DISPROOF OF A CONJECTURE OF JACOBSTHAL

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Abstract. For any integer $n \geq 1$, let $j(n)$ denote the Jacobsthal function, and $\omega(n)$ the number of distinct prime divisors of n. In 1962 Jacobsthal conjectured that for any integer $r \geq 1$, the maximal value of $j(n)$ when n varies over $\mathbb{N}$ with $\omega(n) = r$ is attained when n is the product of the first r primes. We show that this is true for $r \leq 23$ and fails at $r = 24$, thus disproving Jacobsthal's conjecture.

1. Introduction and main results

For $n \geq 1$, the Jacobsthal function $j(n)$ is defined as the smallest integer such that any sequence of $j(n)$ consecutive integers contains an element which is coprime to n. This function was introduced by Jacobsthal in 1960 [6] and was studied by many authors, see e.g. [1], [5] and the references given there. Further, this function was used by Pomerance [9] in connection with the problem of least primes in arithmetic progressions. He applied his result to show the finiteness of integers k having the property that the first $\varphi(k)$ primes coprime to k form a reduced residue system modulo k. In [4] we made the result of Pomerance explicit under some special cases and solved completely a problem of Recaman. In this paper, we consider a conjecture raised by Jacobsthal in 1962 in a letter to Erdős [1]. For any integer $n \geq 1$, let p_n denote the n-th prime and $\omega(n)$ denote the number of distinct prime divisors of n. Note that while dealing with $j(n)$, we may always suppose without loss of generality that n is square-free. Define the functions $h(r)$ and $H(r)$ by

$$h(r) = j(p_1 p_2 \dots p_r)$$

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and

$$H(r) = \max_{\omega(n)=r} j(n).$$

It is clear that $H(r) \geq h(r)$ for all $r \geq 1$. Concerning $H(r)$ we have

$$\frac{c_1 r (\log r)^2 \log\log\log r}{(\log\log r)^2} < H(r) < c_2 r^{c_3}$$

where c_1, c_2, c_3 denote positive absolute constants. Here the left hand side inequality is due to Rankin [10], while the right hand side inequality follows easily from Brun's method (see [1]). By elementary tools Stevens [11] derived the completely explicit estimate

$$H(r) \leq 2r^{2+2e\log r}. \tag{1.1}$$

Further, Jacobsthal himself made a study on the function $H(r)$ in [6].

For $h(r)$ also upper and lower bounds are known. Iwaniec [5] showed that

$$h(r) \ll r^2 \log r.$$

The best known lower bound for $h(r)$ is due to Pintz [8], given by

$$h(r) \geq (e^\gamma + o(1)) \frac{p_r \log p_r \log\log\log p_r}{(\log\log p_r)^2}.$$

Here γ denotes Euler's constant. Recently, Hagedorn [2] has computed the exact values of $h(r)$ for $r < 50$. In a letter to Erdős (see [1], p. 163, ll. 17-19) Jacobsthal formulated the following

Conjecture 1.1. *$H(r) = h(r)$ for all $r \geq 1$.*

He showed that the conjecture is true for $r \leq 10$. In this paper, we show

Theorem 1.2. *We have*

$$H(r) = h(r) \quad \text{for } r \leq 23$$

and the equal values are given in Table 1. Further,

$$236 = H(24) > h(24) = 234.$$

Thus the conjecture of Jacobsthal is true upto $r \leq 23$, but fails at $r = 24$. Thus by Theorem 1.2 and the exact values of $h(r)$ given in [2], we get the exact values of $H(r)$ for $r \leq 23$. The function $j(n)$ seems to behave rather irregularly. It is hard to predict the larger of the two values $j(p_1 \dots p_r)$ and $j(p_1 \dots p_{r-1} p_{r+1})$ when p_r and p_{r+1} are "close". So we feel that Jacobsthal's conjecture should fail infinitely often. In the next result, we show some divisibility property of integers n with $\omega(n) = r$ for which $j(n)$ is maximal, i.e., $j(n) = H(r)$ holds. We shall refer to such integers n as r-maximal integers.

