CONSISTENCY CONDITIONS FOR DISCRETE TOMOGRAPHY

LAJOS HAJDU AND ROB TIJDEMAN

Abstract. For continuous tomography Helgason and Ludwig developed consistency conditions. They were used by others to overcome defects in the measurements. In this paper we introduce a consistency criterion for discrete tomography. We indicate how the consistency criterion can be used to overcome defects in measurements.

1. Introduction

Let A be a finite subset of $\mathbb{Z}^2$, and let D be a finite set of at least two directions. By a discrete tomography problem we mean asking for a function $f : A \to K$ which satisfies prescribed line sums along the directions in D, where K is some integral domain, e.g. $\mathbb{Z}$ or $\mathbb{R}$. The authors and others have developed an algebraic theory of the structure of the solutions of a discrete tomography problem, see [1-13], [17-18]. If $K = \mathbb{R}$, the solutions of a discrete tomography problem form a linear manifold if there is at least one real solution. If $K = \mathbb{Z}$ the solutions form a grid in this linear manifold, provided that at least one integer solution exists.

It is obvious that the sum of the line sums in one direction equals the sum of the line sums in any other direction. So the line sums are linearly dependent. In [9] the authors determined the maximal number of linearly independent line sums in case $K = \mathbb{Z}$ and A has the shape of a full rectangle with sides parallel to the coordinate axes. In [11] they extended this to higher dimensions and in [12] to the case that A is convex. Some years earlier Van Dalen had formulated two conjectures on global dependencies (i.e. dependencies with coefficients independent of A) and proved some special cases of them. An extensive theory on dependencies for dimension two was developed by Stolk and

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Batenburg [17], and for more than two by Stolk [16] in his PhD thesis. Stolk, [16] Chapter 4, gave algorithms to produce bases for the linear space of dependencies. He considered mainly convex A, but also some non-convex A and infinite periodic sets A.

In this paper we restrict our attention to dimension two and global dependencies. We make results of Van Dalen and Stolk and Batenburg more explicit. In Section 2 we introduce notation and in Section 3 we summarize the earlier published results which are relevant to this paper. In Section 4 we give explicit formulas for global dependencies. In Section 5 we provide recurrences for the coefficients of global dependencies and in Section 6 we show that every global dependency is a linear combination of the dependencies constructed in Section 4. Finally, in Section 7, we use our constructions to overcome defects in measurements. This can be considered as the analogue for discrete tomography of the application of the Helgason-Ludwig consistency conditions for continuous tomography, cf. [15] Section II.4, after Helgason [13] and Ludwig [14]. To keep the presentation simple, we shall formulate the new results for $K = \mathbb{R}$. These theorems can be adapted e.g. to $K = \mathbb{Z}, \mathbb{Q}, \mathbb{C}$.

2. Notation

Let A be a finite subset of $\mathbb{Z}^2$. We call A *convex* if every $\mathbf{a} \in \mathbb{Z}^2$ which belongs to the closed convex hull $H(A)$ of A belongs to A itself. A lattice Λ is a set of the form $\mathbb{Z}\mathbf{d}_1 + \mathbb{Z}\mathbf{d}_2$ for some $\mathbf{d}_1, \mathbf{d}_2 \in \mathbb{R}^2$ with $\mathbf{d}_1, \mathbf{d}_2$ linearly independent over $\mathbb{R}$. It has lattice determinant $|\det(\mathbf{d}_1, \mathbf{d}_2)|$. An *affine lattice* of a lattice Λ is a set Λ' such that $\Lambda' - \mathbf{a} = \Lambda$ for some $\mathbf{a} \in \mathbb{Z}^2$. The lattice Λ generates exactly $|\det(\mathbf{d}_1, \mathbf{d}_2)|$ distinct affine lattices and their union is $\mathbb{Z}^2$.

A nonzero vector $\mathbf{d} = (d, e) \in \mathbb{Z}^2$ such that $d \geq 0$ is called a *direction*. If $d = 0$ we require $e > 0$. If d and e are coprime, we call $\mathbf{d}$ a *primitive direction*. Let $D = \{(d_1, e_1), \dots, (d_k, e_k)\}$ be a set of primitive directions. As the case $k = 1$ is trivial, we shall assume $k > 1$ throughout the paper. By Λ_D we denote the lattice generated by the directions of D,

$$\Lambda_D = \{\lambda_1(d_1, e_1) + \cdots + \lambda_k(d_k, e_k) \ : \ \lambda_1, \dots, \lambda_k \in \mathbb{Z}\}. \tag{1}$$

Let K be an *integral domain*, i.e. a commutative ring with 1 and without zero divisors. A function $f : A \to K$ is called a *table*. For $1 \leq i \leq k$ and $j \in \mathbb{Z}$ we denote by $\ell(i, j)$ the *line sum* of the $\mathbb{Z}$-line

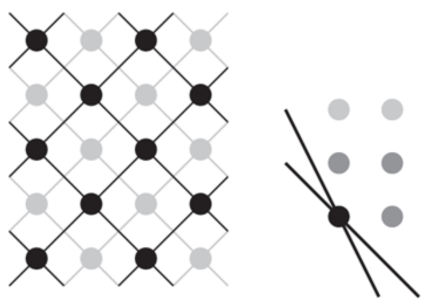
$e_i x - d_i y = j$, given by

$$\ell(i,j) = \sum_{(x,y)\in A,\ e_i x - d_i y = j} f(x,y). \tag{2}$$

We call $\ell(i,j)$ a *line sum in the direction* (d_i, e_i). A (homogeneous linear) *dependency* between the line sums is a relation of the form

$$\sum_{i=1}^{k} \sum_{j\in\mathbb{Z}} c_{i,j}\ell(i,j) = 0 \tag{3}$$

where $c_{i,j}$ are coefficients from K depending only on A, D, i and j. Since $\ell(i,j) = 0$ for all but finitely many j, this sum is well-defined. The dependencies form a module over K (which is a linear space if K is a field). The dimension of this module (or linear space) is denoted by $v(A)$. We define a *global dependency* to be a dependency (3) that is valid for every finite set $A \subset \mathbb{Z}^2$, thus with coefficients $c_{i,j}$ independent of A. The global dependencies form again a module (or a linear space) over K, the dimension of which is denoted by v_g.



(a) (b)

FIGURE 1. Global and local dependencies: (a) global dependencies, (b) a local dependency.

Figure 1 illustrates global and local dependencies. Figure 1 (a) (borrowed from p. 56 of Stolk's thesis [16]) shows a global dependency. The sum of the diagonal sums through the black points equals the sum of the antidiagonal sums through the black points. The same is true for the line sums through the grey points, yielding another global dependency. Figure 1 (b) (taken from p. 23 of [16]) gives an example of a local dependency: the line sums through the bottom left point in the two indicated directions are equal. This is a local property indeed: it is independent of the global dependency which says that the sum of all line sums in the one direction equals that in the other direction.

To keep the presentation simple, from this point on we shall work exclusively with the choice $K = \mathbb{R}$.

3. Conjectures and results of Van Dalen and Stolk-Batenburg

In her Master thesis [5] Van Dalen conjectured:

Conjecture 3.1. *There are* $w := \sum_{1 \leq h < i \leq k} |\det(\mathbf{d}_h, \mathbf{d}_i)|$ *linearly independent global dependencies, that is* $v_g = w$.

Van Dalen proved that for every subset D' of D of cardinality $\kappa \geq 2$ there exists a dependency of the form

$$\sum_{i \in D'} c_i \sum_{j \in \mathbb{Z}} j^{\kappa - 2} \ell(i, j) = 0 \tag{4}$$

where the coefficients $c_i \in \mathbb{R}$ are not all equal to zero. Comparing (4) and (3), we see that in this case the coefficients $c_{i,j}$ in (3) are of the form $c_i j^{\kappa-2}$. She called it a *dependency of the power* $\kappa - 2$. She further proved that there does not exist such a dependency of power larger than $\kappa - 2$ and that the dependency (4) is unique apart from a multiplicative factor. Van Dalen made a more refined conjecture:

Conjecture 3.2. *The maximal number of independent global dependencies of power* κ *is equal to*

$$g_{\kappa+2} := \sum_{j=0}^{k-\kappa-2} (-1)^j \binom{\kappa + j}{j} G_{\kappa+j+2} \tag{5}$$

where

$$G_t = \sum_{|I|=t} \gcd_{h,i \in I} |d_h e_i - d_i e_h|$$

and the sum runs over the sets $I \subset \{1, 2, \ldots, k\}$ *of cardinality* $t \geq 2$.

She checked that the total number of linearly independent global dependencies according to this conjecture equals G_2 as it should according to Conjecture 3.1. She proved Conjecture 3.2 for dependencies of power $k - 2$ and $k - 3$, and from that for $k = 3$ and $k = 4$.

Conjecture 3.1 has been proved by Stolk and Batenburg ([17] Theorem 4.2, cf. [16] Theorem 3.5.1):

Theorem 3.1. *The dimension* v_g *of the linear space of the global dependencies equals* w.

If A is small or thin, then different global dependencies may coincide on A so that $v(A) < v_g = w$. However, under certain assumptions on A, Stolk and Batenburg could guarantee that $v(A) = w$, see [17] Theorem 9.2. Finally, we mention another theorem of Stolk and Batenburg ([17] Lemma 3.4).

Theorem 3.2 (Stolk-Batenburg)**.** *A real-valued vector represents the line sums of a real-valued table precisely if it satisfies all dependencies.*

4. Explicit expressions for global dependencies

Let D be a finite set of primitive directions. In this section we give explicit expressions for global dependencies of any degree.

