

ON A CONJECTURE OF POMERANCE

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Dedicated to Professor Schinzel on the occasion of his 75th birthday

ABSTRACT. We say that k is a P -integer if the first $\varphi(k)$ primes coprime to k form a reduced residue system modulo k . In 1980 Pomerance proved the finiteness of the set of P -integers and conjectured that 30 is the largest P -integer. We prove the conjecture assuming the Riemann Hypothesis. We further prove that there is no P -integer between 30 and 10^{11} and above 10^{3500} .

1. INTRODUCTION

Let $k > 1$ be an integer. We denote Euler's totient function by $\varphi(k)$ and the number of distinct prime divisors of k by $\omega(k)$. We say that k is a P -integer if the first $\varphi(k)$ primes coprime to k form a reduced residue system modulo k . In 1980, Pomerance [8] proved the finiteness of the set of P -integers. The following conjecture was proposed by him in [8].

Conjecture of Pomerance. If k is a P -integer, then $k \leq 30$.

This conjecture is still open. Recently, Hajdu and Saradha [3] and Saradha [12] have given simple conditions under which an integer k is not a P -integer. By their results, it follows that

- no prime is a P -integer except 2;
- no square or a cube of a prime is a P -integer except 4;
- no integer k with its least odd prime divisor $> \log k$ is a P -integer except when $k \in \{2, 4, 6, 12, 18, 30\}$.

It is easy to check that the only P -integers ≤ 30 are 2, 4, 6, 12, 18, 30. It was checked by computation in [3] that if k is another P -integer, then $k \geq 5.5 \cdot 10^5$. In Theorem 4.1 we improve this bound to 10^{11} .

2010 *Mathematics Subject Classification.* Primary 11N13.

Key words and phrases. primes in residue classes, Riemann Hypothesis.

¹ Research supported in part by the OTKA grants K67580 and K75566, and by the TÁMOP 4.2.1./B-09/1/KONV-2010-0007 project. The project is implemented through the New Hungary Development Plan, cofinanced by the European Social Fund and the European Regional Development Fund.

In this paper, we give a quantitative version of the finiteness result of Pomerance and prove the conjecture of Pomerance under the Riemann Hypothesis. We have

Theorem 1.1. *If k is a P -integer, then $k < 10^{3500}$.*

Theorem 1.2. *Suppose the Riemann Hypothesis holds. Then the only P -integers are 2, 4, 6, 12, 18, 30.*

Pomerance's conjecture is closely related to the classical problem about the least primes in arithmetic progressions. Let ℓ be a positive integer with $\gcd(k, \ell) = 1$. Denote by $p(k, \ell)$ the least prime $p \equiv \ell \pmod{k}$. Let $P(k)$ be the maximum value of $p(k, \ell)$ for all ℓ . Linnik [7] has shown that

$$P(k) \ll k^L$$

for some constant L which is known as Linnik's constant. A huge literature is available on finding good values for L (see [4, 15]). In the other direction, Prachar [9] and Schinzel [13] have shown that there is an absolute constant c such that for each ℓ there are infinitely many k with

$$p'(k, \ell) > \frac{ck \log k \cdot \log \log k \cdot \log \log \log \log k}{(\log \log \log k)^2}$$

where $p'(k, \ell)$ is the first prime q with $q \equiv \ell \pmod{k}$. In his proof of the finiteness of P -integers Pomerance [8] used the Jacobsthal function to show that

$$P(k) \geq (e^\gamma + o(1))\varphi(k) \log k$$

where γ is Euler's constant.

In our proofs we applied different tools. We use that the primitive residues modulo k between 0 and k are symmetric around $k/2$. Our arguments are based on results about the zeros of the Riemann zeta function and estimates for the number of primes in intervals.

2. LEMMAS

Throughout the paper, let $p_1 < p_2 < \dots$ be the increasing sequence of prime numbers. For any $x > 1$, let $\pi(x)$ denote the number of prime numbers not exceeding x , and $\text{Li}(x) = \lim_{\epsilon \rightarrow 0^+} \int_{t=0}^{1-\epsilon} \frac{dt}{\log t} + \int_{t=1+\epsilon}^x \frac{dt}{\log t}$. We put $\pi(x) = 0$ for $0 \leq x \leq 1$.

