On the GCD-s of k consecutive terms of Lucas sequences

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Abstract

Let $u = (u_n)_{n=0}^{\infty}$ be a Lucas sequence, that is a binary linear recurrence sequence of integers with initial terms $u_0 = 0$ and $u_1 = 1$. We show that if k is large enough then one can find k consecutive terms of u such that none of them is relatively prime to all the others. We even give the exact values g_u and G_u for each u such that the above property first holds with $k = g_u$; and that it holds for all $k \geq G_u$, respectively. We prove similar results for Lehmer sequences as well, and also a generalization for linear recurrence divisibility sequences of arbitrarily large order. On our way to prove our main results, we provide a positive answer to a question of Beukers from 1980, concerning the sums of the multiplicities of 1 and -1 values in non-degenerate Lucas sequences. Our results yield an extension of a problem of Pillai from integers to recurrence sequences, as well.

Key words: Lucas sequences, greatest common divisor, divisibility sequences, Pillai's problem
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1 Introduction

Let $u = (u_n)_{n=0}^{\infty}$ be a Lucas sequence, that is a binary linear recurrence sequence of integers with initial terms $u_0 = 0$ and $u_1 = 1$. The investigation of the divisibility properties of the terms of such sequences, or more generally, linear recurrence sequences, has a very long history, and a huge literature. Here we only mention a few of the several important and interesting directions, considered by many authors.

One of the most important questions concerns the existence of primitive prime divisors of the terms of Lucas sequences. After several results yielding partial answers to this problem, Bilu, Hanrot and Voutier [9] could completely settle the question. Since there are so many related results, instead of trying to summarize them, for the history of the problem we just refer the reader to [9] and the references given there.

Another important problem which has been closely investigated is to characterize the so-called divisibility sequences. That is, describe all linear recurrence sequences u_n (now of arbitrary order) such that $u_i \mid u_j$ whenever $i \mid j$. After certain partial results of Hall [24] and Ward [45], the complete description of such sequences has been provided by Bézivin, Pethő and Van der Poorten [8]; see also the paper of Győry and Pethő [21]. There are also important related results of Horák and Skula [25] and Schinzel [40], concerning so-called strong divisibility sequences.

The next topic we mention concerns the investigation of the property $n \mid u_n$ for $n \geq 1$; i.e. the determination of terms being divisible by their indices. For related results see e.g. the papers of Smyth [42], Győry and Smyth [22] and Alba et al. [1], and the references therein.

Another problem is to find the prime terms of the sequences studied, or at least prove that there are only finitely many such terms. For related results and references we refer to the book of Guy [20], p. 17, and the papers of Graham [19], Knuth [27], Wilf [47] and Dubickas et al. [17], and the references given there.

There are several results concerning the problem when a term or the product of terms of a sequence u (or even of more sequences) is a perfect power; see e.g. the book of Shorey and Tijdeman [44] and the papers of Bremner and Tzanakis [11–13], Luca and Walsh [35], Kiss [26], Brindza, Liptai and Szalay [14], and Luca and Shorey [32–34], and the references there.

Finally, we mention a problem which is not closely related, but on the one hand deeply investigated, and on the other hand, also important from the viewpoint of the present paper. This problem is the question of zero-multiplicity (or more generally the ω-multiplicity) of linear recurrence sequences. That is, given such a sequence $u = (u_n)_{n=0}^{\infty}$, we are interested in the number of solutions of $u_n = 0$ (or more generally of $u_n = \omega$ with some given number ω). Further, in case of infinitely many solutions, we would like to know the structure of solutions. After the fundamental results of Skolem, Mahler and Lech [30] several much more general results have appeared; see e.g. the papers of Ward [46], Kubota [28,29], Beukers [5,6], Beukers and Tijdeman [7], Brindza, Pintér and Schmidt [15], Allen [2,3], Amoroso and Viada [4] and the references given there.

In this paper we consider a property of linear recurrence sequences which is strongly related to the above ones. To set the problem, first we recall a question considered by Pillai [39]: is it true that for any $k \geq 2$ one can find k consecutive integers such that none of them is relatively prime to all the others? Pillai [39] himself proved that this is not true for $2 \leq k \leq 16$, but holds for $17 \leq k \leq 430$. The question has been completely answered to the affirmative by Brauer [10]. Later, the original problem has been extended and generalized into several directions. For related results, see e.g. the papers of Caro [16], Saradha and Thangadurai [43] and Hajdu and Saradha [23] and the references there. In particular, Ohtomo and Tamari [38] have extended the original problem to arithmetic progressions, i.e. one considers k consecutive terms of an arithmetic progression, rather than k consecutive integers.

In this paper we extend Pillai's problem to linear recurrence sequences. More precisely, we consider the following problem, and also some of its generalizations. Let $u = (u_n)_{n=0}^{\infty}$ be a non-degenerate Lucas sequence. (For exact definitions and notation see Section 2.) Let g_u be the smallest integer such that for $k = g_u$, one can find k consecutive terms in u such that none of these terms is relatively prime to all the others. Similarly, let G_u be the smallest integer k_0 such that for any $k \geq k_0$ one can find k consecutive terms in u such that none of these terms is relatively prime to all the others. Note that *a priori* it is not known that g_u and G_u exist. However, if they both exist, then we obviously have $g_u \leq G_u$. We prove that for any non-degenerate Lucas sequence u, both g_u and G_u exist, and further, we calculate the exact values of these numbers for each u (see Theorem 1). On our way to prove this result, we provide a positive answer to a question of Beukers [5], concerning the sums of the multiplicities of the values 1 and -1 in non-degenerate Lucas sequences (see Corollary 10). Just for curiosity, we also mention that as a special case we obtain that among any 24 consecutive Fibonacci numbers one of them is always coprime to all the others, however, it is possible to find 25 consecutive Fibonacci numbers lacking this property. The index n_0 of the first term of 25 such numbers where this phenomenon first occurs is $n_0 = 208569474$.