We recall that Van Dalen [5] proved that if D' is a subset of cardinality $\kappa \geq 2$, then there exists a global dependency of the form (4) with the integers c_i not all equal to zero and uniquely determined (up to a nonzero factor). Moreover, she proved that there does not exist such a dependency with $\kappa - 2$ replaced by a larger number. We shall express the coefficients c_i of the unique global dependency explicitly.

We denote the determinant of the $m \times m$ matrix with h-th column vector $\mathbf{x}_h = (x_{1,h}, x_{2,h}, \ldots, x_{m,h})$ by $\det(\mathbf{x}_1, \ldots, \mathbf{x}_m)$. Furthermore, we denote the determinant of the matrix which we obtain by omitting its first column vector and its i-th row vector by $\det(\mathbf{x}_2, \ldots, \mathbf{x}_m)_i$.

The following lemma provides explicit expressions for the dependencies. An empty product equals 1.

Lemma 4.1. *Let $f : \mathbb{Z}^2 \to \mathbb{R}$ be a table. Let $D_\kappa = \{\mathbf{d}_1, \ldots, \mathbf{d}_\kappa\}$ be a set of κ primitive directions with $\mathbf{d}_i = (d_i, e_i)$ for $i = 1, \ldots, \kappa$ with $\kappa \geq 2$. Put $\mathbf{d}^s = (d_1^s, d_2^s, \ldots, d_\kappa^s)$ and $\mathbf{e}^s = (e_1^s, e_2^s, \ldots, e_\kappa^s)$ for $s = 0, 1, \ldots, \kappa$. Let E_i be defined by*

$$E_i = \prod_{\substack{1 \leq h < j \leq \kappa \\ h, j \neq i}} (d_h e_j - d_j e_h).$$

Then

$$\sum_{i=1}^{\kappa} (-1)^{i-1} E_i \sum_{j \in \mathbb{Z}} j^{\kappa-2} \ell(i,j) = 0. \tag{6}$$

Proof. Obviously, for $s = 0, 1, \ldots, \kappa - 2$ we have

$$\det(\mathbf{d}^s \mathbf{e}^{\kappa-2-s}, \mathbf{d}^{\kappa-2}, \mathbf{d}^{\kappa-3}\mathbf{e}, \mathbf{d}^{\kappa-4}\mathbf{e}^2, \ldots, \mathbf{e}^{\kappa-2}) = 0.$$

By developing to the first column we obtain, for $s = 0, 1, \ldots, \kappa - 2$,

$$\sum_{i=1}^{\kappa} (-1)^{i-1} d_i^s e_i^{\kappa-2-s} \det(\mathbf{d}^{\kappa-2}, \mathbf{d}^{\kappa-3}\mathbf{e}, \mathbf{d}^{\kappa-4}\mathbf{e}^2, \ldots, \mathbf{e}^{\kappa-2})_i = 0.$$

It follows that, for arbitrary integers x and y,

$$\sum_{i=1}^{\kappa} (-1)^{i-1} (e_i x - d_i y)^{\kappa-2} \det(\mathbf{d}^{\kappa-2}, \mathbf{d}^{\kappa-3}\mathbf{e}, \mathbf{d}^{\kappa-4}\mathbf{e}^2, \ldots, \mathbf{e}^{\kappa-2})_i = 0. \tag{7}$$

Observe that $\det(\mathbf{d}^{\kappa-2}, \mathbf{d}^{\kappa-3}\mathbf{e}, \mathbf{d}^{\kappa-4}\mathbf{e}^2, \ldots, \mathbf{e}^{\kappa-2})_i$ is the Vandermonde determinant E_i. It follows that

$$\sum_{i=1}^{\kappa} \sum_{j \in \mathbb{Z}} \sum_{\substack{x, y \in \mathbb{Z} \\ e_i x - d_i y = j}} f(x, y)\ (-1)^{i-1} (e_i x - d_i y)^{\kappa-2} E_i = 0.$$

Thus

$$0 = \sum_{i=1}^{\kappa} \sum_{j \in \mathbb{Z}} (-1)^{i-1} j^{\kappa-2} E_i \sum_{\substack{x, y \in \mathbb{Z} \\ e_i x - d_i y = j}} f(x, y) =$$

$$\sum_{i=1}^{\kappa} \sum_{j \in \mathbb{Z}} (-1)^{i-1} E_i j^{\kappa-2} \ell(i, j).$$

□

Remark 4.1. Note $E_i \neq 0$ for $i = 1, \ldots, \kappa$.

Corollary 4.1 (Van Dalen [5], Theorem 3.3)**.** *In the notation of Lemma 4.1 there does not exist a dependency of the form*

$$\sum_{i=1}^{\kappa} c_i \sum_{j \in \mathbb{Z}} j^{\kappa-1} \ell(i, j) = 0,$$

with coefficients $c_i \in \mathbb{R}$ not all equal to 0.

Proof. Suppose such a dependency exists. We apply the dependency to the table for which $f(x, y) = 1$ in some point $(x', y') \in \mathbb{Z}^2$ and $f(x, y) = 0$ in all other points. Then we obtain

$$0 = \sum_{i=1}^{\kappa} c_i (e_i x' - d_i y')^{\kappa-1} = \sum_{i=1}^{\kappa} c_i \sum_{j=0}^{\kappa-1} \binom{\kappa-1}{j} e_i^j d_i^{\kappa-1-j} (x')^j (-y')^{\kappa-j-1}.$$

Since the polynomial on the right-hand side is identically 0 on $\mathbb{Z}^2$ and the coefficient determinant is a nonzero constant multiple of $\prod_{h,i;h<i}(d_h e_i - d_i e_h)$, the coefficients c_i have all to be 0. □

An immediate consequence of Corollary 4.1 is that there does not exist a dependency of degree $> \kappa - 2$.

Corollary 4.2 (Van Dalen [5], Corollary 3.4)**.** *The coefficients in* (6) *are uniquely determined apart from a multiplicative factor.*

Proof. Suppose there exist two linearly independent dependencies. Then we can form a linear combination of them with one coefficient equal to 0. Thus we have a nontrivial dependency for $\kappa - 1$ directions of degree $\kappa - 2$. This contradicts Corollary 4.1 applied with $\kappa - 1$ directions. $\square$

If $\gcd_{h<j} |d_h e_j - d_j e_h| = m > 1$, there are even m linearly independent relations of degree m. To see why, we first prove the following lemma.

Lemma 4.2. *Let be given a set of κ primitive directions $D_\kappa = \{(d_1, e_1), \dots, (d_\kappa, e_\kappa)\}$. Set $\gcd_{1 \le h < i \le \kappa} |d_h e_i - d_i e_h| = m$. Then $\mathbb{Z}^2$ splits into m affine lattices of the lattice Λ_{D_κ} such that all the $\mathbb{Z}$-lines through some point $(x_0, y_0) \in \mathbb{Z}^2$ in a direction of D_κ belong to just one affine lattice.*

Proof. Consider any two elements of Λ_{D_κ},
$\mathbf{p} = (p_1, p_2) = \lambda_1(d_1, e_1) + \lambda_2(d_2, e_2) + \dots + \lambda_\kappa(d_\kappa, e_\kappa)$ and
$\mathbf{q} = (q_1, q_2) = \mu_1(d_1, e_1) + \mu_2(d_2, e_2) + \dots \mu_\kappa(d_\kappa, e_\kappa)$, say.
Observe that every $\mathbb{Z}$-line in a direction of D_κ which contains one point of Λ_{D_κ} is entirely contained in Λ_{D_κ}. The determinant of the lattice spanned by $\mathbf{p}$ and $\mathbf{q}$, which equals $p_1 q_2 - p_2 q_1 = \sum_{h \ne j} \lambda_h \mu_j (d_h e_j - d_j e_h)$, is divisible by m. Since $\mathbf{p}$ and $\mathbf{q}$ are arbitrary, the lattice determinant of Λ_{D_κ} is divisible by m. It follows that there are m distinct affine lattices of $\Lambda_{D\kappa}$. Since each is a translate of $\Lambda_{D\kappa}$, every $\mathbb{Z}$-line in a direction of D_κ which contains one point of an affine lattice is entirely contained in that affine lattice. $\square$

Lemma 4.3. *Let $D = \{(d_1, e_1), \dots, (d_k, e_k)\}$ be a finite set of primitive directions. Let $m = \gcd_{1 \le h < i \le k} |d_i e_h - e_i d_h|$. Then every point (x, y) of Λ_D satisfies $m | (d_i y - e_i x)$ for $i = 1, \dots, k$. Moreover, for each $i \in \{1, \dots, k\}$ there is a bijection between the affine lattices of Λ_D and the residue classes modulo m such that for every i and every point (x, y) in a fixed affine lattice of Λ_D the values $d_i y - e_i x$ are in the same equivalence class modulo m.*

Proof. Every point (x, y) of Λ_D can be written as $\lambda_1(d_1, e_1) + \dots + \lambda_k(d_k, e_k)$ with $\lambda_1, \dots, \lambda_k \in \mathbb{Z}$. Since $m | (d_i e_h - e_i d_h)$ for all h, i, we have $m | (d_i y - e_i x)$ for all i. This proves the first statement.