TABLE 1. The values of $H(r)$ and fixed prime divisors of r-maximal integers for $1 \le r \le 24$

r	$H(r)$	S_r	r	$H(r)$	S_r
1	2	$\emptyset$	13	74	$\{2,\dots,31\}$
2	4	$\{2\}$	14	90	$\{2,\dots,23\} \cup \{31\}$
3	6	$\{2\}$	15	100	$\{2,\dots,37\}$
4	10	$\{2,3\}$	16	106	$\{2,\dots,29\}$
5	14	$\{2,3,5\}$	17	118	$\{2,\dots,43\}$
6	22	$\{2,\dots,7\}$	18	132	$\{2,\dots,47\}$
7	26	$\{2,\dots,11\}$	19	152	$\{2,\dots,37\} \cup \{43\}$
8	34	$\{2,\dots,13\}$	20	174	$\{2,\dots,53\}$
9	40	$\{2,\dots,13\}$	21	190	$\{2,\dots,47\} \cup \{59,61\}$
10	46	$\{2,\dots,19\}$	22	200	$\{2,\dots,43\} \cup \{53,61\}$
11	58	$\{2,\dots,23\}$	23	216	$\{2,\dots,61\} \cup \{79,83\}$
12	66	$\{2,\dots,23\}$	24	236	$\{2,\dots,61\} \cup \{73,89,101\}$

Theorem 1.3. *Let $r \le 24$ and S_r be the set appearing in the r-th row of Table 1. If n is an r-maximal integer, then*

$$\prod_{p \in S_r} p \quad \textit{divides} \quad n. \tag{1.2}$$

On the other hand, if p is a prime and $p \notin S_r$, then there exists an r-maximal integer n such that p does not divide n.

Based upon Theorem 1.3, we propose the following problem, which is a weaker version of Jacobsthal's conjecture.

Problem. Fix $r \ge 1$. Is it true that for all sufficiently large R, there is an R-maximal integer divisible by $\prod_{i=1}^{r} p_i$?

Had the original Conjecture 1.1 of Jacobsthal been valid, it would have implied an affirmative answer to this problem with $R = r$.

Our proofs of Theorems 1.2 and 1.3 are mainly based on the methods of Hagedorn used in [2] to compute $h(r)$ for $r < 50$. For any fixed r, computation of $h(r)$ requires the evaluation of the Jacobsthal function $j(n)$ at only one integer $n = p_1 \dots p_r$. Now $H(r)$ is the maximum taken over an infinite set of values. Thus an important step is to convert the calculation of $H(r)$ into a finite problem as done in section 2.3. For a theoretical explicit upper bound for $H(r)$, we refer to (1.1). This bound is rather huge even for small values of r. For instance, for $r = 10$, (1.1) gives

$$H(10) \le 2 \times 10^{14.6}$$

while from Table 1, we now know $H(10) = 46$. Thus the calculation of the exact values of $H(r)$ requires the introduction of new ideas and the modifications of the algorithms used in [2]. At this point we also mention our paper on a problem of Pillai [3] where similar algorithms were developed.

2. Algorithms and auxiliary results

In this section we explain the methods, algorithms and other ingredients which were used in the proofs of our theorems.

2.1. **Sieves and coverings.** Let $2 = p_1 < p_2 < \dots$ be the sequence of all primes. Let $S = \{q_1, \dots, q_t\}$ be a given finite set of primes. Then the set

$$T = \{(q_1, c_1), \dots, (q_t, c_t)\}$$

with some integers $c_i \in \{1, 2, \dots, q_i\}$ $(i = 1, 2, \dots, t)$ is called an S-*sieve.* Let A be a finite set of positive integers. We say that T *covers* A or T is an S-*covering* of A if for every $a \in A$ we can find a pair $(q, c) \in T$ such that $a \equiv c \pmod{q}$. We also say that a is *covered* by q or q *covers* a. In particular, when $A = \{1, 2, \dots, k\}$ we observe that c_i is the least positive integer covered by q_i for $1 \le i \le t$. We call c_i the *position* of q_i. Fix an $i \in \{1, 2, \dots, t\}$. We say that q_i *exclusively* covers $a \in A$ if

$$a \equiv c_i \pmod{q_i} \text{ and } a \not\equiv c_j \pmod{q_j} \text{ for } 1 \le j \le t, j \ne i.$$

It is clear that in the notion of coverings as above, the set S plays the primary role. Hence we say that A can be covered by S if there exist $c_1, \dots, c_t$ as above such that the corresponding T covers A. Note that if A can be covered by some set S, then the same is true for any set S' with $S \subseteq S'$. This leads us to define a *minimal cover* of A as a set T such that T covers A and no proper subset of T covers A. In all the discussions below, by a cover we shall always mean a *minimal cover* without any mention. Further we say that T is an r-*exclusive covering* of the set A if every prime $> p_{r+1}$ in S covers exclusively *at least two elements* of A. We also observe that if S covers A, then S also covers $A + 1 = \{a + 1 : a \in A\}$. If S consists of only odd primes, then S covers A if and only if S covers $2A = \{2a : a \in A\}$. The next statement highlights the importance of coverings.

