \in P_i\ (i = 1, \dots, r)\}$$

for $1 \leq r < l$, and $A = \bigcup_{r=1}^{k-1} A_r$, $B = \bigcup_{r=k}^{l-1} A_r$. Moreover, set

$$C = \{a_{k+1} + \dots + a_l \mid a_i \in P_i\ (i = k+1, \dots, l)\}.$$

Write $R = \sum_{\alpha \in C} |\alpha| + 4 \max_{a \in B} \{|a|\}$, and let Q denote the number of those $p \in \mathbb{Z}^n$ for which $|p| \leq R$ holds. Define the set D by

$$D = \{a + \alpha_1 + \dots + \alpha_c \mid a \in A \cup B,\ 0 \leq c \leq 2Q,\ \alpha_i \in C\ (i = 1, \dots, c)\}.$$

Clearly, D is a finite set. Hence by Theorem 14, we can check whether $W(b) = W(-b)$ holds for every $b \in D \cup (-D)$, or not. If not, then N does not induce a metric. So assume that W is symmetric on $D \cup (-D)$. If W is also symmetric on $\mathbb{Z}^n \setminus (D \cup (-D))$, then we are done. Thus suppose that $W(b) \neq W(-b)$ for some $b \in \mathbb{Z}^n \setminus (D \cup (-D))$. Then, a shortest path from $\mathbf{O}$ to b is of the form $b = a + \alpha_1 + \alpha_2 + \cdots + \alpha_t$ with $a \in B$, $t \geq 2Q + 1$ and $\alpha_i \in C$ for $i = 1, \ldots, t$. Similarly, a shortest path to $-b$ is given by $-b = a' + \alpha'_1 + \alpha'_2 + \cdots + \alpha'_{t'}$ with $a' \in B$, $t' \geq 2Q + 1$ and $\alpha'_i \in C$ for $i = 1, \ldots, t'$. We can further assume that $t + t'$ is minimal with the property $W(b) \neq W(-b)$ ($b \notin D \cup (-D)$). Observe that

$$W(a + \alpha_1 + \alpha_2 + \cdots + \alpha_r) = \ell(a) + \ell(\alpha_1) + \ell(\alpha_2) + \cdots + \ell(\alpha_r) \tag{13}$$

for $t - 2Q < r \leq t$ and similarly for $a' + \alpha'_1 + \alpha'_2 + \cdots + \alpha'_{r'}$. Moreover, observe that in the above formulae we may permute $\alpha_1, \ldots, \alpha_t$ arbitrarily as well as $\alpha'_1, \ldots, \alpha'_{t'}$ without affecting the validity of the statements.

Now apply Corollary 19 with $b_0 = a + a'$ to

$$(a + a') + \alpha_1 + \cdots + \alpha_t + \alpha'_1 + \cdots + \alpha'_{t'} = b + (-b) = \mathbf{O}.$$

Then there exists a sequence $i_1, \ldots, i_{t+t'}$ such that $\alpha^*_1, \ldots, \alpha^*_{t+t'}$ is a permutation of $\alpha_1, \ldots, \alpha_t, \alpha'_1, \ldots, \alpha'_{t'}$ and $\left| a + a' + \sum_{j=1}^{T} \alpha^*_j \right| \leq 4 \max_{a \in B}\{|a|\} + \sum_{\alpha \in C} |\alpha| = R$ for $T = 1, \ldots, t + t'$. Since there are exactly Q vectors $p \in \mathbb{Z}^n$ with $|p| \leq R$ and by $t, t' > 2Q$, we obtain the existence of T and T' with $0 \leq T < T' \leq Q$ such that $a + a' + \sum_{j=1}^{T} \alpha^*_j = a + a' + \sum_{j=1}^{T'} \alpha^*_j$ which implies $\sum_{j=T+1}^{T'} \alpha^*_j = \mathbf{O}$.