Let $(x_0, y_0) \in \mathbb{Z}^2$. Let Γ_D be the affine lattice of Λ_D to which (x_0, y_0) belongs. Then, by the first statement, for every $(x, y) \in \Gamma_D$,

$$d_i y - e_i x \equiv d_i y_0 - e_i x_0 \pmod{m} \text{ for } i = 1, \dots, k.$$

Fix i. Since $\gcd(d_i, e_i) = 1$, for every residue class mod m there exist integers x, y such that $d_i y - e_i x$ belongs to that residue class. As there are exactly m affine lattices of Λ_D, the m affine lattices have distinct values modulo m. (In particular, if $m|(d_i y - e_i x)$ for some i, then $(x, y) \in \Lambda_D$.) □

Theorem 4.1. *Let $f : \mathbb{Z}^2 \to \mathbb{R}$ be a table. Let be given κ primitive directions $D_\kappa = \{(d_1, e_1), \ldots, (d_\kappa, e_\kappa)\}$ with $\gcd_{1\le h<i\le\kappa}|d_h e_i - d_i e_h| = m$. For $(x_0, y_0) \in \mathbb{Z}^2$ let $e_i x_0 - d_i y_0 = j_i$ for $i = 1, \ldots, \kappa$. Then*

$$\sum_{i=1}^{\kappa}(-1)^{i-1}E_i \sum_{j\equiv j_i(\mathrm{mod}\ m)} j^{\kappa-2}\ell(i, j) = 0. \tag{8}$$

Proof. According to Lemmas 4.2 and 4.3 the points (x, y) in the affine lattice to which (x_0, y_0) belongs satisfy $e_i x - d_i y \equiv j_i \pmod m$ for $i = 1, \ldots, \kappa$ and no other points do so. We obtain from (7)

$$\sum_{i=1}^{\kappa} \sum_{e_i x - d_i y\equiv j_i(\mathrm{mod}\ m)} f(x, y)(-1)^{i-1}(e_i x - d_i y)^{\kappa-2}E_i = 0.$$

Thus

$$\begin{aligned} 0 &= \sum_{i=1}^{\kappa} \sum_{j\equiv j_i(\mathrm{mod}\ m)} (-1)^{i-1}j^{\kappa-2}E_i \sum_{e_i x - d_i y = j_i} f(x, y) \\ &= \sum_{i=1}^{\kappa} \sum_{j\equiv j_i(\mathrm{mod}\ m)} (-1)^{i-1}E_i j^{\kappa-2}\ell(i, j). \end{aligned}$$

□

Every linear combination of dependencies (with real coefficients) is also a dependency. We call a linear combination of dependencies constructed by Theorem 4.1 *polynomial*. We shall prove in Theorem 6.2 that the dependencies constructed in Theorem 4.1 generate all the global dependencies. Thus every global dependency is polynomial.

The dependencies generated by Theorem 4.1 are not independent. For example, according to Lemma 4.1 with $k = 3$ we find the three dependencies $\sum_j \ell(1, j) - \sum_j \ell(2, j) = 0, \sum_j \ell(1, j) - \sum_j \ell(3, j) = 0, \sum_j \ell(2, j) - \sum_j \ell(3, j) = 0$ which are dependent, as the third is the difference of the first two. Theorem 3.1 implies that the number of linearly independent global dependencies equals w. According to Conjecture 3.2 the number of linearly independent global dependencies of degree κ generated by Theorem 4.1 is at most $g_{\kappa+2}$. We conjecture that it equals $g_{\kappa+2}$.

Example 4.1. Consider the four directions

$$D = \{\mathbf{d}_1 = (1,0), \mathbf{d}_2 = (0,1), \mathbf{d}_3 = (1,1), \mathbf{d}_4 = (1,-1)\}.$$

Set $\delta_{h,i} = d_h e_i - d_i e_h$ for $1 \le h < i \le 4$. Then $|\delta_I| = 1$ for all $I \subseteq \{1,2,3,4\}$ with $|I| \ge 2$ except for $|\delta_{3,4}| = 2$. Hence according to Theorem 3.1 there is a basis of

$$|\delta_{1,2}| + |\delta_{1,3}| + |\delta_{1,4}| + |\delta_{2,3}| + |\delta_{2,4}| + |\delta_{3,4}| = 7$$

global dependencies. With Theorem 4.1 we find for $\kappa = 2$ the global dependencies $\sum_j \ell(1,j) = \sum_j \ell(2,j), \sum_j \ell(1,j) = \sum_j \ell(3,j), \sum_j \ell(1,j) = \sum_j \ell(4,j), \sum_j \ell(2,j) = \sum_j \ell(3,j), \sum_j \ell(2,j) = \sum_j \ell(4,j), \sum_{j\ even} \ell(3,j) = \sum_{j\ even} \ell(4,j), \sum_{j\ odd} \ell(3,j) = \sum_{j\ odd} \ell(4,j)$.
Since $\sum_j \ell(1,j) = \sum_j \ell(2,j) = \sum_j \ell(3,j) = \sum_j \ell(4,j)$ the equations $\sum_j \ell(2,j) = \sum_j \ell(3,j), \sum_j \ell(2,j) = \sum_j \ell(4,j)$ and $\sum_{j\ odd} \ell(3,j) = \sum_{j\ odd} \ell(4,j)$ depend on the other four equations which are linearly independent. (It follows from Van Dalen's proof of Conjecture 3.2 in case $k = 4$ that there are exactly four linearly independent equations with constant coefficients.)

Two linearly independent global dependence relations with coefficients linear in j can be found with Theorem 4.1. The directions $(1,0), (0,1), (1,1)$ yield the equation

$$\sum_j j\ell(1,j) + \sum_j j\ell(2,j) - \sum_j j\ell(3,j) = 0.$$

The directions $(1,0), (0,1), (1,-1)$ yield the independent equation

$$\sum_j j\ell(1,j) - \sum_j j\ell(2,j) - \sum_j j\ell(4,j) = 0.$$

The other equations with linear coefficients are dependent on these two.

Finally, according to Theorem 4.1 there is a dependency with quadratic coefficients, unique up to a constant factor,

$$2\sum_j j^2\ell(1,j) + 2\sum_j j^2\ell(2,j) - \sum_j j^2\ell(3,j) - \sum_j j^2\ell(4,j) = 0.$$

The $4 + 2 + 1 = 7$ equations are linearly independent and therefore form a basis. Actually they are equivalent with the seven equations found in [9]. This shows that the seven dependencies found in that paper are all global. □

Example 4.2. Consider the four directions

$$D = \{\mathbf{d}_1 = (1,1), \mathbf{d}_2 = (1,-1), \mathbf{d}_3 = (1,-3), \mathbf{d}_4 = (3,2)\}.$$

Set $\delta_{h,i} = d_h e_i - d_i e_h$ for $1 \le h < i \le 4$. We have $\delta_{1,2} = -2, \delta_{1,3} = -4, \delta_{1,4} = -1, \delta_{2,3} = -2, \delta_{2,4} = 5, \delta_{3,4} = 11$. Then according to Theorem 3.1 there is a basis of

$$|\delta_{1,2}| + |\delta_{1,3}| + |\delta_{1,4}| + |\delta_{2,3}| + |\delta_{2,4}| + |\delta_{3,4}| = 25$$

global dependencies. With Theorem 4.1 we find for $\kappa = 2$ the global dependencies
$\sum_{j\equiv h (\mathrm{mod}\ 2)} \ell(1, j) = \sum_{j\equiv h (\mathrm{mod}\ 2)} \ell(2, j)$ for $h = 0, 1$,
$\sum_{j\equiv h (\mathrm{mod}\ 4)} \ell(1, j) = \sum_{j\equiv h (\mathrm{mod}\ 4)} \ell(3, j)$ for $h = 0, 1, 2, 3$,
$\sum_j \ell(1, j) = \sum_j \ell(4, j)$,
$\sum_{j\equiv h (\mathrm{mod}\ 2)} \ell(2, j) = \sum_{j\equiv h (\mathrm{mod}\ 2)} \ell(3, j)$ for $h = 0, 1$,
$\sum_{j\equiv h (\mathrm{mod}\ 5)} \ell(2, j) = \sum_{j\equiv h (\mathrm{mod}\ 5)} \ell(4, j)$ for $h = 0, 1, \dots, 4$,
$\sum_{j\equiv h (\mathrm{mod}\ 11)} \ell(3, j) = \sum_{j\equiv h (\mathrm{mod}\ 11)} \ell(4, j)$ for $h = 0, 1, \dots, 10$.
As in the previous example from each of the last three series of equations one can be omitted, because they are dependent on the others. Moreover another equation is dependent on the remaining equations among the first three series with h even. The remaining 21 equations are linearly independent.

Three linearly independent global dependence relations with non-constant coefficients can be found with Theorem 4.1. The directions $(1, 1), (1, -1), (1, -3)$ yield the linear equation with linear coefficients in j

$$\sum_{j \text{ even}} j\ell(1, j) - 2 \sum_{j \text{ even}} j\ell(2, j) + \sum_{j \text{ even}} j\ell(3, j) = 0,$$

and similarly for j odd. The directions $(1, -1), (1, -3), (3, 2)$ yield the independent linear equation with linear coefficients

$$11 \sum_j j\ell(2, j) - 5 \sum_j j\ell(3, j) - 2 \sum_j j\ell(4, j) = 0.$$

The two equations which can be generated for the other triples of directions depend on these equations.

Finally, according to Theorem 4.1 there is a dependency with quadratic coefficients, unique up to a constant factor,

$$55 \sum_j j^2\ell(1, j) + 22 \sum_j j^2\ell(2, j) - 5 \sum_j j^2\ell(3, j) - 8 \sum_j j^2\ell(4, j) = 0.$$

The $21 + 3 + 1 = 25$ remaining equations form a basis for the global dependencies.

5. Recurrence relations for the coefficients

We start with an assertion which in fact had already been noticed (though not explicitly formulated) by Van Dalen [5], p. 13.

Let $\mathbf{a} = (a, b) \in A$. Consider a global dependency (3). Denote by $c_i(\mathbf{a})$ the value of $c_{i,j}$ for which $e_i a - d_i b = j$. Thus $\sum_{i=1}^{k} c_i(\mathbf{a})$ is the sum of all the coefficients of the lines through $\mathbf{a}$ in the directions of D.