Lemma 2.1. *Let n be an integer with $n > 1$, and write S for the set of prime divisors of n. Let k be the largest positive integer such that the set $A = \{1, 2, \dots, k\}$ can be covered by S. Then $j(n) = k + 1$.*

Proof. The statement immediately follows from the results of Hagedorn [2]. See in particular the proof of Proposition 2.8 of [2]. One may also consult Lemma 5.4 of [3], which is of similar nature. However, for the convenience of the reader we give a proof of the statement.

Write $S = \{q_1, \ldots, q_r\}$ for the set of prime divisors of n, and let k be as in the statement. First we show that $j(n) \geq k + 1$. Let $T = \{(q_1, c_1), \ldots, (q_r, c_r)\}$ be an S-covering of A. Let N be an integer such that

$$N \equiv -c_i \pmod{q_i} \text{ for } 1 \leq i \leq r.$$

By the Chinese Remainder Theorem such an N exists. Since T is a covering of A, for every $1 \leq j \leq k$ there exists a $c_{h(j)}$ with $1 \leq h(j) \leq r$ such that $j \equiv c_{h(j)} \pmod{q_{h(j)}}$. Then $N + j \equiv 0 \pmod{q_{h(j)}}$ implying that $\gcd(n, N + j) > 1$. Hence $j(n) \geq k + 1$. Now suppose that $j(n) > k + 1$. Then there exists a positive integer N such that $\gcd(n, N+i) > 1$ for $i = 1, 2, \ldots, k+1$. For each $q_j \in S$ $(j = 1, 2, \ldots, r)$ let c_j be the smallest positive integer such that q_j divides $N + c_j$. Then one can readily check that $T = \{(q_1, c_1), \ldots, (q_r, c_r)\}$ is an S-covering for $\{1, 2, \ldots, k + 1\}$ which violates the maximality of k. Hence the lemma follows. □

As a consequence of Lemma 2.1 we get the following property of the Jacobsthal function.

Lemma 2.2. *Let m be an odd positive integer. Then we have $j(2m) = 2j(m)$.*

Proof. Let $S = \{q_1, \ldots, q_t\}$ be the set of prime divisors of m. By the definition of $j(m)$, we find that S covers $\{1, 2, \ldots, j(m) - 1\}$ Hence S also covers $\{2, 4, \ldots, 2(j(m)-1)\}$. By covering the integers $\{1, 3, \ldots, 2j(m)-1\}$ by the prime 2, we find that the set $S' = \{2, q_1, \ldots, q_t\}$ covers $\{1, 2, \ldots, 2j(m) - 1\}$. Hence $j(2m) \geq 2j(m)$.

Suppose S' covers $\{1, 2, \ldots, j(2m) - 1\}$. By the maximality of $j(2m)$ and the properties of coverings mentioned in Section 2.1, we may assume that position of the prime 2 is 1 and $j(2m)$ is even. Then $\{2, 4, \ldots, j(2m) - 2\}$ are covered by S. Hence $\{1, 2, \ldots, \frac{j(2m)-2}{2}\}$ is covered by S. Thus $j(m) \geq \frac{j(2m)-2}{2} + 1 = j(2m)$. Now the lemma follows. □

2.2. **Getting rid of the prime 2.** As in [2], it turns out that in fact it is sufficient to work only with odd numbers. Write p_i^* for the i-th odd prime. Obviously, we have $p_i^* = p_{i+1}$. For any $r \geq 1$ define the functions $h^*(r)$ and $H^*(r)$ by

$$h^*(r) = j(p_1^* p_2^* \ldots p_r^*)$$

and

$$H^*(r) = \max_{\substack{\omega(n)=r \\ 2\nmid n}} j(n).$$

Then we clearly have $H^*(r) \geq h^*(r)$ for all $r \geq 1$. Further, Hagedorn proved that $h(r) = 2h^*(r-1)$ holds for all $r \geq 2$ (see Proposition 2.8 of [2]; note that in the notation of [2] we have $w(r) = h^*(r) - 1$). The next lemma provides a similar property for $H(r)$ and $H^*(r)$. We shall call any odd integer n for which $\omega(n) = r$ and $j(n) = H^*(r)$ as $(r, *)$-maximal.