After suitable permutations of $\alpha_1, \ldots, \alpha_t$ and $\alpha'_1, \ldots, \alpha'_{t'}$, we may assume that $\sum_{j=T+1}^{T'} \alpha^*_j = \alpha_{s+1} + \cdots + \alpha_t + \alpha'_{s'+1} + \cdots + \alpha'_{t'}$, where $t \geq s \geq Q+1$, $t' \geq s' \geq Q+1$, $(t - s) + (t' - s') = T' - T \leq Q$. Hence

$$a + \alpha_1 + \cdots + \alpha_s + a' + \alpha'_1 + \cdots + \alpha'_{s'} = \mathbf{O}. \tag{14}$$

From the minimality condition it follows that

$$W(a + \alpha_1 + \cdots + \alpha_s) = W(a' + \alpha'_1 + \cdots + \alpha'_{s'}), \tag{15}$$

hence by (13), $\ell(\alpha_{s+1} + \cdots + \alpha_t) \neq \ell(\alpha'_{s'+1} + \cdots + \alpha'_{t'})$ in view of $W(b) \neq W(-b)$. By applying the above reasoning to (14) we obtain after suitable permutation integers r, r' with $s \geq r \geq 1$, $s' \geq r' \geq 1$, $0 < (s - r) + (s' - r') \leq Q$ and

$$a + \alpha_1 + \cdots + \alpha_r + a' + \alpha'_1 + \cdots + \alpha'_{r'} = \mathbf{O}.$$

From (15) and the minimality condition it follows that

$$W(a+\alpha_1+\cdots+\alpha_r)=W(a'+\alpha'_1+\cdots+\alpha'_{r'}),$$

hence by (13)

$$\ell(\alpha_{r+1}+\cdots+\alpha_s)=\ell(\alpha'_{r'+1}+\cdots+\alpha'_{s'}). \tag{16}$$

However, we can as well consider

$$a+\alpha_1+\cdots+\alpha_r+\alpha_{s+1}+\cdots+\alpha_t+a'+\alpha'_1+\cdots+\alpha'_{r'}+\alpha'_{s'+1}+\cdots+\alpha'_{t'}=\mathbf{O}.$$

By the minimality condition we have

$$W(a+\alpha_1+\cdots+\alpha_r+\alpha_{s+1}+\cdots+\alpha_t)=W(a'+\alpha'_1+\cdots+\alpha'_{r'}+\alpha'_{s'+1}+\cdots+\alpha'_{t'}).$$

Comparing this with $W(b)\neq W(-b)$, by $(s-r)+(s'-r')\leq Q$ we conclude that

$$\ell(\alpha_{r+1}+\cdots+\alpha_s)\neq\ell(\alpha'_{r'+1}+\cdots+\alpha'_{s'})$$

in contradiction with (16). Hence the theorem follows. □

6.3 Metricity

By combining the results obtained for the triangle inequality and symmetry with some additional observations, we obtain the following statement.

Theorem 21 *Let $N=N_1\dots N_k\overline{N_{k+1}\dots N_l}\in S_n^u$ with $N_i=(P_i,w_i)$ for $i\in\{1,\dots,l\}$. Then there is a finite procedure to decide whether N induces a metric on $\mathbb{Z}^n$ or not.*

PROOF. First we have to check whether $\mathbb{Z}^n$ is N-connected. This can be done by a finite procedure according to Theorem 11. From Theorems 17 and 20 we know that the triangular and symmetric behavior of $W(N)$ also can be checked by a finite procedure.

So only the positivity of $W(N)$ which remains to check. To test positivity, we do the following. Let $i\in\{1,\dots,l\}$ be the maximal index for which for $j=1,\dots,i$ there exists a $p_j\in P_j$ such that $w_j(p_j)=0$. Then $W(N)$ is positive if and only if we have $p_j=\mathbf{O}$ whenever $w_j(p_j)=0$ for $j=1,\dots,i$. Clearly, this property can be checked by a finite procedure. Hence the theorem follows. □

The following corollary extends the results of Das et al. [6] and Nagy [25] obtained for periodic octagonal and for general octagonal neighborhood sequences, respectively, in the finite dimensional case.