Lemma 2.1. *For any $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we have*

- (i) $\pi(x) > \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{1.8x}{\log^3 x}$ for $x > 32299$;
- (ii) $\pi(x) < \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2.51x}{\log^3 x}$ for $x > 355991$;
- (iii) $|\pi(x) - \text{Li}(x)| < .4394 \frac{x}{(\log x)^{3/4}} \exp\left(-\sqrt{\frac{\log x}{9.646}}\right)$ for $x \geq 58$;

- (iv) if the Riemann Hypothesis holds, then $|\pi(x) - \text{Li}(x)| < \frac{1}{8\pi}\sqrt{x} \log x$ for $x > 2656$;
- (v) $\text{Li}(x) > \pi(x)$ for $x \leq 10^{14}$;
- (vi) $p_n < n(\log n + \log \log n)$ for $n \geq 6$;
- (vii) $p_n > n \log n$ for $n \geq 1$;
- (viii) $\frac{n}{\varphi(n)} < 1.7811 \log \log n + \frac{2.51}{\log \log n}$ for $n \geq 3$.

Proof. We mention the references where the estimates from Prime Number Theory given in the lemma can be found.

(i), (ii) Dusart [2], p. 36.

(iii) Dusart [2], p. 41.

(iv) Schoenfeld [14], p. 339.

(v) Kotnik [6], p. 59.

(vi), (vii) Rosser and Schoenfeld [10], p. 69.

(viii) Rosser and Schoenfeld [10], p. 72. □

Lemma 2.2. *Let x be a real number with $x > 712000$. Then we have*

$$2\pi\left(\frac{x}{2}\right) - \pi(x) > \frac{.693x}{\log^2 x}.$$

Proof. We have, by Lemma 2.1 (i), (ii), for $x > 712000$,

$$\begin{aligned} & 2\pi(x/2) - \pi(x) > \\ & \frac{x}{\log(x/2)} + \frac{x}{\log^2(x/2)} + \frac{1.8x}{\log^3(x/2)} - \frac{x}{\log x} - \frac{x}{\log^2 x} - \frac{2.51x}{\log^3 x} > \\ & \frac{x}{\log x \left(1 - \frac{\log 2}{\log x}\right)} - \frac{x}{\log x} + \frac{x}{\log^2 x \left(1 - \frac{\log 2}{\log x}\right)^2} - \frac{x}{\log^2 x} - \frac{.71x}{\log^3 x} > \\ & \frac{x}{\log x} \cdot \frac{\log 2}{\log x} + \frac{x}{\log^2 x} \cdot \frac{2 \log 2}{\log x} - \frac{.71x}{\log^3 x} > \frac{.693x}{\log^2 x}. \end{aligned}$$

□

Lemma 2.3. *Let x and y be positive real numbers with $x > y$, $x \geq 59$. Then*

$$\begin{aligned} & 2\pi(x+y) - \pi(x) - \pi(x+2y) > \\ & \frac{y^2}{(x+2y)\log^2(x+2y)} - \frac{1.7576(x+2y)}{(\log x)^{3/4}} e^{-\sqrt{\frac{\log x}{9.646}}}. \end{aligned}$$

Proof. By Lemma 2.1 (iii),

$$\begin{aligned} & 2\pi(x+y) - \pi(x) - \pi(x+2y) > \\ & 2\text{Li}(x+y) - \text{Li}(x) - \text{Li}(x+2y) - 1.7576 \frac{x+2y}{(\log x)^{3/4}} \exp\left(-\sqrt{\frac{\log x}{9.646}}\right). \end{aligned}$$

Observe that

$$\begin{aligned} 2\text{Li}(x+y) - \text{Li}(x) - \text{Li}(x+2y) &= \int_x^{x+y} \frac{dt}{\log t} - \int_{x+y}^{x+2y} \frac{dt}{\log t} \\ &= \int_x^{x+y} \left(\frac{1}{\log t} - \frac{1}{\log(t+y)} \right) dt = \frac{y^2}{\xi \log^2 \xi} \end{aligned}$$

for some ξ with $x < \xi < x+2y$, by the mean value theorem applied twice. Thus

$$\begin{aligned} &2\pi(x+y) - \pi(x) - \pi(x+2y) > \\ &\frac{y^2}{(x+2y) \log^2(x+2y)} - 1.7576 \frac{x+2y}{(\log x)^{3/4}} \exp\left(-\sqrt{\frac{\log x}{9.646}}\right). \end{aligned}$$

□

Lemma 2.4. *Suppose the Riemann Hypothesis holds true.*

Let $x > y > 0$, $x \geq 2657$. Then

$$\begin{aligned} &2\pi(x+y) - \pi(x) - \pi(x+2y) > \\ &\frac{y^2}{(x+2y) \log^2(x+2y)} - \frac{\log(x+2y)}{\theta} \sqrt{x+2y} \end{aligned}$$

where

$$\theta = \begin{cases} 2\pi & \text{if } x+2y > 10^{14} \\ 4\pi & \text{if } x+2y \leq 10^{14}. \end{cases}$$