Lemma 2.3. *For any $r \geq 2$ we have*

$$H(r) = \max(H^*(r), 2H^*(r-1)).$$

Proof. The proof is similar to that of the above mentioned statement concerning $h(r)$ and $h^*(r)$ from [2]. However, for the convenience of the reader we provide a complete argument.

Observe that

$$H(r) = \max(H^*(r), H'(r)) \tag{2.1}$$

where

$$H'(r) = \max_{\substack{\omega(n)=r \\ 2\mid n}} j(n).$$

Let N be an even square-free integer with $\omega(N) = r$ such that $j(N) = H'(r)$. Then by Lemma 2.2 we get that $j(N) = 2j(N/2)$, which gives

$$2H^*(r-1) \geq j(N) = H'(r).$$

On the other hand, let m be an $((r-1), *)$-maximal integer. Then using again Lemma 2.2, we get $j(2m) = 2H^*(r-1)$. This yields

$$H'(r) \geq 2H^*(r-1).$$

Thus we obtain $H'(r) = 2H^*(r-1)$, and the lemma follows by (2.1). $\square$

It is important to note that for all the r values occurring in the present paper we have $H'(r) \geq H^*(r)$, that is $H(r) = 2H^*(r-1)$. It is very much likely that this equality is valid for all $r > 1$.

2.3. **Making the problem finite.** As noted in the Introduction, it is important to make the calculation of $H(r)$ a finite problem for a given r. Obviously, we have $H(1) = H^*(1) = 2$. Further, (1.1) provides a completely explicit upper bound for $H(r)$. However, to calculate the exact values of $H(r)$ we need another tool. In fact, by Lemma 2.3 it is sufficient to deal with $H^*(r)$ instead of $H(r)$. The next lemma provides important information about "large" prime factors of n in calculating $j(n)$.

Lemma 2.4. *Let $n > 1$ be a square-free odd integer with $\omega(n) = r$ and write S for the set of prime divisors of n. Further, put $A = \{1, 2, \dots, j(n) - 1\}$. Then we have the following properties.*

i) If q is a prime divisor of n with $q > H^*(r-1)$ then in any S-covering of A, q covers exactly one element.

ii) Let q be a prime divisor of n with $q > p_r^*$. Suppose that there exists an S-covering of A in which q covers one element exclusively. Then there exists an odd prime $p \le p_r^*$ such that $j(pn/q) \ge j(n)$.

Proof. i) Suppose to the contrary that there is an S-covering T of A in which q covers at least two elements. Let $(q, c) \in T$ be the corresponding pair. Then the set $\{c + 1, \dots, c + q - 1\}$ is covered by $T \setminus \{(q, c)\}$. However, this is clearly possible only if $q - 1 < H^*(r - 1)$. Thus we get a contradiction, and the statement follows.

ii) Let T be an S-covering of A in which q covers at most one element exclusively; write a for this element. Such an element exists, since otherwise q may be used to cover $j(n)$, giving that T covers $\{1, \dots, j(n)\}$. This contradicts the definition of $j(n)$. Take an odd prime p such that $p \nmid n$ and $p \le p_r^*$. Since $\omega(n) = r$ and $q > p_r^*$, such a prime exists. Let c be the smallest positive integer $\equiv a \pmod p$ and replace the pair corresponding to q in T by (p, a). Then we get a covering of A, which by Lemma 2.1 shows that $j(pn/q) \ge j(n)$, and the statement follows. □

As a simple consequence of the previous lemma, the next statement inductively shows that from $r \ge 2$ on, it is sufficient to consider only finitely many integers to obtain the value of $H^*(r)$. We need the following notation: for an integer $m \ge 2$ let $P(m)$ denote the largest prime divisor of m.