Corollary 22 *Suppose that* $N = (N_i)_{i=1}^{\infty} \in S_n$ *with* $N_i = (P_i, w_i)$*, such that* $N^{(j)} \sqsupseteq^* N$ *for every* $j \in \mathbb{N}$*,* $\mathbb{Z}^n$ *is* N*-connected,* N_i *is symmetric, and* $w_i(x) > 0$ *for all* $x \in P_i \setminus \{\mathbf{O}\}$ $(i \in \mathbb{N})$*. Then* $W(N)$ *is a metric on* $\mathbb{Z}^n$*.*

PROOF. Let W denote the distance function induced by N. The second part of Theorem 15 guarantees that the triangle inequality holds for W. All the other necessary properties of metricity follow from our assumptions. $\square$

Corollary 23 *Let* $N = \overline{N_1} \in S_n$ *with* $N_1 = (P_1, w_1)$*. If* $\mathbb{Z}^n$ *is* N*-connected,* N_1 *is symmetric, and* $w_1(x) > 0$ *for all* $x \in P_1 \setminus \{\mathbf{O}\}$*, then* $W(N)$ *is a metric on* $\mathbb{Z}^n$*.*

PROOF. The statement easily follows from Corollary 22 by noting that $W(q, r; N^{(j)}) = W(q, r; N)$ for all $q, r \in \mathbb{Z}^n$ and $j \in \mathbb{N}$ in this case. $\square$

Corollary 24 *Let* $N = (N_i)_{i=1}^{\infty} \in S_n^{\mathbf{O}}$ *with* $N_i = (P_i, w_i)$*, such that for every* $i \in \mathbb{N}$*,* N_i *occurs infinitely often in* N*. Suppose that* $\mathbb{Z}^n$ *is* N*-connected,* N_i *is symmetric, and* $w_i(x) > 0$ *for all* $x \in P_i \setminus \{\mathbf{O}\}$ $(i \in \mathbb{N})$*. Then* $W(N)$ *is a metric on* $\mathbb{Z}^n$*.*

PROOF. As by our assumption there is no neighborhood N_i occurring only finitely many times in N, Proposition 5 and its proof show that there exists a constant neighborhood sequence M such that $W(N)$ is identical with $W(M)$. Hence the statement follows from Corollary 23. $\square$

7 Approximating the Euclidean metric

The approximation of the Euclidean distance by digital metrics is a key problem in digital geometry. In this section we present some results towards this direction.

Yamashita and Ibaraki [30] showed that for any $N \in S_n^p$, if $W(N)$ is a metric then there exist $c_1, c_2 \in \mathbb{R}_{>0}$ such that

$$c_1 d(x; N) \leq ||x||_2 \leq c_2 d(x; N) \text{ for any } x \in \mathbb{Z}^n. \tag{17}$$

Now we investigate whether the Euclidean distance can be minorated/majorated or not in our more general model.

Proposition 25 *Let $N \in S_n$ and suppose that $W(N)$ is a metric. Then there exists a $c_1 \in \mathbb{R}_{>0}$, such that*

$$c_1 d(x; N) \leq ||x||_2 \text{ for any } x \in \mathbb{Z}^n.$$

PROOF. In fact the proof of Theorem 5 of Yamashita and Ibaraki [30] for periodic neighborhood sequences can be extended to this case. However, for the convenience of the reader, we recall the main steps of the proof.

Let $c_0 = \max\left\{d(e; N) \,\middle|\, e = (e_1, \ldots, e_n),\ e_i \in \{0, 1\},\ \sum_{i=1}^{n} e_i = 1\right\}$. Note that $c_0 > 0$, since $W(N)$ is a metric. Then $d(x; N) \leq c_0 \sum_{i=1}^{n} |x_i| = c_0 ||x||_1$ for any $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$. It is well-known that $\frac{1}{\sqrt{n}} ||x||_1 \leq ||x||_2$. Thus $c_1 d(x; N) \leq ||x||_2$ with $c_1 = \frac{1}{c_0 \sqrt{n}}$. $\square$

From the following example we can see that there exists N such that the Euclidean distance cannot be majorated in terms of $d(x; N)$, even not with metrics generated by ultimately periodic neighborhood sequences.