We provide a similar result also for Lehmer sequences (cf. Theorem 3).

We prove a more general statement concerning divisibility sequences of arbitrarily large order as well, where instead of the gcd-s the S-free parts of them are calculated, with S being a finite set of primes. It turns out that the corresponding numbers g_u and G_u still exist (see Theorem 4), and they can be bounded in terms of the cardinality of S and the order of u.

We also handle the case of degenerate Lucas and Lehmer sequences in Theorem 5.

It is important to mention that the existence of g_u and G_u is very far from being automatic. We do not claim that the existence of these numbers would characterize say linear recurrence divisibility sequences, however, it seems that it is still a very special property. This is supported by the fact that g_u and G_u in general do not exist - this is the case already for the so-called associated Lucas and Lehmer sequences (see Theorem 6).

Finally, we also note that the motivation of Pillai in considering the original problem roots in the famous diophantine equation

$$x(x+1)\dots(x+k-1) = y^n,$$

which has been resolved later by Erdős and Selfridge [18]. As we mentioned above, Luca and Shorey [32–34] have several nice related results for the products of terms of a linear recurrence sequence yielding a perfect power. We hope that our results and methods may find some applications concerning this problem.

2 Notation

In this section we introduce the notation which is necessary to formulate our results.

2.1 Linear recurrence sequences

We emphasize that throughout the paper we work with *integral* sequences. A sequence $u = (u_n)_{n=0}^{\infty}$ of integers is called a linear recurrence sequence of order r if $u_0, \dots, u_{r-1}$ are not all zero and it satisfies a relation of the form

$$u_{n+r} = c_1 u_{n+r-1} + c_2 u_{n+r-2} + \dots + c_r u_n \quad (n \geq 0) \tag{1}$$

with $c_1, \dots, c_r \in \mathbb{Z}, c_r \neq 0$, and r is minimal with this property. The polynomial

$$p(x) = x^r - c_1 x^{r-1} - \dots - c_r$$

is called the companion polynomial of u. Denote the distinct roots of this polynomial with $\alpha_1, \ldots, \alpha_t$. We say that u is non-degenerate if $i \neq j$ implies that α_i/α_j is not a root of unity. As is well-known, writing

$$p(x) = (x - \alpha_1)^{e_1} \ldots (x - \alpha_t)^{e_t},$$

for any $n \geq 0$ we have the representation

$$u_n = \sum_{i=1}^{t} P_i(n)\alpha_i^n \tag{2}$$

where the polynomial P_i is of degree $e_i - 1$ and has coefficients from the number field $\mathbb{Q}(\alpha_1, \ldots, \alpha_t)$ for $i = 1, \ldots, t$.

2.2 Divisibility sequences

A linear recurrence sequence $u = (u_n)_{n=0}^{\infty}$ of integers is called a divisibility sequence, if $i \mid j$ implies $u_i \mid u_j$. For the complete characterization of such sequences we refer to the nice and deep paper of Bézivin, Pethő and Van der Poorten [8]. If further on, we have $\gcd(u_i, u_j) = u_{\gcd(i,j)}$ for all $i, j \geq 0$ then we say that u is a strong divisibility sequence. For the characterization of such binary sequences see the paper of Horák and Skula [25] in the integral case, and the paper of Schinzel [40] in the case where the elements of u are algebraic integers.

2.3 Lucas sequences and their associated sequences

We shall be particularly interested in Lucas sequences. A binary linear recurrence sequence $u = (u_n)_{n=0}^{\infty}$ is called a Lucas sequence (or sometimes as generalized Fibonacci sequence) corresponding to the parameters $M, N \in \mathbb{Z}$ with $N \neq 0$ if $u_0 = 0$, $u_1 = 1$ and for any $n \geq 2$ we have

$$u_{n+2} = Mu_{n+1} - Nu_n. \tag{3}$$

Note that the role of M and N corresponds to the choices $c_1 = M$ and $c_2 = -N$ in (1). Obviously, with $(M, N) = (1, -1)$ we just get the Fibonacci sequence.

If $v_0 = 2$, $v_1 = M$ and for $n \geq 2$ the sequence $v = (v_n)_{n=0}^{\infty}$ satisfies (3) then v is called an associated Lucas sequence (corresponding to u).

2.4 Lehmer sequences and their associated sequences

We shall also work with Lehmer sequences. A sequence $\tilde{u} = (\tilde{u}_n)_{n=0}^{\infty}$ is called a Lehmer sequence corresponding to the parameters $M, N \in \mathbb{Z}$ with $N \neq 0$ if $\tilde{u}_0 = 0$, $\tilde{u}_1 = 1$ and for any $n \geq 2$ we have

$$\tilde{u}_{n+2} = \begin{cases} \tilde{u}_{n+1} - N\tilde{u}_n, & \text{if } n \text{ is even,} \\ M\tilde{u}_{n+1} - N\tilde{u}_n, & \text{if } n \text{ is odd.} \end{cases}$$

We say that the sequence $\tilde{u}$ is non-degenerate, if α/β is not a root of unity, where α and β are the roots of the polynomial $x^2 - \sqrt{M}x + N$.