Lemma 2.5. *Let $r \ge 2$, and set $M = \max(H^*(r - 1), p_r^*)$. Then we have*

$$H^*(r) = \max_{\substack{\omega(n)=r \\ 2\nmid n,\ P(n)\le M}} j(n).$$

Further, we can restrict the values of n on the right hand side to numbers for which any covering of $\{1, 2, \dots, j(n) - 1\}$ by the prime divisors of n is r-exclusive.

Proof. Let $r \ge 2$, and let n be a square-free $(r, *)$-maximal integer. Suppose that n is such that $P(n)$ is minimal with these properties, and write q for the largest prime divisor of n. Let $q > M$. Then by part i) of Lemma 2.4 we get that q covers at most one element of $A = \{1, 2, \ldots, j(n) - 1\}$ in any covering by the set S of prime divisors of n. Then part ii) of Lemma 2.4 gives that with some odd prime $p \nmid n$ and $p \leq p_r^*$, we have $j(pn/q) \geq j(n)$. However, this contradicts the minimality of $P(n)$.

Suppose now that we have an S-covering of A which is not r-exclusive. By part ii) of Lemma 2.4 on replacing a prime divisor $> p_r^*$ of n which covers at most one element exclusively with a prime $\leq p_r^*$, and repeating the process if necessary, ultimately we get an r-exclusive covering of A by the prime divisors of an appropriate n. Thus the statement follows. □

We note that this lemma proves to be very useful later on. Indeed, for a fixed r, to compute $H^*(r)$ we need only to check all the possible r-tuples consisting of odd primes $\leq M$ with M given in Lemma 2.5.

2.4. **The Principal Algorithm.**

Aim. We develop an algorithm to prove Theorems 1.2 and 1.3. In view of Lemma 2.3, it is sufficient to calculate the exact value of $H^*(r)$ for $r \leq 23$, and to get an upper bound for $H^*(24)$ which is less than $2H^*(23)$. To obtain the exact values of $H^*(r)$ we shall use Lemma 2.5. This involves calculating $j(n)$ with n odd and $P(n) \leq M$. For this we need to cover a set $A = \{1, 2, \ldots, k\}$ with a set $S = \{q_1, \ldots, q_r\}$ of r odd primes for suitably chosen k.

Simplifications and Modifications. Our algorithm is based on a modified version of an algorithm of Hagedorn [2]. The modifications are necessary due to the important difference that we need to consider several r-tuples of odd primes to find the value of $H^*(r)$ - in contrast with the calculation of $h^*(r)$, where only the primes $p_1^*, \ldots, p_r^*$ are needed. This causes a "combinatorial explosion" in the number of cases to be considered for a fixed r. Fortunately, since the conjecture fails already for a relatively small value of r, this does not yield a serious problem. However, to speed up the calculations, we apply the following considerations.

(a) If $H^*(r) > h^*(r)$ for some r, then for any r^*-maximal integer n, we necessarily have $P(n) > p_r^*$. Thus by part ii) of Lemma 2.4, when we consider coverings with the set of prime divisors of an odd number n, we can assume that every prime $q \mid n$ with $q > p_r^*$ exclusively covers at least two elements, i.e. we need to consider only r-exclusive coverings.

(b) We use the following ideas of Hagedorn.

(b.1) If we find that a subset S' of S with $|S'| = r'$ covers a subset A' of A with $|A \setminus A'| \le r - r'$, then the S'-covering of A' can be extended to an S-covering of A. Indeed, we use each of the remaining $r - r'$ primes in S for each of the elements of $A \setminus A'$ in a one-to-one manner to get an S-covering of A.
(b.2) Let A' be the largest subset of A which is covered by some set T' belonging to a subset S' of S. Let m_l be the maximal number of elements of $A \setminus A'$ which can be covered by a prime q_l in $S \setminus S'$. It is easy to see that if $\sum\limits_{q_l \in S \setminus S'} m_l < |A \setminus A'|$, then T' cannot be extended to an S-covering of A.

Main Steps of the Algorithm.
(i) We consider all possible positions in $A = \{1, 2, \dots, k\}$ of the primes in S exceeding p_r^* so that each such prime *exclusively* covers at least two elements of A.
(ii) We fix all possible positions of the other primes in S successively so that we get r-exclusive coverings.
(iii) When we find a covering satisfying (i) and (ii), we check that S does not cover $A \cup \{k+1\}$.
(iv) We list all possible coverings of A with S satisfying the properties (i)-(iii).