Example 26 *Let $n = 1$, $N_1 = (P, w_1)$ with $P = \{\pm 1\}$ and $w_1(\pm 1) = 1$, and let $N_2 = (P, w_2)$ with $w_2(\pm 1) = 0$. Consider $N = N_1 \overline{N_2} \in S_1^u$. Then $d(x; N) = 1$ for any $x \in \mathbb{Z}$ $(x \neq 0)$, and thus there exists no $c_2 \in \mathbb{R}_{>0}$ such that $||x||_2 \leq c_2 d(x; N)$ for any $x \in \mathbb{Z}$.*

The next two propositions show that property (17) holds under suitable mild conditions.

Proposition 27 *If $N \in S_n^{\mathbf{O}}$ induces a metric on $\mathbb{Z}^n$, then there exists a $c_2 \in \mathbb{R}_{>0}$ such that*

$$||x||_2 \leq c_2 d(x; N) \text{ for any } x \in \mathbb{Z}^n.$$

PROOF. Theorem 8 and its proof guarantee that there exists a constant (periodic) neighborhood sequence, which generates the same metric as N. Since the statement holds for periodic sequences (see [30]), the proof is complete. $\square$

Proposition 28 *Let $N = N_1 \ldots N_k \overline{N_{k+1} \ldots N_l} \in S_n^u$ with $N_i = (P_i, w_i)$ for $i \in \{1, \ldots, l\}$ such that $W(N) =: d(N)$ is a metric. Then there exists a $c_3 \in \mathbb{R}_{>0}$ such that*

$$||x||_2 \leq c_3 d(x; N) \text{ for every } x \in \mathbb{Z}^n$$

if and only if $\ell(s) > 0$ for every $s \in Q_\infty \setminus \{\mathbf{O}\}$, where Q_∞ is defined in Theorem 11.

PROOF. We may assume without loss of generality that $N = N_1 \ldots N_k\overline{N_{k+1}}$ with $N_i = (P_i, w_i)$ for $i = 1, \ldots, k+1$ using Proposition 7. Thus to prove the statement, we can replace the condition "$\ell(s) > 0$ for every $s \in Q_\infty \setminus \{\mathbf{O}\}$" by "$w_{k+1}(y) > 0$ for every $y \in P_{k+1} \setminus \{\mathbf{O}\}$".

First we prove necessity. Suppose that there exists a $y \in P_{k+1} \setminus \{\mathbf{O}\}$ such that $d(y) = 0$. Consider an arbitrary path $s = [\mathbf{O}, x_1, \ldots, x_k]$ (the empty path if $k = 0$) and continue this path by always selecting $y \in P_{k+1}$ from the neighborhood N_{k+1}. Using this path we can move arbitrarily far from the origin with respect to the Euclidean distance. However, all the points on the path have length at most $\ell(s)$. This means that we cannot majorate the Euclidean distance, and proves the necessity part.

To prove sufficiency assume that $w_{k+1}(y) > 0$ for every $y \in P_{k+1} \setminus \{\mathbf{O}\}$. Put

$$b_1 = \min \left\{ \frac{\ell(s)}{||s||_2} \;\middle|\; s \in \bigcup_{t=1}^{k} \left\{ \sum_{i=1}^{t} u_i \;\middle|\; u_i \in P_i,\ i = 1, \ldots, t \right\} \right\},$$

$$b_2 = \min \left\{ \frac{w_{k+1}(y)}{||y||_2} \;\middle|\; y \in P_{k+1} \setminus \{\mathbf{O}\} \right\},$$

and put $b_3 = \min\{b_1, b_2\}$. Note that $b_3 > 0$, since $d(N)$ is a metric and $w_{k+1}(y) > 0$ for every $y \in P_{k+1} \setminus \{\mathbf{O}\}$. Let x be an arbitrary point of $\mathbb{Z}^n$, and consider a shortest N-path from $\mathbf{O}$ to x, $\mathbf{O} = q_0, \ldots, q_r = x$ say. If $r \leq k$ then $||x||_2 \leq \frac{1}{b_1} d(x; N)$. Otherwise, $d(x; N) = \ell([\mathbf{O}, \ldots, q_k]) + \sum\limits_{i=k+1}^{r} w_{k+1}(q_i - q_{i-1}) \geq b_1 ||q_k||_2 + b_2 \sum\limits_{i=k+1}^{r} ||q_i - q_{i-1}||_2 \geq b_3 ||x||_2$. Hence $c_3 d(x; N) \geq ||x||_2$ with $c_3 = \frac{1}{b_3}$, and the proof is complete. □