Proof. By Lemma 2.1 (iv), (v),

$$\begin{aligned} &2\pi(x+y) - \pi(x) - \pi(x+2y) > \\ &2\text{Li}(x+y) - \text{Li}(x) - \text{Li}(x+2y) - \frac{\log(x+2y)}{\theta} \sqrt{x+2y}. \end{aligned}$$

The lemma follows in the same way as in the proof of Lemma 2.3. □

3. A CRITERION FOR AN INTEGER k TO BE NOT A P -INTEGER

Suppose k is a P -integer > 30 . Further, due to results from [3] and [12] mentioned in the introduction, we may also assume that neither k nor $k/2$ is prime. Let $\varphi(k) + \omega(k) = T$. Then there are exactly $\varphi(k)$ primes belonging to the set $\{p_1, \dots, p_T\}$ which are coprime to k and form a reduced residue system mod k . The remaining $\omega(k)$ primes in this set divide k . Let

$$\begin{aligned} D'_k &= \left\{ i \leq T : p_i \pmod{k} < \frac{k}{2} \right\}, \\ D''_k &= \left\{ i \leq T : p_i \pmod{k} \geq \frac{k}{2} \right\} \end{aligned}$$

and

$$D_k''' = \{i \leq T : p_i | k\}.$$

Note that $|D_k''| = \omega(k)$ where $|A|$ denotes the number of elements of a set A . By the symmetry of the primitive residues about $k/2$, we get

$$|D_k' \setminus D_k''| = |D_k'' \setminus D_k''|$$

which implies

$$(1) \quad |D_k'| - |D_k''| \leq |D_k'''| = \omega(k).$$

Let t be an integer such that $tk < p_T < (t+1)k$. We observe that if $p_T \in (tk, tk + \frac{k}{2})$ we have

$$|D_k'| = \sum_{n=0}^{t-1} \left(\pi \left(nk + \frac{k}{2} \right) - \pi(nk) \right) + T - \pi(tk),$$

$$|D_k''| = \sum_{n=0}^{t-1} \left(\pi(nk + k) - \pi \left(nk + \frac{k}{2} \right) \right)$$

and if $p_T \in (tk + \frac{k}{2}, tk + k)$, then

$$|D_k'| = \sum_{n=0}^t \left(\pi \left(nk + \frac{k}{2} \right) - \pi(nk) \right),$$

$$|D_k''| = \sum_{n=0}^{t-1} \left(\pi(nk + k) - \pi \left(nk + \frac{k}{2} \right) \right) + T - \pi \left(tk + \frac{k}{2} \right).$$

Thus we get

$$|D_k'| - |D_k''| = \sum_{n=0}^{t-1} \left(2\pi \left(nk + \frac{k}{2} \right) - \pi(nk) - \pi(nk + k) \right) + T - \pi(tk)$$

in the former case, and in the latter case

$$|D_k'| - |D_k''| = \sum_{n=0}^t \left(2\pi \left(nk + \frac{k}{2} \right) - \pi(nk) - \pi(nk + k) \right) + \pi(tk + k) - T.$$

Let $L(k) = t - 1$ in the former case and $L(k) = t$ in the latter. Let $L := L(k)$. We shall use this parameter L later on without any further mentioning. Noting that $T - \pi(tk)$ and $\pi(tk + k) - T$ are both non-negative and that $\omega(k) < \log k$, we find by (1) the following criterion.

Lemma 3.1. *The integer k is not a P -integer, if*

$$S_L := \sum_{n=0}^L \left(2\pi \left(nk + \frac{k}{2} \right) - \pi(nk) - \pi(nk + k) \right) - \log k > 0.$$

We note that

$$tk < p_T \leq p_k \leq k \log(k \log k)$$

by Lemma 2.1 (vi). Thus

$$(2) \quad L \leq t < \log(k \log k).$$

On the other hand, using Lemma 2.1 (vii), (viii), putting $h(k) = 1.7811 \log \log k + \frac{2.51}{\log \log k}$, we get

$$(3) \quad L + 2 \geq t + 1 > \frac{p_T}{k} \geq \frac{p_{\varphi(k)}}{k} > \frac{\log k - \log h(k)}{h(k)}.$$

4. A COMPUTATIONAL RESULT

Theorem 4.1. *If $30 < k \leq 10^{11}$, then k is not a P -integer. Further, if k is even with $30 < k \leq 2 \cdot 10^{11}$ then k is not a P -integer.*

Proof. In [3] it has been computationally verified that no integer k with $30 < k < 5.5 \cdot 10^5$ is a P -integer. Hence we may assume henceforth that

$$5.5 \cdot 10^5 \leq k \leq 2 \cdot 10^{11}.$$

To cover this interval, we apply a modified version of the algorithm used in [3].