Conclusion. If the list in (iv) is empty, we conclude that no such covering exists. This implies that $j(n) \le k$. Otherwise, the list gives all possible r-exclusive coverings of A. Further, if in (iii) we get that these coverings do not cover $A \cup \{k+1\}$, then $j(n) = k+1$. Collecting the appropriate lists we can construct the set S_r of those primes which must divide any n which is r-maximal. (This is explained in the proof of Theorem 1.3.) Table 1 is prepared from these lists.

Implementation of the Principal Algorithm.

Initialization. Fix k and r to be positive integers. Let $L = \emptyset$; $A = \{1, 2, \dots, k\}$ and

$$S = \{q_1 < \cdots < q_u < q_{u+1} < \cdots < q_r\}$$

where the q_i's are odd primes and $q_{u+1} > p_r^* \ge q_u$.

(PA.1)
(a) Take a tuple $(c_{u+1}, \dots, c_r)$ with $1 \le c_j \le q_j$ $(j = u+1, \dots, r)$. Let

$$X_j = \{x \in A \mid x \text{ is exclusively covered by } q_j\}.$$

(b) If $|X_j| \ge 2$ for all j with $u+1 \le j \le r$, then put

$$T = \{(q_{u+1}, c_{u+1}), \dots, (q_r, c_r)\},$$

$T' = \emptyset$ and $r' = 1$, and go to **(PA.2)**.
(c) If (b) fails, execute (a) and (b) with another tuple $(c_{u+1}, \dots, c_r)$. If all the possible tuples are checked already, then stop.

(PA.2)
(a) If $r' = 0$ then go to **(PA.1)**.
(b) Take a new $c_{r'}$ with $1 \le c_{r'} \le q_{r'}$ and

$$|\{x \in X_j \mid x \not\equiv c_{r'} \pmod{q_{r'}}\}| \ge 2$$

for all $j \in \{u+1, \dots, r\}$. Replace the pair in T' corresponding to $q_{r'}$ by $(q_{r'}, c_{r'})$, and go to **(PA.3)**.
(c) If no such $c_{r'}$ exists or all of them have been considered already, then remove the pair corresponding to $q_{r'}$ from T', put $r' = r' - 1$ and go to (a).

(PA.3)
(a) Let A' be the maximal subset of A which is covered by $T \cup T'$.
(b) If $|A \setminus A'| \le u - r'$, then list into L all the appropriate S-coverings of A containing $T \cup T'$ as a subset, and return to step **(PA.2)**.
(c) For $l = r'+1, \dots, u$ put $m_l = \max\limits_{c_l \in M_l} |\{x \in A \setminus A' \ : \ x \equiv c_l \pmod{q_l}\}|$ where M_l is the set of integers c with $1 \le c \le q_l$ and

$$|\{x \in X_j \mid x \not\equiv c \pmod{q_l}\}| \ge 2$$

for all $j = u+1, \dots, r$. If $|A \setminus A'| > m_{r'+1} + \dots + m_u$ or $r' = u$ then return to step **(PA.2)**.
(d) In all the other cases put $r' = r' + 1$ and return to step **(PA.2)**.

Output. After some time the algorithm terminates at part (c) of **(PA.1)**. Its output is the set L of the appropriate r-exclusive coverings of A.

3. Proofs

We start with the proof of our first theorem.

Proof of Theorem 1.2. It is easy to see that $H(1) = H^*(1) = 2$. So we assume that $r \ge 2$. By Lemma 2.3 we have

$$H(r) = \max(H^*(r), 2H^*(r-1)) \text{ for any } r \ge 2. \tag{3.1}$$

Thus in order to compute the values of $H(r)$, we need only to compute $H^*(r)$ and use the relation (3.1). So we restrict to computing $H^*(r)$ for $r \ge 2$. Note that $h^*(1) = 2$ and as mentioned already, by Proposition 2.8 of [2], we have

$$h^*(r) = h(r+1)/2 \text{ for } r \ge 2.$$

Further, if $H^*(r-1) < p^*_{r+1}$ holds then we have $M < p^*_{r+1}$ in Lemma 2.5, i.e. the calculation of $H^*(r)$ is restricted to odd values n with $\omega(n) = r$ and $P(n) \le p^*_r$. This gives $n = p^*_1 \dots p^*_r$. That is, we have $H^*(r) = h^*(r)$ in this case. Combining these equalities we obtain that