8 Applications

Neighborhood sequences have already been applied successfully for practical image processing purposes like segmentation [20], and retrieval [22]. In this section we show how our current results can be used in the theory of distance transformations. On one hand we indicate how ultimately periodic neighborhood sequences can provide a new tool in some well-known image processing procedures. On the other hand we present an application scheme for neighborhood sequences from $S_n^{\mathbf{O}}$.

Distance transformations provide a very useful basis for many image processing problems. To indicate how widely this technique is applied, we refer to [2,4,14,28] and the references given there as characteristic examples. Usually the classical families of distance transformations (n-Neighbor, chamfer, octagonal, D-Euclidean) are considered in applications. These families are deeply investigated by Borgefors in [1]. These distance transformations are based on a mask of given size (e.g. 3×3 or 5×5 in $\mathbb{Z}^2$), with certain non-negative weights assigned to the entities of the mask. For example, Fig. 2 shows the general 3×3 mask used by Borgefors in [1] to compose various distance transformations.

d_2	d_1	d_2
d_1	**O**	d_1
d_2	d_1	d_2

Fig. 2. Classical 3×3 mask operator for distance transformations with non-negative weights d_1, d_2.

Using this mask, the distance of two points of the domain is calculated just as in our model, by considering a constant neighborhood sequence $N = \overline{M}$, where the neighborhood M is the mask of the distance transformation together with the assigned weights.

Note that the distance transformation families investigated by Borgefors [1] are special cases of the model given by Yamashita and Ibaraki in [30], who used periodic neighborhood sequences. However, Yamashita and Ibaraki [30] showed that if the distance function generated by a periodic neighborhood sequence is a metric then the periodic sequence is equivalent to a constant sequence (with respect to the generated distance functions). Thus we yet again have a constant sequence if the important property of metricity is required.

Ultimately periodic sequences presented in our model obviously cover periodic ones and thus also the classical families in [1]. Hence the use of such sequences opens up new possibilities to achieve more general distance transformations. We underline Example 6 which shows that ultimately periodic sequences cannot be replaced by constant ones, even if metricity is required. Especially, as a key problem, we mention the famous results of Borgefors [1] about finding suitable weights for a distance transformation to approximate the Euclidean distance. It is natural to expect that in our more general model better approximations can be found for the Euclidean metric than in case of using constant sequences.

We show an application scheme to demonstrate the importance and applicability of our results about $S_n^{\mathbf{O}}$. Using sequences belonging to $S_n^{\mathbf{O}}$, intuitively we have the opportunity to ignore undesired elements of the sequence, by doing nothing for no cost at a step (by using $\mathbf{O}$ with weight 0). Moreover, we do not have to deal with the order of the neighborhoods in the sequence when finding a shortest path between two points, as according to Remark 4 and Proposition 5 the elements of such sequences can be freely permuted.

In general, we can interpret this case as if we have resources with given costs, such that some of the resources can be used only a prescribed number of times, while others infinitely often. More precisely, we can think of moving in the space. Suppose that our task is to find an optimal path to our destination. If we know which path would be optimal then regardless of the order, we can pick up the desired vectors freely and build up our path.

To make our ideas more clear we consider a concrete example. Let the neighborhoods N_1, N_2, N_3, N_4 be as shown in Figure 3(a), where the black discs represent the neighborhood vectors and the values inside their weights. Let $N = N_1N_2N_3N_4N_1\{N_i\}_{i=5}^{\infty}$ be any sequence such that $N_i \in \{N_3, N_4\}$ for $i \geq 5$, with both $N_i = N_3$ and $N_i = N_4$ infinitely often. Then clearly, $N \in S_2^{\mathbf{O}}$. Now, as it can be seen also in Figure 3(b), to reach $(2,2)$ from the origin we can take $\mathbf{O}$, $(1,0)$, $\mathbf{O}$ and $(1,2)$ from N_1, N_2, N_3 and N_4, respectively, to obtain the shortest path. In other words we can "skip" N_1 and N_3, and use N_2 and N_4 freely to build up the path.