If $\tilde{v}_0 = 2$, $\tilde{v}_1 = 1$ and for $n \geq 2$ the sequence $\tilde{v} = (\tilde{v}_n)_{n=0}^{\infty}$ satisfies

$$\tilde{v}_{n+2} = \begin{cases} M\tilde{v}_{n+1} - N\tilde{v}_n, & \text{if } n \text{ is even,} \\ \tilde{v}_{n+1} - N\tilde{v}_n, & \text{if } n \text{ is odd,} \end{cases}$$

then $\tilde{v}$ is called an associated Lehmer sequence (corresponding to $\tilde{u}$). These sequences have been introduced by Lehmer [31].

Finally, note that as is well-known, both Lehmer- and associated Lehmer sequences are linear recurrence sequences, of order at most four.

2.5 Pillai sequences

In this paper we investigate the problem of Pillai for linear recurrence sequences. Since already the original version of the problem shall be important for our purposes, we give a complete introduction of the topic. For a general and more detailed overview see e.g. the papers [43] and [23], and the references there.

For a given integer $k \geq 2$ let S_k denote a set of k consecutive integers. Pillai [39] proved that in any set S_k with $k < 17$ one can find an integer x which is coprime to all the other elements of S_k. On the other hand, he also showed that for any $17 \leq k \leq 430$ there are sets S_k having no such element x. The latter result was proved to hold for all $k \geq 17$ by Brauer [10].

The problem of Pillai has been generalized by Caro [16] by relaxing the coprimality condition to $\gcd(x, y) \leq d$ for some $d \geq 1$. Since this point is not important in the present paper, we suppress the details, and just refer to [16] and [43] for related results. However, we shall use a further generalization due to Hajdu and Saradha [23]. Let T be a non-empty set of positive integers. We say that S_k has property $P(T)$ if there is an $x \in S_k$ such that for all

$y \in S_k$, $y \neq x$ we have $\gcd(x, y) \in T$. Note that the choice $T = \{1\}$ gives back the original definition. Write $g(T)$ for the minimal $k \geq 2$ such that property $P(T)$ does not hold for some S_k, and $G(T)$ for the smallest integer k_0 such that for every $k \geq k_0$ property $P(T)$ does not hold for some S_k. Obviously, these values do not exist for all T. However, under certain assumptions Hajdu and Saradha [23] proved the existence of $g(T)$ and $G(T)$. Moreover, they have calculated the exact values of these functions for several particular choices of T, including $T = \{1, 2\}$ and $\{1, 2, 3\}$. Note that the original results of Pillai [39] and Brauer [10] imply that $g(\{1\}) = G(\{1\}) = 17$.

As another direction of generalization, Ohtomo and Tamari [38] extended the problem of Pillai from consecutive integers to arithmetic progressions. For details and related results see [38] and [23]. In this paper we extend the investigations to recurrence sequences. For the sake of generality, we set the problem for arbitrary sequences of integers. Let $A = (A_n)_{n=0}^{\infty}$ be a sequence of integers and let T be a non-empty set of positive integers. For a given integer $k \geq 2$ let

$$A_n, \ldots, A_{n+k-1} \quad (n \geq 0) \tag{4}$$

be k consecutive terms of A. We say that these k terms have property $P_A(T)$ if there is an $i \in \{0, 1, \ldots, k-1\}$ such that for all $j \in \{0, 1, \ldots, k-1\}$ with $i \neq j$ we have $\gcd(A_{n+i}, A_{n+j}) \in T$. Further, similarly as above we shall write $g_A(T)$ for the minimal k such that property $P_A(T)$ does not hold for some k terms (4), and $G_A(T)$ for the smallest integer k_0 such that for every $k \geq k_0$ property $P_A(T)$ does not hold for some k terms (4). Obviously, these values do not exist for every choice of A and T. Note that, however, if $G_A(T)$ exists, then so does $g_A(T)$, and we obviously have $g_A(T) \leq G_A(T)$. If $G_A(T)$ exists, then we shall call A a T-Pillai sequence. To simplify our notation in the most interesting and most frequently used situation $T = \{1\}$, instead of $P_A(\{1\})$, $g_A(\{1\})$ and $G_A(\{1\})$ we shall write P_A, g_A and G_A, respectively, and if G_A exists, then we shall call A a Pillai sequence. Note that by the results mentioned above, any arithmetic progression of integers different from $1, 1, 1, \ldots$ and $-1, -1, -1, \ldots$ is a Pillai sequence.

3 New results

We separate our main results into three blocks. First we formulate theorems for non-degenerate Lucas and Lehmer sequences and divisibility sequences of arbitrarily large order. Then we provide results concerning the degenerate case. Finally, we give a statement showing that apparently Pillai sequences are rather "rare" among linear recurrence sequences. Namely, the assumption that u is a Lucas (resp. Lehmer) sequence is necessary in Theorem 1 (resp. in Theorem 3) - at least the desired property is not valid already for associated Lucas (resp. Lehmer) sequences.

Our first result shows that in case of non-degenerate Lucas sequences u, the values of g_u and G_u always exist. Furthermore, the statement provides the exact values of g_u and G_u explicitly for all such u.

Theorem 1 *Every non-degenerate Lucas sequence $u = (u_n)_{n=0}^{\infty}$ is a Pillai sequence. Further, if the corresponding parameters M, N are not coprime, then $g_u = G_u = 2$. Otherwise, if $\gcd(M, N) = 1$, then we have $g_u = G_u = 17$ except for the cases given in Table 1.*

(M, N)	g_u	G_u
$(\pm 1, N)$, $N \neq 1, 2, 3, 5$	25	25
$(M, M^2 \pm 1)$, $\|M\| \geq 2$	43	43
$(\pm 12, 55)$	31	31
$(\pm 12, 377)$	31	31
$(\pm 1, 3)$	45	45
$(\pm 1, 5)$	49	51
$(\pm 1, 2)$	107	107

Table 1
The values of g_u and G_u for exceptional Lucas sequences.