$$H^*(r) = h(r+1)/2 \tag{3.2}$$

whenever

$$H^*(r-1) < p^*_{r+1}. \tag{3.3}$$

From the values of $h(r)$ given in Table 1 of [2], we check that (3.3) holds and then find the value in (3.2) for $2 \le r \le 18$. For example, when $r = 18$ then

$$H^*(r-1) = H^*(17) = 66 < 71 = p^*_{19} = p^*_{r+1}$$

and hence

$$H^*(18) = h(19)/2 = 76.$$

Next we take $r = 19$. Then Lemma 2.5 gives

$$H^*(19) = \max_{\substack{\omega(n)=r \\ 2\nmid n,\ P(n)\le 73}} j(n).$$

That is, the set of prime divisors of n can be any 19 element subset U of the set $S = \{3, 5, \dots, 73\}$ of the first 20 odd primes. We take $k = h^*(19) - 1 = 86$, i.e. $A = \{1, 2, \dots, 86\}$. Note that by the definition of $h^*(r)$, A can be covered by the first 19 odd primes. Further, by part ii) of Lemma 2.4 it is sufficient to check the possible r-exclusive coverings of A. For each U as above, we find all such possible coverings of A, by our Principal Algorithm. Then we check that these coverings do not cover the set $\{1, 2, \dots, 86, 87\}$. This shows that $H^*(19) = h^*(19) = 87$.

Let now $r = 20, 21, 22$. We use a similar method as above. In these cases the set S equals $\{3, 5, \dots, 83\}$, $\{3, 5, \dots, 89\}$ and $\{3, 5, \dots, 97\}$, respectively. Thus $|S| = r+2$. We take $A = \{1, 2, \dots, 94\}, \{1, 2, \dots, 99\}, \{1, 2, \dots, 107\}$, respectively. Then we consider all subsets $U \subset S$ with $|U| = r$ and all possible r-exclusive coverings T of the corresponding set A. By the same method as above, in each case we get that $H^*(r) = h^*(r)$. Note that as we need to choose subsets having r elements from a set having $r+2$ elements and then check all the possible coverings for each subset, the amount of computation increases considerably.

Let now $r = 23$. Then $S = \{3, 5, \dots, 103\}$ with $|S| = 26$. Now we take $A = \{1, 2, \dots, 117\}$. Here we need to consider subsets $U \subset S$ with

$|U| = 23$ and the possible sievings. We find the following covering of A:

$$\{(3,2),(5,4),(7,3),(11,4),(13,7),(17,8),(19,2),(23,13),$$
$$(29,3),(31,26),(37,30),(41,22),(43,12),(47,6),(53,43),(59,16),$$
$$(61,51),(67,60),(73,18),(79,27),(83,58),(89,28),(101,1)\}.$$

Note that here we use the first 23 odd primes, but with 71 replaced by 101. We find all the r-exclusive coverings of A and check that they cannot be extended to $\{1,2,\dots,118\}$. Hence we get $H^*(23) = 118$.

Lastly, let $r = 24$. From $H^*(23) = 118$ and Lemma 2.5 we get

$$H^*(24) = \max_{\substack{\omega(n)=r \\ 2\nmid n,\ P(n)\leq 113}} j(n).$$

Since $113 = p^*_{29}$, we obviously get $H^*(24) \leq h^*(29)$. As $h^*(29) = h(29)/2 = 165$ by Table 1 of [2], this yields $H^*(24) \leq 165$.

Having the exact values of $H^*(r)$ for $r \leq 23$ and the inequality $H^*(24) \leq 165$, by (3.1) we get the values of $H(r)$ for $r \leq 24$ appearing in Table 1. Hence the statement follows. $\square$

Now we give the proof of our second result.

Proof of Theorem 1.3. We need to show that the sets S_r given in Table 1 have property (1.2), and further that they are maximal with this property. We shall denote by $S^*_r = S_r \setminus \{2\}$ where S_r is given by Table 1. For $r = 1$ and for any odd n with $\omega(n) = 1$, $H(r) = j(n) = 2$. This yields that $S_1 = \emptyset$. For $r \geq 2$ we explain how the set S_r is obtained with an example.