Lemma 5.1. *For every i, j with $i = 1, \ldots, k$ and $j \in \mathbb{Z}$ we have a global dependency* (3) *if and only if $\sum_{i=1}^{k} c_i(\mathbf{a}) = 0$ for every $\mathbf{a} \in \mathbb{Z}^2$.*

Proof. If (3) holds for every set A, then it holds for every singleton $\{\mathbf{a}\}$. All the lines in the directions of D through $\{\mathbf{a}\}$ yield the same line sum $f(\mathbf{a})$. Hence we find from (3) for all $\mathbf{a} \in \mathbb{Z}^2$ that $0 = \sum_{i=1}^{k} \sum_{j \in \mathbb{Z}} c_{i,j} \ell(i, j) = f(\mathbf{a}) \sum_{i=1}^{k} c_i(\mathbf{a})$. We are free to choose $f(\mathbf{a}) \neq 0$.

On the other hand, suppose for every $\mathbf{a} \in \mathbb{Z}^2$ we have $\sum_{i=1}^{k} c_i(\mathbf{a}) = 0$. Let $A \subset \mathbb{Z}^2$. Let $f : A \to \mathbb{R}$ be any table. Then

$$\sum_{i=1}^{k} \sum_{j \in \mathbb{Z}} c_{i,j} \ell(i, j) = \sum_{i=1}^{k} \sum_{(a,b) \in A,\ e_i a - d_i b = j} c_{i,j} f((a, b)) = \sum_{\mathbf{a} \in A} f(\mathbf{a}) \sum_{i=1}^{k} c_i(\mathbf{a}) = 0.$$

□

The following theorem displays the structure of the coefficients $c_{i,j}$. It extends Van Dalen's result on the degrees of the coefficients of global dependencies. For a less explicit existence result see [17], Theorem 5.2 or [16], Theorem 3.5.5.

Theorem 5.1. *Let $D = \{(d_1, e_1), \ldots, (d_k, e_k)\}$ be a finite set of primitive directions. For any i with $1 \leq i \leq k$ let $c_{i,j}$ be the (real) coefficient of a global dependency corresponding to the line $e_i x - d_i y = j$ for $j \in \mathbb{Z}$. Set $\delta_{h,i} = d_h e_i - d_i e_h$ for all h and*

$$\prod_{h=1,\ h \neq i}^{k} (1 - x^{\delta_{h,i}}) = \sum_{n \in \mathbb{Z}} r_{i,n} x^n. \tag{9}$$

Then $(c_{i,j})_{j \in \mathbb{Z}}$ satisfies the recurrence relation

$$\sum_{n \in \mathbb{Z}} r_{i,n} c_{i,j+n} = 0 \tag{10}$$

of order $\sum_{h=1,h \neq i}^{k} |\delta_{h,i}|$.

The following consequence of Theorem 5.1 makes a remark of Stolk at page 131 of [16] explicit.

Corollary 5.1. *Under the conditions of Theorem 5.1 let be given a global dependence relation* (3) *and some* $i \in \{1, \dots, k\}$. *Then* $c_{i,j} = O(j^{k-2})$ *for* $|j| \to \infty$. *If* $\gcd_{h \neq i}|\delta_{h,i}| = 1$, *then there is a constant* $c_i \in \mathbb{R}$ *such that* $c_{i,j} = c_i j^{k-2} + O(j^{k-3})$ *for* $|j| \to \infty$.

The following variant of Corollary 5.1 states that restricted to the lattice Λ_D for every $i \in \{1, \dots, k\}$ the sequence $(c_{i,j})_{j \in \mathbb{Z}}$ has a dominant term of degree $k-2$.

Corollary 5.2. *In the notation of Theorem 5.1, let* $m = \gcd_{h,i}|\delta_{h,i}|$. *Then for* $i = 1, \dots, k; \mu = 0, 1, \dots, m-1$ *there is a constant* $c^*_{i,\mu} \in \mathbb{R}$ *such that* $c_{i,jm+\mu} = c^*_{i,\mu} j^{k-2} + O(j^{k-3})$ *for* $|j| \to \infty$.

In the proof of Theorem 5.1 we consider the hull of the projection of a parallelopiped in $\mathbb{Z}^k$ to $\mathbb{Z}^2$ with sides of the projection in the directions of D. The only two integer points on a side of the projection have opposite values so that the corresponding line sum vanishes.

Proof of Theorem 5.1. Without loss of generality we may assume $i = 1$. Let $(x_0, y_0) \in \mathbb{Z}^2$ with $e_1 x_0 - d_1 y_0 = j$. Let $H \subset \{2, \dots, k\}$. We apply Lemma 5.1 to each point of the form $(x_0, y_0) + \sum_{h \in H}(d_h, e_h)$ and obtain

$$c_{1,j+\sum_{h\in H}\delta_{1,h}} + \sum_{\kappa=2}^{k} c_{\kappa, e_\kappa x_0 - d_\kappa y_0 + \sum_{h\in H}\delta_{\kappa,h}} = 0.$$

Hence,

$$\sum_H (-1)^{|H|} \left(c_{1,j+\sum_{h\in H}\delta_{1,h}} + \sum_{\kappa=2}^{k} c_{\kappa, e_\kappa x_0 - d_\kappa y_0 + \sum_{h\in H}\delta_{\kappa,h}} \right) = 0. \tag{11}$$

Observe that for $\kappa \notin H$, by $\delta_{\kappa,\kappa} = 0$,

$$c_{\kappa, e_\kappa x_0 - d_\kappa y_0 + \sum_{h\in H}\delta_{\kappa,h}} = c_{\kappa, e_\kappa x_0 - d_\kappa y_0 + \sum_{h\in H\cup\{\kappa\}}\delta_{\kappa,h}},$$

and that these terms in (11) have opposite signs, since H contains one term less than $H \cup \{\kappa\}$. It follows that the terms for $\kappa > 1$ in (11) cancel in pairs. Thus we obtain the recurrence

$$\sum_H (-1)^{|H|} c_{1,j+\sum_{h\in H}\delta_{1,h}} = 0 \text{ for } j \in \mathbb{Z}. \tag{12}$$

Note that

$$\sum_{n\in\mathbb{Z}} r_{1,n} x^n = \prod_{h=2}^{k} (1 - x^{\delta_{1,h}}) = \sum_H (-1)^{|H|} x^{\sum_{h\in H}\delta_{1,h}}.$$

Hence $r_{1,n} = \sum_{\sum_{h\in H} \delta_{1,h}=n} (-1)^{|H|}$. It follows from (12) that, for $j \in \mathbb{Z}$,

$$\sum_{n\in\mathbb{Z}} r_{1,n} c_{1,j+n} = \sum_{n\in\mathbb{Z}} \sum_{\sum_{h\in H} \delta_{1,h}=n} (-1)^{|H|} c_{1,j+n} = 0.$$

□

Example 5.1. [Continuation of Example 4.2]
We consider the case $i = 1, (x_0, y_0) = (0, 0)$ of the proof of Theorem 5.1. We find by Lemma 5.1 for $H = \{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}$, respectively, using that $\delta_{h,i} = -\delta_{i,h}, \delta_{h,h} = 0$ for all h, i and $\delta_{1,2} = -2, \delta_{1,3} = -4, \delta_{1,4} = -1, \delta_{2,3} = -2, \delta_{2,4} = 5, \delta_{3,4} = 11$:

$$\begin{gathered}
c_{1,0} + c_{2,0} + c_{3,0} + c_{4,0} = 0 \\
c_{1,-2} + c_{2,0} + c_{3,2} + c_{4,-5} = 0 \\
c_{1,-4} + c_{2,-2} + c_{3,0} + c_{4,-11} = 0 \\
c_{1,-1} + c_{2,5} + c_{3,11} + c_{4,0} = 0 \\
c_{1,-6} + c_{2,-2} + c_{3,2} + c_{4,-16} = 0 \\
c_{1,-3} + c_{2,5} + c_{3,13} + c_{4,-5} = 0 \\
c_{1,-5} + c_{2,3} + c_{3,11} + c_{4,-11} = 0 \\
c_{1,-7} + c_{2,3} + c_{3,13} + c_{4,-16} = 0
\end{gathered}$$

(The second indices can (also) be obtained by applying Lemma 5.1 to the functions $x-y, -x-y, -3x-y, 2x-3y$ at the points $(0, 0), (-1, 1), (-1, 3), (-3, -2), (-2, 4), ($ respectively.) By adding these expressions with coefficients 1,-1,-1,-1,1,1,1,-1, respectively, we find that the terms with first index 2, 3 or 4 cancel and that

$$c_{1,0} - c_{1,-2} - c_{1,-4} - c_{1,-1} + c_{1,-6} + c_{1,-3} + c_{1,-5} - c_{1,-7} = 0.$$

We conclude that for $j \in \mathbb{Z}$ the following recurrence holds:

$$c_{1,j} - c_{1,j-1} - c_{1,j-2} + c_{1,j-3} - c_{1,j-4} + c_{1,j-5} + c_{1,j-6} - c_{1,j-7} = 0.$$

Its characteristic power sum is

$$1 - x^{-1} - x^{-2} + x^{-3} - x^{-4} + x^{-5} + x^{-6} - x^{-7} = (1 - x^{\delta_{1,2}})(1 - x^{\delta_{1,3}})(1 - x^{\delta_{1,4}})$$

in accordance with Theorem 5.1. □

Proof of Corollary 5.1. Fix i. From (9) we see that $\sum_{m=0}^{M_i} r_{i,m} x^m$ has a $(k-1)$-multiple root 1 and that every other root of this polynomial is a root of unity of multiplicity at most $k-1$. Moreover, the multiplicity of a root different from 1 can only be $k-1$ if $\gcd_{h\neq i} |\delta_{h,i}| > 1$. It follows from (10) that there exist a positive integer R_i, roots of unity

$\chi_{i,1} = 1, \chi_{i,2}, \dots, \chi_{i,R_i}$ and positive integers $\rho_{i,\mu}$ with $\rho_{i,1} = k-1$ and $\rho_{i,\mu} \leq k-1$ for all μ such that

$$c_{i,j} = \sum_{\mu=1}^{R_i} P_{i,\mu}(j)\chi_{i,\mu}^j \tag{13}$$

where $P_{i,\mu}(x) \in \mathbb{C}[x]$ is a polynomial of degree less than $\rho_{i,\mu}$ for $\mu = 1, \dots, R_i$. Hence $c_{i,j} = O(j^{k-2})$ as $|j| \to \infty$.