Just for curiosity, we give a slightly more precise statement concerning the Fibonacci sequence.

Proposition 2 *The Fibonacci sequence F is a Pillai sequence with $g_F = G_F = 25$. Further, the first index n_0 such that among the Fibonacci numbers $F_{n_0}, F_{n_0+1}, \ldots, F_{n_0+24}$ none of them is coprime to all the others, is given by $n_0 = 208569474$.*

In the next theorem we extend Theorem 1 to Lehmer sequences.

Theorem 3 *Every non-degenerate Lehmer sequence $\tilde{u} = (\tilde{u}_n)_{n=0}^{\infty}$ is a Pillai sequence. Further, if the corresponding parameters M, N are not coprime then $g_{\tilde{u}} = G_{\tilde{u}} = 2$. Otherwise, if $\gcd(M, N) = 1$, then $g_{\tilde{u}} = G_{\tilde{u}} = 25$ except for the cases listed in Table 2. In the third row of the table F_n stands for the n-th Fibonacci number.*

Our final result in this subsection yields a significant generalization of Theorem 1, into two directions. On the one hand, we consider divisibility recurrence sequences u of arbitrary order, and on the other hand, we investigate the much more general property $P_u(T)$ where T is a set of integers having no

(M, N)	$g_{\tilde{u}}$	$G_{\tilde{u}}$
$\pm(N \pm 1, N)$, $N \geq 3$	49	53
$\pm(2N \pm 1, N)$, $N \geq 3$	47	47
$\pm(F_{n\pm 2}, F_n)$, $n \geq 4$	45	45
$\pm(1, 5)$	49	51
$\pm(13, 4)$	49	51
$\pm(14, 9)$	49	51
$\pm(5, 2)$	61	69
$\pm(3, 2)$	81	81
$\pm(1, 2)$	107	107

Table 2
The values of $g_{\tilde{u}}$ and $G_{\tilde{u}}$ for exceptional Lehmer sequences.

prime divisors outside some finite set of primes S. We emphasize that our upper bounds provided for $G_u(T)$ (and $g_u(T)$) depend only on the size of S and the order of u.

Theorem 4 *Let S be an arbitrary finite set of primes having s elements, and T be an arbitrary set of integers having no prime divisors outside S. Let $u = (u_n)_{n=0}^{\infty}$ be a non-degenerate divisibility recurrence sequence of order r. In case of $r = 1$, assume further that u_1 has a prime divisor outside S. Then u is a T-Pillai sequence. Further,*

$$g_u(T) \leq G_u(T) \leq C(s, r)$$

holds with

$$C(s, r) = \begin{cases} 2 & \text{if } r = 1, \\ 20(s + 30)\log(s + 30) & \text{if } r = 2, \\ r^{2^{8(s+r)}} & \text{if } r \geq 3. \end{cases}$$

Remark 1. In view of Theorems 5 and 6, we cannot omit neither the assumption that u is non-degenerate, nor that it is a divisibility sequence. Further, in the (trivial) case when $r = 1$ and all prime divisors of u_1 belong to S, taking T to be the set of all integers having no prime divisors outside S, we clearly get that $u_i \in T$ for all $i \geq 0$, whence $G_u(T)$ and $g_u(T)$ do not exist.

3.2 Degenerate Lucas and Lehmer sequences

Our next statement gives a complete characterization of Pillai sequences among degenerate Lucas and Lehmer sequences.

Theorem 5 *If u is a degenerate Lucas sequence with parameters M, N then u is a Pillai sequence if and only if either* $\gcd(M, N) > 1$, *when* $g_u = G_u = 2$, *or* $(M, N) = (\pm 2, 1)$, *when* $g_u = G_u = 17$.

Similarly, if $\tilde{u}$ is a degenerate Lehmer sequence with parameters M, N then $\tilde{u}$ is a Pillai sequence if and only if either $\gcd(M, N) > 1$, *when* $g_{\tilde{u}} = G_{\tilde{u}} = 2$, *or* $(M, N) = \pm(4, 1)$, *when* $g_{\tilde{u}} = G_{\tilde{u}} = 25$.

3.3 Associated Lucas and Lehmer sequences

Our final result shows that linear recurrence Pillai sequences are rather "rare" - at least already associated Lucas and Lehmer sequences do not have the required properties in general.

Theorem 6 *Let $\hat{u} = (\hat{u}_n)_{n=0}^{\infty}$ be an associated Lucas or Lehmer sequence with coprime parameters M, N such that M is odd and N is even. Then $\hat{u}$ is not a Pillai sequence. Further, even $g_{\hat{u}}$ does not exist.*

4 Lemmas and auxiliary results

To prove Theorems 1 and 3 we need several lemmas. The first one shows that investigating Pillai sequences, the case $\gcd(c_1, \ldots, c_r) > 1$ in (1) can be easily treated.