Using Proposition 5 we can determine an ultimately constant neighborhood sequence N' equivalent to N. This sequence is given by $N' = N_1N_1N_2\overline{N_0}$, where N_0 is shown in Figure 3(c).

When investigating metricity, we can restrict our attention from the general case to ultimately periodic sequences. To see this, consider a (not necessarily ultimately periodic) neighborhood sequence from $S_n^{\mathbf{O}}$. Let $N_1, N_2, \ldots, N_k$ be the neighborhoods which occur only finitely often with the right multiplicities, and $N_{k+1}, \ldots, N_l$ the neighborhoods which occur infinitely often. Then, as shown in the paper, the metrical properties of the sequence are the same as of $N_1N_2\ldots N_k\overline{N_{k+1}\ldots N_l}$. So we can apply the theory in the paper for ultimately periodic sequences to study the metrical properties of such neighborhood sequences. We may even combine $N_1 \ldots N_k$ to one neighborhood and $N_{k+1} \ldots N_l$ to another to simplify formulas, but then we loose control on the sizes of the neighborhoods as shown above.

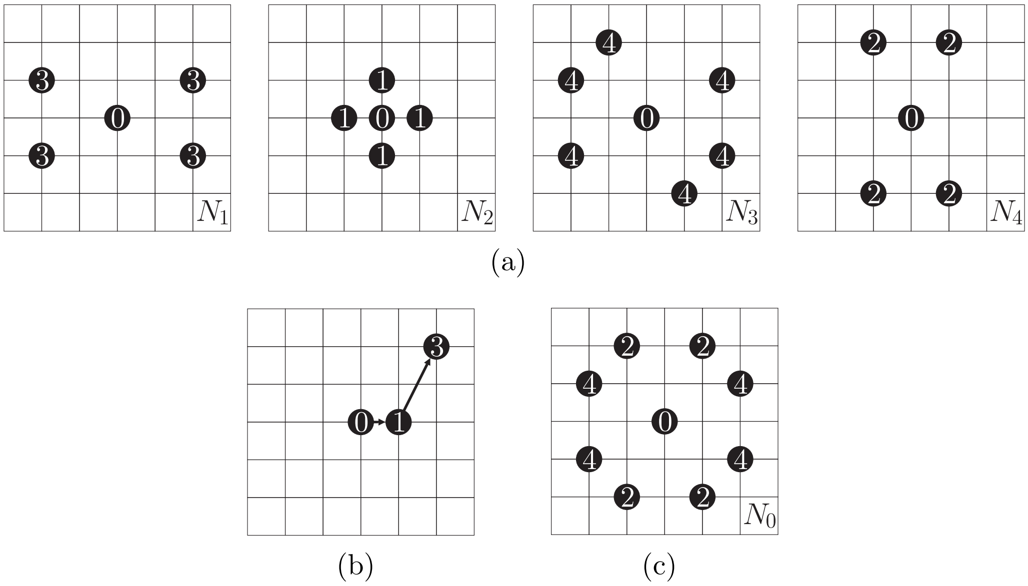

Fig. 3. Choosing optimal neighborhood elements to obtain a shortest path using a sequence $N \in S_2^{\mathbf{O}}$; (a) the elements of N, (b) the shortest path to $(2, 2)$ using N_2 and N_4, (c) the neighborhood to compose the equivalent ultimately constant sequence $N' = N_1N_1N_2\overline{N_0}$.

9 Conclusion

Since the first submission of the paper, the authors went on with their research regarding neighborhood sequences. As corresponding results, we highlight the derivation of integer values to be used in chamfering to approximate the Euclidean metric [19,29], the application of ultimately periodic neighborhood sequences in image retrieval [22], and the application of weighted neighborhoods to approximate non metrical Minkowski distances [21]. We also note that the complexity analysis of the finite procedures we have given to decide on several properties regarding neighborhood sequences has been left as an open issue.

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