If $\gcd_{h\neq i}|\delta_{h,i}| = t$ for some t, then t divides every $\delta_{h,i}$ for $1 \leq h \leq k$. Thus, if $\gcd_{h\neq i}|\delta_{h,i}| = 1$, then 1 is the only root of multiplicity $k-1$. It follows that there is a $c_i \in \mathbb{R}$ such that $c_{i,j} = c_i j^{k-2} + O(j^{k-3})$ as $|j| \to \infty$. □

Proof of Corollary 5.2. We first prove that for $i = 1, \dots, k$ we have

$$\gcd_{h;h\neq i} |\delta_{h,i}| = \gcd_{h,j;h\neq j} |\delta_{h,j}|. \tag{14}$$

Set $v = \gcd_{h,j;h\neq j} |\delta_{h,j}|$ and $v_i = \gcd_{h\neq i} |\delta_{h,i}|$ for $i = 1, \dots, k$. We shall show that $v = v_1 = \dots = v_k$. Let $h, j \in \{2, \dots, k\}$ with h, j distinct. Since v_h divides both $d_he_1 - e_hd_1$ and $d_he_j - e_hd_j$ we deduce that v_h divides both $d_h(d_1e_j - e_1d_j)$ and $e_h(d_1e_j - e_1d_j)$. By $\gcd(d_h, e_h) = 1$ we obtain that v_h divides $d_1e_j - e_1d_j$. Since this is valid for all $j > 1$, we find that v_h divides v_1. By symmetry v_1 divides v_h. Hence $v_1 = \dots = v_k$. Since $v = \gcd(v_1, \dots, v_k)$, we also have $v = v_1$.

Fix $i \in \{1, \dots, k\}$. Then

$$\prod_{h=1,h\neq i}^{k} (1 - x^{\delta_{h,i}/m}) =: \sum_{n\in\mathbb{Z}} r_{i,n}^* x^n$$

is a finite sum with integer coefficients $r_{i,n}^*$. Hence, by Theorem 5.1, we have the finite recurrence relation

$$\sum_{n\in\mathbb{Z}} r_{i,n}^* c_{i,(j+n)m} = 0 \text{ for every } j \in \mathbb{Z}.$$

By (14) we have $(\gcd_{h\neq i}|\delta_{h,i}|)/m = 1$. As in the second statement of Corollary 5.1, it follows that there is a constant $c_{i,0}^* \in \mathbb{R}$ such that $c_{i,jm}^* = c_{i,0}^* j^{k-2} + O(j^{k-3})$ for $|j| \to \infty$. In view of Lemma 4.3 we find that for every $\mu \in \{0, 1, \dots, m-1\}$ there is a constant $c_{i,\mu}^* \in \mathbb{R}$ such that $c_{i,jm+\mu} = c_{i,\mu}^*\, j^{k-2} + O(j^{k-3})$ for $|j| \to \infty$. □

It follows from Theorem 5.1 (cf. (13)) that $c_{i,j}$ is polynomial in j on the arithmetic progressions modulo m for a certain m. The following theorem makes this explicit. We denote by $j(\text{mod } m)$ the number $j_0 \in \{0, 1, \dots, m-1\}$ such that $j - j_0$ is divisible by m.

Theorem 5.2. *Let $D = \{(d_1, e_1), \ldots, (d_k, e_k)\}$ be a finite set of primitive directions. For any i with $1 \le i \le k$ let $c_{i,j}$ be the coefficient of a global dependency corresponding to the line $e_i x - d_i y = j$ for $j \in \mathbb{Z}$. Set $\delta_{h,i} = d_h e_i - d_i e_h$. For $\kappa = 0, 1, \ldots, k-2$ let $\delta_{H,i} = \gcd_{h \in H} |\delta_{h,i}|$ where the* gcd *is taken over all $H \subset \{1, 2, \ldots, k\} \setminus \{i\}$.*

a) *Then there exist constants $c_{i,\kappa,\delta_{H,i},m}$ for $m = 0, \ldots, \delta_{H,i} - 1$ and all $H \subset \{1, \ldots, k\} \setminus \{i\}$ with $|H| = \kappa + 1$ such that*

$$c_{i,j} = \sum_{\kappa=0}^{k-2} \sum_{\delta_{H,i}} c_{i,\kappa,\delta_{H,i},j} j^{\kappa} \text{ where } c_{i,\kappa,\delta_{H,i},j} = c_{i,\kappa,\delta_{H,i},j (\mathrm{mod}\ \delta_{H,i})}. \tag{15}$$

Here the second summation is over all distinct values $\delta_{H,i}$ with $|H| = \kappa + 1$.

b) *Set $\Delta_{i,\kappa} = \mathrm{lcm}\ \delta_{H,i}$ where the* lcm *is taken over all H with $|H| = \kappa + 1$ and $i \notin H$. Then there exist constants $c_{i,\kappa,m}$ for $m = 0, \ldots, \Delta_{i,\kappa} - 1$ such that*

$$c_{i,j} = \sum_{\kappa=0}^{k-2} c_{i,\kappa,j} j^{\kappa} \text{ where } c_{i,\kappa,j} = c_{i,\kappa,j (\mathrm{mod}\ \Delta_{i,\kappa})} \tag{16}$$

for $i = 1, \ldots, k$ and $j \in \mathbb{Z}$.

Proof of Theorem 5.2. Without loss of generality we may assume $i = 1$. Hence, by Theorem 5.1, for all $j \in \mathbb{Z}$,

$$\sum_{n \in \mathbb{Z}} r_{1,n} c_{1,j+n} = 0 \text{ where } \prod_{h=2}^{k} (1 - x^{\delta_{h,1}}) = \sum_{n \in \mathbb{Z}} r_{1,n} x^n. \tag{17}$$

All the roots of $\sum_{n \in \mathbb{Z}} r_{1,n} z^n$ are roots of unity. Let χ be a t-th root of unity which is a zero of $\sum_{n \in \mathbb{Z}} r_{1,n} x^n$ of multiplicity at least $\kappa + 1$. Since $1 - x^{\delta_{h,1}}$ has simple roots, t divides $\delta_{h,1}$ for at least $\kappa + 1$ values $h > 1$. Hence t divides $\delta_{H,1}$ for some $H \subset \{2, \ldots, k\}$ with $|H| = \kappa + 1$. Conversely, suppose t divides $\delta_{H,1}$ for some $H \in \{2, \ldots, k\}$ with $|H| = \kappa + 1$. Then t divides $\delta_{h,1}$ for at least $\kappa + 1$ values $h \in \{2, \ldots, k\}$ and every t-th root of unity is a zero of $\sum_{n \in \mathbb{Z}} r_{1,n} x^n$ of multiplicity at least $\kappa + 1$. Thus the set of orders t of the zeros of $\sum_{n \in \mathbb{Z}} r_{1,n} x^n$ with multiplicity at least $\kappa + 1$ equals the set of the divisors of the values of $\delta_{H,1}$ which occur for some H with $|H| = \kappa + 1$.

Let $(\chi_s)_{s \in S}$ be set of the roots of $\sum_{n \in \mathbb{Z}} r_{1,n} x^n$. Denote the multiplicity of χ_s by m_s for $s = 1, \ldots, S$. Then, by (17), $\prod_{h=2}^{k} (1 - x^{\delta_{h,1}})$ is the characteristic polynomial of the recurrence and therefore there are complex numbers $c_{1,\kappa,s}$ such that, for $j \in \mathbb{Z}$,

$$c_{1,j} = \sum_{s=1}^{S} \sum_{\kappa=0}^{m_s-1} c_{1,\kappa,s}\ j^\kappa \chi_s^j.$$

Hence,

$$c_{1,j} = \sum_{s=1}^{S} \sum_{\kappa=0}^{m_s-1} (c_{1,\kappa,s}\ \chi_s^j) j^\kappa = \sum_{\kappa=0}^{k-2} j^\kappa \sum_{s=1, m_s>\kappa}^{S} c_{1,\kappa,s}\ \chi_s^j. \tag{18}$$

Note that $(\chi_s^j)_{j\in\mathbb{Z}}$ is periodic modulo the order t_s of the root of unity χ_s. Recall that the set of numbers t_s corresponding to roots of unity χ_s which are zeros of $\sum r_{1,n} x^n$ with multiplicity greater than κ equals the set of the divisors of $\delta_{H,1}$ for all the sets $H \subset \{2, \ldots, k\}$ with $|H| = \kappa+1$. So for the coefficient of j^κ we can combine terms with χ_s's of which the order t_s divides the same $\delta_{H,1}$ with $|H| = \kappa + 1$. Thus there are constants $c_{1,\kappa,\delta_{H,1},m}$ for $m = 0, \ldots, \delta_{H,1} - 1$ and all $H \subset \{2, \ldots, k\}$ with $|H| = \kappa + 1$ such that

$$c_{1,j} = \sum_{\kappa=0}^{k-2} \sum_{\delta_{H,1}} c_{1,\kappa,\delta_{H,1},j (\mathrm{mod}\ \delta_{H,1})}\ j^\kappa. \tag{19}$$