Lemma 7 *Let $u = (u_n)_{n=0}^{\infty}$ be a linear recurrence sequence of order $r \geq 2$ with* $\gcd(c_1, \ldots, c_r) > 1$ *in* (1). *Then u is a Pillai sequence, with $g_u = G_u = 2$.*

PROOF. Write $D := \gcd(c_1, \ldots, c_r)$ and observe that under the assumptions of the lemma for any $i > j \geq r$ by $D \mid \gcd(u_i, u_j)$ we have $\gcd(u_i, u_j) > 1$. Thus u is obviously a Pillai sequence, with $g_u = G_u = 2$. $\square$

The following lemma yields that Lucas and Lehmer sequences are strong divisibility sequences.

Lemma 8 *Let $\hat{u} = (\hat{u}_n)_{n=0}^{\infty}$ be a Lucas or a Lehmer sequence. Then for any $i, j \geq 0$ we have*

$$\gcd(\hat{u}_i, \hat{u}_j) = \hat{u}_{\gcd(i,j)}.$$

PROOF. This is a classical property of such sequences, see e.g. [36] and [31]. $\square$

The next result completely describes the terms with $|u_n| = 1$ and $|\tilde{u}_n| = 1$ of non-degenerate Lucas and Lehmer sequences, respectively. Though the statement easily follows from a well-known, deep theorem of Beukers [5] and the celebrated result of Bilu, Hanrot and Voutier [9] describing the terms u_n and $\tilde{u}_n$ having no primitive prime divisors, we prefer to call it a theorem since it can be of independent interest. In particular, as a simple consequence we get a positive answer to a question of Beukers [5] concerning the sums of multiplicities of 1 and -1 values in non-degenerate Lucas sequences.

Theorem 9 *Let $u = (u_n)_{n=0}^{\infty}$ and $\tilde{u} = (\tilde{u}_n)_{n=0}^{\infty}$ be a non-degenerate Lucas and Lehmer sequence, respectively, both corresponding to the parameters M, N. Then the only term of u with $|u_n| = 1$ is $u_1 = 1$, except for the cases given in Table 3. Similarly, the only terms of $\tilde{u}$ with $|\tilde{u}_n| = 1$ are $\tilde{u}_1 = \tilde{u}_2 = 1$, except for the cases given in Table 4. In the third row of Table 4, F_n stands for the n-th Fibonacci number.*

(M, N)	all indices with $\|u_n\| = 1$
$(\pm 1, N)$, $N \neq 1, 2, 3, 5$	$1, 2$
$(M, M^2 \pm 1)$, $\|M\| \geq 2$	$1, 3$
$(\pm 12, 55)$	$1, 5$
$(\pm 12, 377)$	$1, 5$
$(\pm 1, 3)$	$1, 2, 5$
$(\pm 1, 5)$	$1, 2, 7$
$(\pm 1, 2)$	$1, 2, 3, 5, 13$

Table 3
Non-degenerate Lucas sequences with more than one terms satisfying $|u_n| = 1$.

PROOF. Using the result of Bilu, Hanrot and Voutier [9], it is clear that the equations $|u_n| = 1$ and $|\tilde{u}_n| = 1$ have no solutions for $n > 30$. Moreover, using Tables 1 and 2 of [9] giving all sequences and indices such that the corresponding terms of the corresponding sequences have no primitive prime divisors, one can explicitly find all the ± 1 values in the sequences under investigation. For

(M,N)	all indices with $\lvert\tilde{u}_n\rvert = 1$
$\pm(N\pm 1, N)$, $N \geq 3$	$1,2,3$
$\pm(2N\pm 1, N)$, $N \geq 3$	$1,2,4$
$\pm(F_{n\pm 2}, F_n)$, $n \geq 4$	$1,2,5$
$\pm(1,5)$	$1,2,7$
$\pm(13,4)$	$1,2,7$
$\pm(14,9)$	$1,2,7$
$\pm(5,2)$	$1,2,4,5$
$\pm(3,2)$	$1,2,3,4,7$
$\pm(1,2)$	$1,2,3,5,13$

Table 4
Non-degenerate Lehmer sequences with more than two terms satisfying $|\tilde{u}_n| = 1$.

example, assume that we are interested in Lucas sequences u with $|u_{13}| = 1$. From Table 1 of [9] it follows that $\alpha = \pm\frac{1+\sqrt{-7}}{2}$ and $\beta = \pm\frac{1-\sqrt{-7}}{2}$. Thus we have $M = \pm 1$, $N = 2$. In case of some "small" indices n, the corresponding terms are not explicitly listed in the tables of [9]. In these cases a little more (but rather simple) calculation is needed. We illustrate this by an example. Assume that u is a Lucas sequence with $|u_3| = 1$. Then using $u_0 = 0$, $u_1 = 1$, $u_2 = Mu_1 - Nu_0$ and $u_3 = Mu_2 - Nu_1$, we get $u_3 = M^2 - N$. So $|u_3| = 1$ yields that M is arbitrary and $N = M^2 \pm 1$. However, $M \neq 0$ because u is non-degenerate. (Further note that if $M = \pm 1$ then $|u_2| = 1$ is also valid.) A completely similar argument works for Lehmer sequences, too.

In case of Lucas sequences one can make these calculations much simpler using Theorem 4 of Beukers [5], which explicitly gives all cases where the sequence contains more than two ± 1 values. Note that Beukers [5] calls Lucas sequences with rational integer roots α, β also degenerate. However, as one can easily check, such sequences (assuming that α/β is not a root of unity) contain at most two ± 1 values.

By the help of the above explained method we could determine all ± 1 terms of non-degenerate Lucas and Lehmer sequences, and we just obtained Tables 3 and 4. $\square$

As a simple corollary of Theorem 9 we get the following statement, that provides an affirmative answer to a question of Beukers (see [5] pp. 251 and 252). For any recurrence sequence u and integer ω, denote by $m(\omega)$ the number of occurrences of ω in u.