To prove the statement for a given k we apply the following strategy. We find a prime p such that $k < p < p_{\varphi(k)}$ and $p \pmod{k}$ is also a prime. Then k is not a P -integer. To make this strategy work on the whole range for k under consideration, we shall make use of the following two properties. Let k be an integer with $k \geq 5.5 \cdot 10^5$. Then we have

$$(4) \quad \pi(k+1) + 100 < \varphi(k)$$

and

$$(5) \quad p_{\pi(k+1)+100} < 1.5k.$$

These assertions can be easily checked e.g. by Magma [1], using parts (ii), (vi), (viii) of Lemma 2.1.

First we prove the statement for the even values of k . This is done by the algorithm below, which is based on the strategy indicated above.

Initialization. Let $k_0 = 5.5 \cdot 10^5$. Let H be the list of the first 100 primes larger than $k_0 + 1$, i.e. $H = [p_{\pi(k_0+1)+1}, \dots, p_{\pi(k_0+1)+100}]$.

Step 1. Check successively for the primes $p \in H$ whether $p \pmod{k_0}$ is also a prime. When such a p is found then, by (4), k_0 is not a P -integer – proceed to the next step.

Step 2. Check if $k_0 + 3$ is a prime. If not, then proceed to Step 3. If so, this is the first element of H . Remove this prime from H , and append to H the prime $p_{\pi(k_0+1)+101}$ which is the next prime to the last element of H .

Step 3. If $k_0 < 2 \cdot 10^{11}$ then put $k_0 := k_0 + 2$, and go to Step 1.

Using this procedure we could check by a Magma program that there is no even P -integer in the interval $[5.5 \cdot 10^5, 2 \cdot 10^{11}]$.

Let now k be odd with $5.5 \cdot 10^5 < k < 10^{11}$. Then by our algorithm above, using (4) and (5), we know that there exists a prime p satisfying $2k < p < \min\{3k, p_{\varphi(2k)}\}$ such that $q := p \pmod{2k}$ is also a prime. Observe that $q < k$. Thus, as $\varphi(k) = \varphi(2k)$, p is a prime such that $k < p < p_{\varphi(k)}$ and $q = p \pmod{k}$ is also a prime. Hence k is not a P -integer and the theorem follows. \square

5. PROOFS OF THEOREMS 1.1 AND 1.2

Proof of Theorem 1.1. Let k be an integer with $k \geq 10^{3500}$. Then by (3), $L > 500$. We apply Lemma 2.1 (i), (ii) to get

$$2\pi(k/2) - \pi(k) > \frac{k}{\log(k/2)} + \frac{k}{\log^2(k/2)} + \frac{1.8k}{\log^3(k/2)} - \frac{k}{\log k} - \frac{k}{\log^2 k} - \frac{2.51k}{\log^3 k}.$$

For $n \geq 1$ we apply Lemma 2.3 with $x = nk$, $y = k/2$ to find

$$2\pi(nk + k/2) - \pi(nk) - \pi(nk + k) > \frac{k}{4(n+1)\log^2(nk+k)} - 1.7576 \frac{nk+k}{(\log nk)^{3/4}} \exp\left(-\sqrt{\frac{\log(nk)}{9.646}}\right)$$

Put

$$f_0(k) := \frac{k}{\log \frac{k}{2}} + \frac{k}{\log^2 \frac{k}{2}} + \frac{1.8k}{\log^3 \frac{k}{2}} - \frac{k}{\log k} - \frac{k}{\log^2 k} - \frac{2.51k}{\log^3 k} - \log k,$$

$$f_n(k) := \frac{k}{4(n+1)\log^2(nk+k)} - 1.7576 \frac{nk+k}{(\log nk)^{3/4}} \exp\left(-\sqrt{\frac{\log(nk)}{9.646}}\right)$$

for $n \geq 1$. A simple calculation shows that S_L , defined in Lemma 3.1, satisfies

$$S_L \geq f_0(k) + \sum_{n=1}^L f_n(k) > 0$$

for $L \leq 1500$. This shows that k is not a P -integer for such L . Hence we may assume that $L > 1500$.