This proves part a).

b) Since $\Delta_{i,\kappa}$ is a multiple of all $\delta_{H,1}$ with $|H| = \kappa + 1$, there are constants $c_{1,\kappa,m} \in \mathbb{R}$ for $m = 0, \ldots, \delta_{1,\kappa} - 1$ such that

$$c_{1,j} = \sum_{\kappa=0}^{k-2} c_{1,\kappa,j (\mathrm{mod}\ \Delta_{1,\kappa})}\ j^\kappa. \tag{20}$$

for all integers j. □

Example 5.2. In Example 4.1 the following complete set of 7 linearly independent dependencies is given for $D = \{(1, 0), (0, 1), (1, 1), (1, -1)\}$:

nr.	$c_{1,j}$	$c_{2,j}$	$c_{3,j}$	$c_{4,j}$
(1)	1	-1	0	0
(2)	1	0	-1	0
(3)	1	0	0	-1
(4)	0	0	$(-1)^j$	$(-1)^{j+1}$
(5)	j	j	$-j$	0
(6)	j	$-j$	0	$-j$
(7)	$2j^2$	$2j^2$	$-j^2$	$-j^2$

We find that the left-hand side of (9) gives $x^{-2}(1-x)^3$ for $i = 1$, $(1-x)^3$ for $i = 2$, $-x^{-1}(1-x)^2(1-x^2)$ for $i = 3$ and $-x^{-4}(1-x)^2(1-x^2)$ for

$i = 4$. In this example we have $|\delta_{3,4}| = 2$ and $|\delta_{h,j}| = 1$ for all other values $h \neq j$. According to Theorem 5.2 a) with $\kappa = 0$ the sequences $(c_{1,j})$ and $(c_{2,j})$ are constant and the sequences $(c_{3,j})$ and $(c_{4,j})$ are periodic modulo 2 as in dependencies (1)-(4). According to Theorem 5.2 a) with $\kappa = 1$ all the sequences $(c_{i,j})$ are constant multiples of j as in dependencies (5)-(6) and applied with $\kappa = 2$ all the sequences $(c_{i,j})$ are constant multiples of j^2 as in dependency (7).

6. Sets of generators for global dependencies

In this section we show that Theorem 4.1 yields a set of generators for the global dependencies. First we show that every global dependency is a linear combination of global dependencies of distinct powers.

Theorem 6.1. *Let $D = \{(d_1, e_1), \dots, (d_k, e_k)\}$ be a set of primitive directions. Let be given an arbitrary global dependency*

$$\sum_{i=1}^{k} \sum_{j \in \mathbb{Z}} c_{i,j} \ell(i,j) = 0. \tag{21}$$

Then

$$\sum_{i=1}^{k} \sum_{j \in \mathbb{Z}} c_{i,\kappa,j} j^{\kappa} \ell(i,j) = 0$$

is a global dependency for $\kappa = 0, 1, \dots, k-2$, where the numbers $c_{i,\kappa,j}$ are defined in Theorem 5.2 and are periodic in j with period $\Delta_{i,\kappa}$.

Proof. Let be given an arbitrary global dependency (21). Then we have, by (2),

$$\sum_{(x,y) \in A} f(x,y) \sum_{i=1}^{k} \sum_{j : e_i x - d_i y = j} c_{i,j} = 0.$$

Since this is true for every finite set $A \subset \mathbb{Z}^2$, we may apply it to $A_0 := \{(x_0, y_0)\}$ with $f(x_0, y_0) \neq 0$. This gives

$$\sum_{i=1}^{k} \sum_{j : e_i x_0 - d_i y_0 = j} c_{i,j} = 0.$$

According to Theorem 5.2 b) there exist numbers $c_{i,\kappa,m}$ for $m = 0, \dots, \Delta_{i,\kappa} - 1$ such that

$$c_{i,j} = \sum_{\kappa=0}^{k-2} c_{i,\kappa,j (\mathrm{mod}\ \Delta_{i,\kappa})} j^{\kappa}. \tag{22}$$

Hence

$$\sum_{\kappa=0}^{k-2}\sum_{i=1}^{k}(e_ix_0-d_iy_0)^\kappa c_{i,\kappa,e_ix_0-d_iy_0 (\text{mod } \Delta_{i,\kappa})}=0.$$

We may replace (x_0, y_0) by $(x_0 + s\Delta_{i,\kappa}, y_0 + t\Delta_{i,\kappa})$ for any $s, t \in \mathbb{Z}$ and the coefficients do not change. It follows that the polynomial $\sum_{\kappa=0}^{k-2}\sum_{i=1}^{k} c_{i,\kappa,j}(e_ix-d_iy)^\kappa = 0$ on the affine lattice $(x_0, y_0)+(\mathbb{Z}\Delta_0, \mathbb{Z}\Delta_0)$ where $\Delta_0 = \text{lcm}_{i,\kappa}\Delta_{i,\kappa}$. Therefore the homogeneous polynomial $\sum_{i=1}^{k} c_{i,k-2,j}j^{k-2}$ is of order $O(\max(|x|, |y|)^{k-3})$ as $\max(|x|, |y|) \to \infty$ on this affine lattice. We conclude that $\sum_{i=1}^{k} c_{i,k-2,j} = 0$ on the affine lattice. Since (x_0, y_0) is arbitrary and $c_{i,k-2,j}$ depends only on i, $k-2$ and j (mod $\Delta_{i,\kappa}$), we have $\sum_{i=1}^{k} c_{i,k-2,j} = 0$ for all j. Thus

$$\sum_{i=1}^{k}\sum_{j\in\mathbb{Z}} c_{i,k-2,j}j^{k-2}\ell(i,j) = \sum_{(x,y)\in A} f(x,y)\sum_{j\in\mathbb{Z}} j^{k-2}\sum_{i=1}^{k} c_{i,k-2,j} = 0. \tag{23}$$

Since the difference of two global dependencies is a global dependency, $\sum_{\kappa=0}^{k-3}\sum_{i=1}^{k} c_{i,\kappa,j}(e_ix - d_iy)^\kappa = 0$ is a global dependency. By subsequently applying the above argument to the homogeneous parts of degree $k-3, k-4, \dots, 0$ we obtain that $\sum_{i=1}^{k} c_{i,\kappa,j}j^\kappa\ell(i,j) = 0$ is a global dependency for $\kappa = k-2, k-3, \dots, 0$. □

Theorem 6.2. *Let $D = \{(d_1, e_1), \dots, (d_k, e_k)\}$ be a finite set of primitive directions. The set of dependencies constructed in Theorem 4.1 provides a set of generators for the linear space of the global dependencies.*

Proof. Let be given an arbitrary global dependency (21). Then according to Theorem 6.1 $\sum_{i=1}^{k} c_{i,\kappa,j}j^\kappa\ell(i,j) = 0$ is a global dependency for $\kappa = 0, 1, \dots, k-2$. Fix the degree κ. According to Theorem 5.2 a) the coefficient $c_{1,\kappa,j}j^\kappa$ of this homogeneous dependency has the form

$$\sum_{\delta_{H,1}} c_{1,\kappa,\delta_{H,1},j(\text{mod } \delta_{H,1})}\ j^\kappa, \tag{24}$$

where the summation is over all distinct values $\delta_{H,1}$ of $H \subset \{2, \dots, k\}$ with $|H| = \kappa+1$. Consider such an H and put $m_H = \delta_{H,1}$. Let Λ_H be the lattice generated by the directions $\mathbf{d}_h$ with $h \in H \cup \{1\}$. Then this lattice has m_H affine lattices by (14). For each affine lattice Theorem 4.1 provides a dependency with nonzero coefficients for directions $\mathbf{d}_1$ and $\mathbf{d}_h$ with $h \in H$ and zero coefficients for the other directions. By subtracting suitable multiples of these dependencies from the original homogeneous dependency of degree κ the coefficients $c_{1,\kappa,m_H,j(\text{mod } m_H)}$ in (24) cancel. We do so for all such values $\delta_{H,1}$. Thus we are left with a homogeneous dependency of power κ such that all coefficients $c_{1,j}$ are zero. Next we apply the same procedure to this dependency for all values $\delta_{H,2}$ where $H \subset \{3,\ldots,k\}$ with $|H| = \kappa+1$. After subtraction of suitable multiples of degeneracies constructed in Theorem 4.1 this yields a homogeneous dependency of power κ where the coefficients $c_{1,j}$ and $c_{2,j}$ are all zero. We proceed with $3,4,\ldots$ until we are left with a dependency where all the coefficients $c_{i,j}$ for $i = 1,2,\ldots,k-\kappa-1$ are zero. Thus this is a dependency of degree κ restricted to the $\kappa+1$ directions $\mathbf{d}_{k-\kappa},\ldots,\mathbf{d}_k$. By Corollary 4.2 this dependency is identically zero. We conclude that the homogeneous part of degree κ of the originally given dependency is a linear combination of dependencies constructed in Theorem 4.1. By Theorem 6.1 this implies that the original global dependency is a linear combination of global dependencies constructed in Theorem 4.1. $\square$

Remark 6.1. If, for some κ and i, $\delta_{H',i}$ divides $\delta_{H,i}$, then it suffices to consider $\delta_{H,i}$ only, since by choosing the coefficients for $j \pmod{\delta_{H,i}}$ the values of the coefficients for $j \pmod{\delta_{H',i}}$ are implied. If $\delta_{H',i} = \delta_{H,i}$, then it suffices to consider only one of them for the same reason. The not considered dependencies of type Theorem 4.1 are dependent on the remaining dependencies and therefore superfluous. This induces a considerable reduction in the number of dependencies to be considered, cf. Examples 6.1 and 7.1.