Corollary 10 *Let u be a non-degenerate Lucas sequence with parameters M, N. Then $m(1)+m(-1) = 1$ (or, more precisely $m(1) = 1$ and $m(-1) = 0$), unless $(M, N) = (\pm 1, N)$ with $N \neq 1, 2, 3, 5$, $(M, M^2 \pm 1)$ with $|M| \geq 2$, $(\pm 1, 2)$, $(\pm 1, 3)$, $(\pm 1, 5)$, $(\pm 12, 55)$, $(\pm 12, 377)$.*

PROOF. Using Table 3, the statement easily follows from Theorem 9. □

The following lemma is Theorem 2.1 of Hajdu and Saradha [23]. It explicitly implies that if T is finite then the original Pillai-type function $G(T)$ (and hence also $g(T)$) exists. We shall use the following notation. For any set T of positive integers let $T(X)$ denote the set of elements t of T with $t \leq X$.

Lemma 11 *Suppose that*

$$|T(X)| \leq \frac{X}{10 \log X} \tag{5}$$

holds for all $X \geq X_1$. Then $g(T)$ and $G(T)$ exist and

$$g(T) \leq G(T) \leq \max(425, 2X_1 + 1).$$

The next lemma provides the values of $g(T)$ and $G(T)$ for certain special choices of T. Note that the case $T = \{1\}$ is covered by the classical result of Brauer [10], while the choices $T = \{1, 2\}$ and $\{1, 2, 3\}$ are settled by Hajdu and Saradha [23].

Lemma 12 *For the sets T occurring in the first column of Table 5, the values of $g(T)$ and $G(T)$ are those occurring in the second and third columns of the table, respectively.*

PROOF. As we have mentioned above, the case $T = \{1\}$ is the original result of Pillai [39], while $T = \{1, 2\}$ and $\{1, 2, 3\}$ are already given by Hajdu and Saradha [23]. In all the other cases we have used the same algorithm as in [23]. Since explaining the whole process in detail would require a lot of preparation, we only illustrate and give the main steps of the method, and refer to [23] for detailed explanation and description. Further, we only write about the case $T = \{1, 2, 5\}$, since all the other cases are similar.

As it has been explained in [23], property $P(T)$ (with the actual choice of T) is *not* valid for some set S_k of k consecutive integers if and only if $K := \{1, \ldots, k\}$ can be covered by the set $L := \{p \;:\; p \text{ prime},\ p \neq 2, 5,\ p < k\} \cup \{4, 10, 25\}$, i.e., if there exists a function $f :\ L \to K$ with the following properties:

T	$g(T)$	$G(T)$
$\{1\}$	17	17
$\{1,2\}$	25	25
$\{1,3\}$	43	43
$\{1,5\}$	31	31
$\{1,2,3\}$	49	53
$\{1,2,4\}$	47	47
$\{1,2,5\}$	45	45
$\{1,2,7\}$	49	51
$\{1,2,4,5\}$	61	69
$\{1,2,3,4,7\}$	81	81
$\{1,2,3,5,13\}$	107	107

Table 5
The values of $g(T)$ and $G(T)$ for some particular sets T.

- for every $\ell \in L$ we have $f(\ell) \leq \ell$,
- $2 \mid f(10) - f(4)$ and $5 \mid f(10) - f(25)$,
- for every $i \in K$ there exists a $j \in K$ with $i \neq j$ and an $\ell \in L$ such that $i \equiv j \equiv f(\ell) \pmod{\ell}$.

Indeed, suppose that we have such a function f. (Note that it is worth to think of f such that it defines the places $f(\ell)$ of the elements of $\ell \in L$ in K. Then $\ell \mid i \in K$ if and only if $\ell \mid i - f(\ell)$.) Then using the Chinese Remainder Theorem, we can find a set $S_k = \{n+1, \ldots, n+k\}$ of k consecutive integers such that for any $\ell \in L$ and $i \in K$, we have $\ell \mid i$ if and only if $\ell \mid n+i$. That is, in this case for any $n+i \in S_k$ we can find an $n+j \in S_k$ such that $n+i \neq n+j$, and $\gcd(n+i, n+j)$ has a divisor from L, whence $\notin T$. In other words, the property $P(T)$ is violated for this set S_k, implying $g(T) \neq k$ and certainly also $G(T) > k$. On the other hand, if $P(T)$ does not hold for some set $S_k = \{n+1, \ldots, n+k\}$, then for any $n+i \in S_k$ we can find an $n+j \in S_k$ such that $n+i \neq n+j$, and $\gcd(n+i, n+j) \notin T$, i.e., it has a divisor from L. Now sending the elements of $\ell \in L$ to the first i such that $\ell \mid n+i$, we clearly obtain a function f with the above required properties.

Thus to find $g(T)$, we need to check all k-s from $k_0 = 17$ up. More precisely, we have to cover, or prove that it is impossible to cover the sets $K = \{1, \ldots, k\}$ in the above sense, for $k \geq 17$. (We know that $g(T) \geq g(\{1\}) = 17$.) For this we apply the corresponding algorithm from [23]. Then we find $g(T) = 45$. Now since $|T| = 3$, by Lemma 11 we obtain that $G(T) \leq 425$. Thus we need to check for coverings of K for the values of k in the interval $45 < k < 425$.