We first check by Maple that $f_n(k)$ is a strictly monotone decreasing function of n . By (2) it is therefore enough to show that

$$f_0(k) + \sum_{i=1}^{1500} f_i(k) + (L - 1500)f_n(k) > 0$$

for $k = 10^{3500}$ and $n = \lfloor \log(k \log k) \rfloor$. We check this again with Maple to get the final contradiction. \square

Remark. The constant 9.646 which occurs in Lemma 2.1 (iii) originates from a zero-free region of the Riemann-zeta function derived by Rosser and Schoenfeld ([11] Theorem 11), where the constant appears as R . The zero-free region has been widened by Kadiri [5] where the corresponding constant R is 5.69693. If this constant would be substituted into Lemma 2.1 (iii) instead of the constant 9.646 and we follow our argument, we obtain that if k is a P -integer, then $k < 10^{1000}$. However, we do not know if this substitution is justified.

Proof of Theorem 1.2. Suppose the Riemann Hypothesis is true. Let k be an integer with $k \geq 3 \cdot 10^{13}$. By Lemma 2.2, we get

$$2\pi \left(\frac{k}{2} \right) - \pi(k) > \frac{.693k}{\log^2 k} > \log k > \omega(k).$$

For $n = 1, 2, \dots, \lfloor \log(k \log k) \rfloor - 1$ we apply Lemma 2.4 with $x = nk$, $y = k/2$ to find

$$2\pi \left(nk + \frac{k}{2} \right) - \pi(nk) - \pi(nk + k) > \frac{k}{4(n+1)\log^2(nk+k)} - \frac{\log(nk+k)}{2\pi} \sqrt{nk+k}.$$

The term on the right hand side of the above inequality is positive if

$$\pi\sqrt{k} > 2(n+1)^{1.5} \log^3(nk+k).$$

This is satisfied, since $n < \log(k \log(k)) - 1$ and $k \geq 3 \cdot 10^{13}$. Hence by Lemma 3.1, we find that k is not a P -integer.

Next we take $k < 3 \cdot 10^{13}$. By Theorem 4.1, we may assume $k > 10^{11}$. Note that $L < \log(k \log k) \leq 34$. Further

$$L < \log k + \log \log k < 1.13 \log k$$

giving

$$k > e^{.88L} > 10^{.38L}.$$

Define

$$k_L = \lfloor 10^{\{.38L\}} \rfloor 10^{[.38L]}.$$

where $[x]$ and $\{x\}$ denote the integral and fractional part of any real number x . Note that for any fixed L with $L \leq 34$ if $L(k) \geq L$, then $k \in [k_L, 3 \cdot 10^{13}]$. Applying Lemma 2.4 with $x = nk$, $y = k/2$ we find

$$S_L > 2\pi(k/2) - \pi(k) + \sum_{n=1}^L \left(\frac{k}{4(n+1)\log^2(nk+k)} - \frac{\log(nk+k)}{4\pi} \sqrt{nk+k} \right).$$

For $n = 1, \dots, L$, put

$$\begin{aligned} F_n(k) := & \frac{1}{L} \left(\frac{k}{\log(k/2)} + \frac{k}{\log^2(k/2)} + \frac{1.8k}{\log^3(k/2)} \right) \\ & - \frac{1}{L} \left(\frac{k}{\log k} + \frac{k}{\log^2 k} + \frac{2.51k}{\log^3 k} + \log k \right) \\ & + \frac{k}{4(n+1)\log^2(nk+k)} - \frac{\log(nk+k)}{4\pi} \sqrt{nk+k}. \end{aligned}$$

We have, by Lemma 2.1 (i), (ii),

$$S_L - \log k > \sum_{n=1}^L F_n(k).$$

So it is sufficient to show that the right hand side is positive. For this, we proceed as follows. First, let $29 \leq L \leq 34$. We calculate the value k_L from its definition above. Thus (L, k_L) is one of the pairs from $\{(29, 10^{11}), (30, 2 \cdot 10^{11}), (31, 6 \cdot 10^{11}), (32, 10^{12}), (33, 3 \cdot 10^{12}), (34, 8 \cdot 10^{12})\}$.

We check by Maple that all functions $F_n(k)$ are strictly monotone increasing on $[k_L, 3 \cdot 10^{13}]$, and further

$$\sum_{n=1}^L F_n(k_L) > 0.$$

Hence by Lemma 3.1, there is no P -integer k with $L(k) \in [29, 34]$. Now we consider $k \in [10^{11}, 3 \cdot 10^{13}]$. Then obviously $L(k) > 0$. We may assume $1 \leq L \leq 28$. We check that all functions $F_n(k)$ are strictly monotone increasing and the preceding inequality also holds. Hence we conclude that no integer $k \in [10^{11}, 3 \cdot 10^{13}]$ is a P -integer. \square

6. ACKNOWLEDGEMENTS

The authors are grateful to the referee for her/his helpful and useful remarks.

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