Example 6.1. [Continuation of Example 4.2.]
We consider the four directions $D = \{\mathbf{d}_1 = (1,1), \mathbf{d}_2 = (1,-1), \mathbf{d}_3 = (1,-3), \mathbf{d}_4 = (3,2)\}$. We know from Example 4.2 that there is a basis of 25 degeneracies, viz. one of power 2, three of power 1 and 21 of power 0. Theorem 6.2 gives

$$c_{1,j} = \sum_{\kappa=0}^{2} \sum_{\delta_{H,1}} c_{1,\kappa,\delta_{H,1},j \pmod{\delta_{H,1}}} \; j^{\kappa}.$$

We have $|\delta_{2,1}| = 2, |\delta_{3,1}| = 4, |\delta_{4,1}| = 1$. Hence, in shortened notation, $\delta_{23,1} = 2, \delta_{24,1} = 1, \delta_{34,1} = 1, \delta_{234,1} = 1$. In view of Remark 6.1 we only consider $\delta_{23,1} = 2$ for $\kappa = 1$ and $\delta_{3,1} = 4$ for $\kappa = 0$. Thus $c_{1,j} = c_{1,2,j}j^2 + c_{1,1,j}j + c_{1,0,j}$ where $c_{1,2,j} = c_{1,2,0}$ for all $j, c_{1,1,j} = c_{1,1,0}$ for j even and $c_{1,1,j} = c_{1,1,1}$ for j odd, whereas $c_{1,0,j} = c_{1,0,j(\mathrm{mod} 4)}$ for all j. Thus 7 degeneracies of type Theorem 4.1 are necessary to neutralize direction $\mathbf{d}_1$.

We continue with direction $\mathbf{d}_2$ with respect to directions $\mathbf{d}_3, \mathbf{d}_4$. We have $|\delta_{3,2}| = 2, |\delta_{4,2}| = 5$ and therefore $|\delta_{34,2}| = 1$. Hence we use one dependency of power 1 and $2+5=7$ degeneracies of power 0 of type Theorem 4.1 to eliminate the dependence on direction $\mathbf{d}_2$.

Finally we consider direction $\mathbf{d}_3$ with respect to direction $\mathbf{d}_4$. We have $|\delta_{3,4}|=11$. Hence we use 11 degeneracies of type Theorem 4.1 to neutralize the directions $\mathbf{d}_3$ and $\mathbf{d}_4$.

We conclude that any given dependency can be written as a linear combination of one dependency of power 2, three of power 1 and $4+7+11=22$ of power 0 of type Theorem 4.1. The latter number is just one more than strictly needed according to Theorem 3.1 and 9 dependencies less than the 25+5+1=31 dependencies which have to be considered if we do not take Remark 6.1 into account. □

By Theorem 6.2 the following conjecture is a reformulation of Conjecture 3.2.

Conjecture 6.1. *The dimension of the subspace of global dependencies of power $\kappa-2$ constructed in Theorem 4.1 is g_κ for $\kappa=2,3,\dots,k$.*

7. Correction of noise

Now we show how to use our results for correcting noise, thus when, for fixed A and D, the measured line sums deviate from the actual ones. As before, we work over $K=\mathbb{R}$. Our purpose is to find and explicitly construct the table best fitting the measured line sums. In fact, in [12] we have worked out such a scheme. However, here we can use the information provided in Sections 4 and 6 to introduce a more efficient method.

First we briefly sketch our settings. Let A be any nonempty, finite subset of $\mathbb{Z}^2$. Let $D=\{(d_1,e_1),\dots,(d_k,e_k)\}$ be a finite set of primitive directions. For $i\in\{1,\dots,k\}$ write R_i for the set of integers j for which at least one point $(x,y)\in A$ exists with $e_ix-d_iy=j$. Suppose that as an approximation of the line sums $\ell(i,j)$ of an unknown function $f_0:\ A\to\mathbb{R}$, we measured the real numbers $b(i,j)$ for $i=1,\dots,k$ and $j\in R_i$.

Now one can formulate the challenge.

Problem. Minimize the error

$$\sum_{i=1}^{k}\sum_{j\in R_i}|\ell(i,j)-b(i,j)|^2.$$

To handle the problem, our method in Section 4 of [12] has been the following. Write B for the $t\times s$ incidence matrix, that is, the (p,q)-th entry of B is 1 if the p-th point of A (in an arbitrary, but fixed ordering) belongs to the q-th line through A taken in the directions in

D and 0 otherwise. Here $t = |R_1| + \cdots + |R_k|$ and $s = |A|$. Then the set of possible line sums is just the column space $C(B)$ of B, which is a subspace of $\mathbb{R}^t$. So to solve the above problem, we only need to calculate the orthogonal projection b' of the measured vector b onto $C(B)$, which can be done by standard techniques. The first step is to find a basis for $C(B)$, that is, we need to find a maximal independent system out of the s columns of B. Since s represents the cardinality of A, it may be of order the diameter of A squared.

For a more efficient, but possibly less accurate approach, recall that by Theorem 3.2, the orthogonal complement of $C(B)$ is just the linear subspace S of dependencies generated by the rows of B. Thus to find b', we have an alternative way: we may choose a basis for S, and then (by standard techniques) we can find the orthogonal projection b'' of b onto S. Then we plainly have $b' = b - b''$. Since the number of line sums is of order the product of $|D|$ and the diameter of A, this approach is more efficient than the one in [12] when the diameter of A is large compared to $|D|$. Moreover, Theorem 4.1 enables us to compute a basis for the global dependencies very efficiently.

We follow the above ideas in our treatment for reducing noise in the following method.

Method for improving the noisy measurements.

Input. A finite set D of primitive directions, a finite set $A \subset \mathbb{Z}^2$, and the measured vector $b \in \mathbb{R}^t$ of the line sums of A in the directions in D.

Step 1. The D-part.
Find a basis for the subset S_g of global dependencies.

Details and background. The dependencies given in Theorem 4.1 form a generating system for the subspace S_g of S corresponding to the global dependencies. By standard techniques, we can find a basis for S_g. The dependencies depend only on D, so are independent of A. If different sets A for the same set D are considered, this step has to be done only once.

Step 2. The A-part.
Find a basis of the global dependencies restricted to A.

Details and background. The global dependencies found in Step 1 are now expressed in terms of A. The restriction of the basis obtained in Step 1 to A may form a linearly dependent system. Next compute the orthogonal projection of an arbitrary point onto the hyperplane generated by the generating system (by standard techniques, e.g. by the Gram-Schmidt method). If for fixed D and A different functions f and measured values b are considered, this step has to be done only once.

Step 3. The b-part.
Calculate the orthogonal projection b_g of b onto the subspace generated by the global dependencies restricted to A.

Details and background. This is just a substitution.

Output. The vector $b - b_g$.

If Step 3 has to be performed only once, then it is simpler to compute the orthogonal projection not in Step 2, but in Step 3 for the numerically given point b in place of for a general point.

Global dependencies form the essential part of all dependencies. In some cases all dependencies come from global ones, e.g. if A is convex and D-rounded (see results of Stolk and Batenburg [17]). In that case, $b' = b - b_g$, that is we found the best possible line sum vector closest to b. In general, $b - b_g$ does not belong to $C(B)$, but it is closer to this subspace than b, and by that the noise is significantly reduced. To see this, let L be the orthogonal complement of S_g in $\mathbb{R}^t$. Then we clearly have $C(B) \subseteq L$. Let

$$\mathbf{s}_1, \dots, \mathbf{s}_\alpha, \mathbf{s}_{\alpha+1}, \dots, \mathbf{s}_{\alpha+\beta}, \mathbf{s}_{\alpha+\beta+1} \dots, \mathbf{s}_{\alpha+\beta+\gamma}$$

be an orthonormal basis for $\mathbb{R}^t$ (with $\alpha+\beta+\gamma = t$) such that $\mathbf{s}_1, \dots, \mathbf{s}_\alpha$ and $\mathbf{s}_1, \dots, \mathbf{s}_{\alpha+\beta}$ form bases for $C(B)$ and L, respectively. Write

$$b = \lambda_1 \mathbf{s}_1 + \dots + \lambda_{\alpha+\beta+\gamma} \mathbf{s}_{\alpha+\beta+\gamma}$$

with some real coefficients $\lambda_1, \dots, \lambda_{\alpha+\beta+\gamma}$. Then we have

$$b' = \lambda_1 \mathbf{s}_1 + \dots + \lambda_\alpha \mathbf{s}_\alpha,$$

and

$$b - b_g = \lambda_1 \mathbf{s}_1 + \dots + \lambda_{\alpha+\beta} \mathbf{s}_{\alpha+\beta}.$$

Thus

$$b - b_g - b' = \lambda_1 \mathbf{s}_{\alpha+1} + \dots + \lambda_{\alpha+\beta} \mathbf{s}_{\alpha+\beta},$$

while

$$b - b' = \lambda_1 \mathbf{s}_{\alpha+1} + \dots + \lambda_{\alpha+\beta+\gamma} \mathbf{s}_{\alpha+\beta+\gamma}.$$

This shows that in general $|b - b'|$ is larger than $|b - b_g - b'|$ indeed.

We illustrate our approach by two examples.