For $k < 60$ one can easily and quickly find coverings just as previously. For the larger values of k, the algorithm gets slower and slower, and some other tool is needed. For these values of k we used the heuristic algorithm from [23], to find a covering for K. Finally, we could produce a covering for all k with $45 < k < 425$, which gives $G(T) = 45$, too. Hence the statement is proved in this particular case. In all the other cases a similar method has been used, and we have just obtained the values of $g(T)$ and $G(T)$ occurring in Table 5. □

To prove Theorem 4, we need the finiteness of the number of elements composed of fixed primes in a non-degenerate recurrence sequence of order ≥ 2. This information easily follows from a deep result of Schlickewei and Schmidt [41] concerning polynomial exponential equations, based upon the subspace theorem.

Lemma 13 *Let $u = (u_n)_{n=0}^{\infty}$ be a non-degenerate linear recurrence sequence of order $r \geq 2$ and $p_1, \ldots, p_s$ be distinct primes. Then the equation*

$$u_n = p_1^{z_1} \ldots p_s^{z_s} \tag{6}$$

has at most $r^{2^{7(s+r)}}$ solutions in non-negative integers $n, z_1, \ldots, z_s$.

PROOF. Using (2) we can rewrite (6) as

$$\sum_{i=1}^{t} P_i(n)\alpha_i^n - p_1^{z_1} \ldots p_s^{z_s} = 0.$$

Hence the statement follows from Theorem 1 of [41] by a simple calculation, similarly to the proof of Theorem 2.1 in [41]. □

To prove Theorem 6, we need a further lemma, due to McDaniel [37].

Lemma 14 *Let $\hat{v} = (\hat{v}_n)_{n=0}^{\infty}$ be an associated Lucas or Lehmer sequence. Choose indices $i, j \geq 0$ and write $i = 2^a i'$, $j = 2^b j'$ with $a, b \geq 0$ and i', j' odd, and put $d = \gcd(i, j)$. Then we have*

$$\gcd(\hat{v}_i, \hat{v}_j) = \begin{cases} \hat{v}_d, & \textit{if } a = b, \\ 1 \textit{ or } 2, & \textit{otherwise.} \end{cases}$$

PROOF. For associated Lucas sequences the statement is part (ii) of the Main Theorem in [37]. As noted on p. 28 in [37], the formula remains valid for associated Lehmer sequences, as well. □

5 Proofs of Theorems 1, 3, and 4

In this section we give the proofs of our results, in the order of the statements.

Proof of Theorem 1. Let $u = (u_n)_{n=0}^{\infty}$ be a non-degenerate Lucas sequence. Observe that the case $\gcd(M, N) > 1$ immediately follows from Lemma 7. Thus from this point on we shall assume that $\gcd(M, N) = 1$.

Consider k consecutive terms $u_n, \ldots, u_{n+k-1}$ of u. Using Lemma 8 we deduce that one of these terms, say u_{n+i} $(0 \le i \le k-1)$ is coprime to all the others if and only if $\gcd(u_{n+i}, u_{n+j}) = u_{\gcd(n+i,n+j)} = \pm 1$ hold for all $j \ne i$ with $0 \le j \le k-1$. Conversely, the above set does not have property P_u if and only if for every $i \in \{0, \ldots, k-1\}$ there exists a $j \in \{0, \ldots, k-1\}$ with $j \ne i$ such that $u_{\gcd(n+i,n+j)} \ne \pm 1$. Put

$$T := \{n \mid u_n = \pm 1 \ (n \ge 0)\}.$$

In view of the above argument, finding $k \ge 2$ consecutive elements of u not having property P_u is equivalent to finding k consecutive integers not having property $P(T)$. Further, if $g(T)$ and $G(T)$ exist, then g_u and G_u also exist, and we have $g_u = g(T)$ and $G_u = G(T)$. Hence using Tables 3 and 5 from Theorem 9 and Lemma 12, respectively, the theorem follows. □

Proof of Proposition 2. Since in case of the Fibonacci sequence we have $(M, N) = (1, -1)$, from Theorem 1 we immediately obtain that F is a Pillai sequence with $g_F = G_F = 25$. Further, using the method illustrated in the proof of Lemma 12, corresponding to the choice $T = \{1, 2\}$ we obtain that for any n, among the numbers $F_n, F_{n+1}, \ldots, F_{n+24}$ one of them is coprime to all the others if and only if we have that either

$$3 \mid n, \quad 4 \mid n+2, \quad 5 \mid n+3, \quad 7 \mid n+4, \quad 11 \mid n+5,$$

$$13 \mid n+7, \quad 17 \mid n, \quad 19 \mid n, \quad 23 \mid n+1$$

or

$$3 \mid n, \quad 4 \mid n+2, \quad 5 \mid n+1, \quad 7 \mid n+6, \quad 11 \mid n+8,$$
$$13 \mid n+4, \quad 17 \mid n+7, \quad 19 \mid n+5, \quad 23 \mid n.$$

Hence the statement easily follows from the Chinese remainder theorem. □

Proof of Theorem 3. When $\gcd(M, N) > 1$, similarly to the proof of Lemma 7 one can easily check that $\tilde{u}$ is a Pillai sequence with $g_{\tilde{u}} = G_{\tilde{u}} = 2$. When $\gcd(M, N) = 1$, in view of Lemma 8, using Table 4 in place of Table 3 from

Theorem 9, one can follow the argument in the proof of Theorem 1 to get the statement. $\square$

Proof of Theorem 4. Assume first that $r = 1$ and u_1 has a prime divisor outside S. Then since for any $i \geq 1$ we have $u_1 \mid \gcd(u_i, u_j)$, we obtain $\gcd(u_i, u_j) \notin T$, so u is a T-Pillai sequence and $g_u(T) = G_u(T) = 2$.