Let $r = 13$. Then by (3.2), $H^*(13) = \dots$ and $2H^*(12) = 74$. Hence $H(r) = 74$. Thus the $13-$ maximal integers are even. We take $k = h^*(12) - 1 = H^*(12) - 1 = 36$, and again, we would like to find all coverings of the set $A = \{1,2,\dots,36\}$ with any twelve odd primes. As (3.3) holds in this case, it is sufficient to consider the set of the first twelve odd primes $S = \{3,5,\dots,41\}$. By Lemma 2.4(i), we need to consider those coverings in which the prime 41 covers at most one element. Using our Principal Algorithm we get that there are only two coverings of A by S, given by

$$\{(3,2),(5,1),(7,1),(11,2),(13,12),(17,10),$$
$$(19,9),(23,7),(29,4),(31,3),(37,18),(41,19)\}$$

and

$$\{(3,2),(5,1),(7,1),(11,2),(13,12),(17,10),$$
$$(19,9),(23,7),(29,4),(31,3),(37,19),(41,18)\}.$$

As one can easily check, the primes $3, 5, \ldots, 31$ exclusively cover at least two elements in both cases (e.g. 31 exclusively covers 3 and 34), while the primes 37 and 41 cover only one element each. Hence the primes 37 and 41 could be replaced by any other primes > 41, and we get

$$S_{12}^* = \{3, 5, \ldots, 31\}.$$

That is, if n is $(r, *)$-maximal with $r = 12$, then all the primes in S_{12}^* divide n, but n has no more fixed prime factors. Then following the argument of Lemmas 2.2 and 2.3, one can easily check that $S_{13} = S_{12}^* \cup \{2\}$, just as indicated in Table 1.

The method is similar for the other values of r. When $r \geq 19$ we need to check several coverings corresponding to many subsets $U \subset S$ with $|U| = r$ and $|S| > r$. In particular, given an r-exclusive covering T of A corresponding to some $U \subset S$, we have to take into consideration all possible coverings derived from T where some primes in U are replaced by elements of S which are $> p_r^*$. We explain this step by an example again. Let $r = 20$ and take $k = 94$, $A = \{1, 2, \ldots, 94\}$. Now $S = \{3, 5, \ldots, 83\}$ is the set of the first 22 odd primes and we take U to be a subset of S having $|U| = 20$. Then, using our Principal Algorithm we obtain all coverings T of A using such sets U. One of these coverings is given by

$$T = \{(3,1), (5,2), (7,2), (11,4), (13,11), (17,3), (19,18), (23,14),$$
$$(29,10), (31,4), (37,8), (41,33), (43,41), (47,6), (53,16),$$
$$(59,21), (61,29), (67,36), (71,38), (73,5)\}.$$

Now we need to find all coverings of A which can be derived from T. By part i) of Lemma 2.4 we know that every prime $> H^*(19) = 87$ can cover at most one element in each covering of A. Thus we have two spare primes 79 and 83 from S. We may use them to replace at most two pairs in T as follows. Take the pair $(53, 16)$. Then 53 covers 16 and 69. Note that 16 is also covered by 7 while 69 is covered exclusively by 53. Similarly, the primes 67 and 71 cover exclusively the numbers 36 and 38. Hence we can derive new coverings from T by replacing at most any two pairs in T corresponding to the primes $53, 67, 71$ by 79 and 83 and there are no other possible covers. For example, we get the covering

$$(3,1), (5,2), (7,2), (11,4), (13,11), (17,3), (19,18), (23,14),$$
$$(29,10), (31,4), (37,8), (41,33), (43,41), (47,6), (59,21),$$
$$(61,29), (71,38), (73,5), (79,69), (83,36).$$

This shows that $53, 67, 71 \notin S^*_{20}$. Checking all the other coverings of A with the appropriate sets U and combining the information obtained, we get that $S^*_{20} = \{3, \ldots, 47\} \cup \{59, 61\}$ and

$$S_{21} = \{2, 3, \ldots, 47\} \cup \{59, 61\}$$

just as indicated in Table 1.

Executing these steps, for each value of r we could find the set S^*_r of fixed prime factors of integers n which are $(r, *)$-maximal. Then similarly as above, we get $S_{r+1} = S^*_r \cup \{2\}$ in each case, just as given in Table 1. □

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