Example 7.1. This is the continuation of Examples 4.1 and 5.2. As input, let

$$D = \{\mathbf{d}_1 = (1,0), \mathbf{d}_2 = (0,1), \mathbf{d}_3 = (1,1), \mathbf{d}_4 = (1,-1)\},$$

and

$$A = \{(a,b) \in \mathbb{Z}^2 \ : \ 0 \le a, b \le 3\}.$$

Step 1. Using Theorem 4.1 we found in Example 4.1 that the seven dependencies

$$\sum_j \ell(1,j) = \sum_j \ell(2,j), \sum_j \ell(1,j) = \sum_j \ell(3,j),$$

$$\sum_j \ell(1,j) = \sum_j \ell(4,j), \sum_{j\ even} \ell(3,j) = \sum_{j\ even} \ell(4,j),$$

$$\sum_j j\ell(1,j) + \sum_j j\ell(2,j) - \sum_j j\ell(3,j) = 0,$$

$$\sum_j j\ell(1,j) - \sum_j j\ell(2,j) - \sum_j j\ell(4,j) = 0,$$

$$2\sum_j j^2\ell(1,j) + 2\sum_j j^2\ell(2,j) - \sum_j j^2\ell(3,j) - \sum_j j^2\ell(4,j) = 0$$

form a basis of S_g.

Step 2. We order the points of A in lexicographically increasing order. Recalling that by lines in direction (d,e) we mean lines of the form $ex - dy = j$, we have

$$R_1 = \{-3,-2,-1,0\}, \ R_2 = \{0,1,2,3\},$$

$$R_3 = \{-3,-2,-1,0,1,2,3\}, \ R_4 = \{-6,-5,-4,-3,-2,-1,0\}$$

which in this order correspond to the rows of the incidence matrix B (which we suppress).

The following seven vectors are the restrictions of the above dependencies to A:

$$g_1 = (1,1,1,1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,0,0,0,0,0,0),$$

$$g_2 = (1,1,1,1,0,0,0,0,-1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,0),$$

$$g_3 = (1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,-1,-1,-1,-1,-1,-1,-1),$$

$$g_4 = (0,0,0,0,0,0,0,0,0,1,0,1,0,1,0,-1,0,-1,0,-1,0,-1),$$

$$g_5 = (-3,-2,-1,0,0,1,2,3,3,2,1,0,-1,-2,-3,0,0,0,0,0,0,0),$$

$$g_6 = (-3,-2,-1,0,0,-1,-2,-3,0,0,0,0,0,0,0,6,5,4,3,2,1,0),$$

$$g_7 = (18,8,2,0,0,2,8,18,-9,-4,-1,0,-1,-4,-9,-36,-25,-16,-9,-4,-1,0).$$

Step 3. Suppose that for the line sums of a table $f : \ A \to \mathbb{R}$ we measured

$$b = (3.9,2.9,1.9,1.1,4.1,3.1,2.1,1.1,1.1,2.1,3.1,4.1,0.1,0.1,0.1,0.9,0.9,1.9,2.1,2.1,1.1,1.1).$$

Now a simple calculation gives that the orthogonal projection of b onto S, up to two digit precision, is

$$b'' = (-0.08, -0.10, -0.11, -0.13, 0.00, 0.03, 0.06, 0.10, 0.09, 0.09,$$
$$0.08, 0.08, 0.06, 0.06, 0.04, -0.09, -0.06, -0.04, -0.01, 0.01, 0.04, 0.05).$$

Thus the projection of b onto L (again up to two digit precision) is

$$b' := b - b'' = (3.98, 3.00, 2.01, 1.23, 4.10, 3.07, 2.04, 1.00, 1.01, 2.01,$$
$$3.02, 4.02, 0.04, 0.04, 0.06, 0.99, 0.96, 1.94, 2.11, 2.09, 1.06, 1.05).$$

Actually we find exactly
$601160b' = (2391599, 1801261, 1209593, 736827, 2464371, 1845557, 1225413, 603939, 608506, 1208826, 1817658, 2419308, 23670, 25490, 35822, 596754, 576930, 1166778, 1268516, 1258534, 640210, 631558)$.
Since A is rectangular, it follows from [9] that all dependencies are global. Thus b' is the orthogonal projection on $C(B)$. The tables with line sums b' can be found by solving a linear equation system (with matrix B). $\square$

Example 7.2. We work with similar settings, only with a small change. On the one hand, it will underline the uniformity of our method with respect to D, and we can also illustrate that the error is reduced, even if the restricted global dependencies do not generate the subspace of all dependencies over A.

We take the same set of directions

$$D = \{\mathbf{d}_1 = (1, 0), \mathbf{d}_2 = (0, 1), \mathbf{d}_3 = (1, 1), \mathbf{d}_4 = (1, -1)\},$$

as in Example 7.1, but we put

$$A = \{(a, b) \in \mathbb{Z}^2 \ : \ 0 \le a, b \le 3\} \cup \{(4, 0)\}.$$

Step 1. This is the same as in Example 7.1.

Step 2. We order the points of A in lexicographically increasing order. Now we have

$$R_1 = \{-3, -2, -1, 0\}, \ R_2 = \{0, 1, 2, 3, 4\},$$
$$R_3 = \{-3, -2, -1, 0, 1, 2, 3, 4\}, \ R_4 = \{-6, -5, -4, -3, -2, -1, 0\}$$

which in this order correspond to the rows of the incidence matrix B (which we suppress again). Note that the difference with respect to Example 7.1 is that now we have an extra 4 in R_2 and in R_3.

The global dependencies for A are now given by

$$g_1 = (1, 1, 1, 1, -1, -1, -1, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$g_2 = (1, 1, 1, 1, 0, 0, 0, 0, 0, -1, -1, -1, -1, -1, -1, -1, -1, 0, 0, 0, 0, 0, 0, 0),$$
$$g_3 = (1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, -1, -1, -1),$$
$$g_4 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, -1, 0, -1, 0, -1, 0, -1),$$

$$g_5 = (-3, -2, -1, 0, 0, 1, 2, 3, 4, 3, 2, 1, 0, -1, -2, -3, -4, 0, 0, 0, 0, 0, 0, 0),$$
$$g_6 = (-3, -2, -1, 0, 0, -1, -2, -3, -4, 0, 0, 0, 0, 0, 0, 0, 0, 6, 5, 4, 3, 2, 1, 0),$$
$$g_7 = (18, 8, 2, 0, 0, 2, 8, 18, 32, -9, -4, -1, 0, -1, -4, -9, -16, -36, -25, -16, -9, -4, -1, 0).$$

Step 3. Suppose that for the line sums of a table $f : A \to \mathbb{R}$ we measured

$$b = (3.9, 2.9, 1.9, 1.1, 4.1, 3.1, 2.1, 1.1, 0.1, 1.1, 2.1, 3.1, 4.1, 0.1, 0.1, 0.1, 0.1, 0.9, 0.9, 1.9, 2.1, 2.1, 1.1, 1.1).$$

Then we get for b_g as the orthogonal projection of b onto S, up to two digit precision,

$$b_g = (-0.08, -0.09, -0.11, -0.13, 0.00, 0.03, 0.06, 0.09, 0.10, 0.09, 0.09, 0.07,$$
$$0.08, 0.06, 0.07, 0.06, 0.08, -0.09, -0.05, -0.05, -0.01, -0.00, 0.04, 0.05).$$

Thus the projection of b onto S_g (again up to two digit precision) is

$$b - b_g = (3.98, 2.99, 2.01, 1.23, 4.10, 3.07, 2.04, 1.01, 0.00, 1.01, 2.01, 3.03,$$
$$4.02, 0.04, 0.03, 0.04, 0.02, 0.99, 0.95, 1.95, 2.11, 2.10, 1.06, 1.05).$$

Now the important difference is that $S_g \neq S$, since in this case there are local dependencies as well. A simple Maple calculation with the incidence matrix B shows that there is one more linearly independent dependence relation (which has to be a local one). Clearly, one can choose the identity that the sums in the directions $(0, 1)$ and $(1, 1)$ through the point $(4, 0)$ must be equal. This belongs to the vector

$$g_8 = (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0).$$

Thus as the orthogonal projection b'' of b onto S we obtain (again with two digit precision)

$$b'' = (-0.08, -0.10, -0.11, -0.13, 0.00, 0.03, 0.06, 0.10, 0.09, 0.09, 0.09,$$
$$0.07, 0.08, 0.06, 0.06, 0.04, 0.09, -0.10, -0.06, -0.04, -0.01, 0.00, 0.04, 0.05).$$

Hence the projection of b onto $C(B)$ (again up to two digit precision) is

$$b' = b - b'' = (3.82, 3.00, 2.01, 1.23, 4.10, 3.07, 2.04, 1.00, 0.01, 1.01, 2.01, 3.03,$$
$$4.02, 0.04, 0.04, 0.06, 0.01, 1.00, 0.96, 1.94, 2.11, 2.10, 1.06, 1.05).$$

Actually we obtain exactly
$39514 80b' = 15717908, 11817210, 7937643, 4869470, 16197196, 12111700, 8047324, 4004068, -18068, 3994068, 7933685, 11980262, 15898759, 166816, 112713, 165570, 90347, 3894065, 3769306, 7702987, 8347404, 8308489, 4190006, 4129967).$
This b' is the closest consistent (possible) line sum to the measured b.

We conclude the example by pointing out that though $b - b_g \neq b'$, the vector $b - b_g$ is much closer to b' than the original b. Indeed, we have (up to two digit precision)

$$|b - b_g - b'| = 0.16 \quad \text{and} \quad |b - b'| = 0.36.$$

□

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Institute of Mathematics, University of Debrecen, H-4010 Debrecen, P.O. Box 12, Hungary
E-mail address: hajdul@science.unideb.hu

Mathematical Institute, Leiden University, 2300 RA Leiden, P.O. Box 9512, The Netherlands
E-mail address: tijdeman@math.leidenuniv.nl