Let now $r \geq 2$, and put

$$H := \{n \mid u_n \in T \ (n \geq 0)\}.$$

Suppose first that $r = 2$. Then by a result of Győry and Pethő [21] we get that u, being a non-degenerate binary divisibility sequence, is a constant multiple of a Lucas sequence. Write $u = tU$ where U is a Lucas sequence, and t is a non-zero integer. Observe that if U is a T-Pillai sequence then so is u, and $g_u(T) \leq g_U(T)$ and $G_u(T) \leq G_U(T)$. Hence without loss of generality we may assume that $t = 1$, or, equivalently, that u is a Lucas sequence. Since by [9] we now that then u_n has a primitive prime divisor for any $n > 30$, and any of the s primes $p_1, \ldots, p_s$ can be a primitive prime divisor of at most one term of u, we obtain that $|H| \leq s + 30$ in this case. Now using Lemma 11, we get that the "classical" Pillai numbers $g(H)$ and $G(H)$ exist. Moreover, a simple calculation shows that

$$G(H) \leq 20(s + 30) \log(s + 30).$$

Let k be chosen such that either $k = g(H)$, or $k \geq G(H)$. Then there exists a non-negative integer n such that the set $\{n, \ldots, n + k - 1\}$ does not have property $P(H)$. That is, for any i with $0 \leq i \leq k-1$ there is a $j \in \{0, \ldots, k-1\}$ with $j \neq i$ such that $\gcd(n+i, n+j) = d \notin H$. However, then by the divisibility property of u and by the definition of H, we have $\gcd(u_{n+i}, u_{n+j}) = u_d \notin T$. This shows that the set $\{u_n, \ldots, u_{n+k-1}\}$ does has property P_u, whence

$$g_u \leq g(H) \quad \text{and} \quad G_u \leq G(H),$$

and the statement follows for $r = 2$.

Assume next that $r \geq 3$. Then by Lemma 13 we get that $|H| \leq r^{2^{7(s+r)}}$ holds. From this point on the proof goes along the same lines as in case of $r = 2$, and after some simple calculations the theorem follows. $\square$

Remark 2. Note that if one is interested only in the original functions g_u and G_u, then in the proof the theorem of Schlickewei and Schmidt [41] could be replaced by a result of Amoroso and Viada [4] concerning the ω-multiplicities of recurrence sequences (applied for the cases $\omega = \pm 1$).

Proof of Theorem 5. Suppose that u (resp. $\tilde{u}$) is a degenerate Lucas (resp. Lehmer) sequence with parameters (M, N). Since the cases where M and N are not coprime follow from Lemma 7 for Lucas sequences, and can be easily checked for Lehmer sequences, we may assume that $\gcd(M, N) = 1$. Let α, β denote the roots of the polynomial $x^2 - Mx + N$ (resp. $x^2 - \sqrt{M}x + N$). Then α/β is a root of unity. One can easily check that α/β is a rational or a quadratic algebraic integer in both cases. Hence α/β is one of the following numbers:

$$\pm 1, \pm i, \pm \varepsilon, \pm \varepsilon^2,$$

where $\varepsilon = (1 + i\sqrt{3})/2$. We pick up only one possibility, the proof goes along the same lines in all the other cases. Suppose that u is a Lucas sequence with $\alpha/\beta = -\varepsilon$. Then we have $M = (1 - \varepsilon)\beta$ and $N = -\varepsilon\beta^2$, whence $M^2 = N$. However, this by the coprimality of M and N yields $(M, N) = (1, 1)$. (When $(M, N) = (-1, 1)$ then $\alpha/\beta \neq -\varepsilon$.) In this case the sequence is given by $0, 1, 1, 0, -1, -1, 0, 1, \ldots$, thus u is not a Pillai sequence, and neither g_u nor G_u exists. By similar calculations, recalling that $N \neq 0$, we obtain that the degenerate Lucas sequences with coprime parameters correspond to one of the pairs

$$(M, N) = (0, 1), (0, -1), (\pm 1, 1), (\pm 2, 1).$$

Now checking these sequences one by one, we get the statement for Lucas sequences.

By a rather similar argument we obtain that the degenerate Lehmer sequences with coprime parameters correspond to one of the pairs

$$(M, N) = \pm(0, 1), \pm(1, 1), \pm(2, 1), \pm(3, 1), \pm(4, 1).$$

Checking these sequences one by one again, the statement follows also for Lehmer sequences. $\square$

Proof of Theorem 6. Let $\hat{v}$ be an associated Lucas or Lehmer sequence, and let $\hat{v}_n, \ldots, \hat{v}_{n+k-1}$ be k consecutive elements of $\hat{v}$ with $k \geq 2$. Take that $i \in \{0, \ldots, k-1\}$ for which $\nu_2(n+i) > \nu_2(n+j)$ for all $j \in \{0, \ldots, k-1\}$ with $j \neq i$. (Here $\nu_2(m)$ denotes the exponent of 2 in the prime factorization of the non-negative integer m, with the convention $\nu_2(0) = \infty$.) Obviously, such an i always exists. Then in view of Lemma 14 we get that $\gcd(\hat{v}_{n+i}, \hat{v}_{n+j}) \leq 2$ for all j as above. Observe that by the choices of M and N, apart from $\hat{v}_0 = 2$, all terms of $\hat{v}$ are odd. Hence we get that in fact $\gcd(\hat{v}_{n+i}, \hat{v}_{n+j}) = 1$ for all $j \in \{0, \ldots, k-1\}$ with $j \neq i$, and the statement follows. $\square$

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