PROPERTIES OF GENERALIZED NEIGHBOURHOOD SEQUENCES IN FINITE DIMENSION

Attila Fazekas, András Hajdu and Lajos Hajdu

Abstract. In this paper we investigate the set (and some of its subsets) of the generalized nD-neighbourhood sequences. We give a natural ordering relation on the sets examined. Unfortunately, this relation is proved to have some unkind properties, so we introduce another (not so natural) relation. We can show that the investigated sets form a complete distributive lattice with respect to this relation. Our analyses generalize the 2D and 3D results of Das [1] and Fazekas [3], respectively, besides summarising the existing ones.

1. INTRODUCTION

In digital image processing the space, where the analyses take place usually is the 2-dimensional digital plane $\mathbb{Z}^2$. We can move on the square grid in horizontal, vertical or diagonal directions. Using these basic steps Rosenfeld and Pfaltz defined two kind of motions in two-dimensional digital geometry (see [4]). In cityblock motion horizontal and vertical movements are allowed, while in chessboard motion we can use diagonal directions only. The mixed use of these motions led to the octagonal distance.

The definition of octagonal distance was extended by Das, Chakrabarti and Chatterji (see [2]), when they introduced the concept of neighbourhood sequence. Neighbourhood sequences were defined as arbitrarily long periodic sequences of cityblock and chessboard movements. The first problem which required the use of these sequences was of drawing a circle in the 2D digital plane. Neighbourhood sequences are also can be used in approximating Cartesian distance, since a distance function can be associated to every neighbourhood function, which is however, does not give a metric in the nD digital plane. To compute this distance, the authors constructed a formula.

After this, Das in [1] introduced a "natural" partial ordering relation on the set of periodic 2D-neighbourhood sequences. With respect to this ordering, he investigated the structure of this set and some of its subsets. As for the 3D results, Fazekas recently proved that a similar partial ordering can also be introduced for neighbourhood sequences in 3D (see [3]).

Our aim was to summarise the results about neighbourhood sequences, perform some generalization and extend the existing results. We generalize the concept of neighbourhood sequences, which can be done by allowing not periodic sequences only. The previous results about ordering the set of periodic neighbourhood sequences can be extended to arbitrary dimension. The structure of the set and some subsets of these generalized neighbourhood sequences is also investigated under this ordering in n-dimension. Since the "natural" ordering do not result nice structures in general, we introduce another relation, for which we can usually obtain lattices with some nice properties.

2. BASIC CONCEPTS AND NOTATIONS

First, we give the basic concepts and notations that we used in our analyses. As we usually work in n-dimensional (nD, briefly) spaces, n will denote an arbitrary positive integer throughout the paper. Our investigations take place in $\mathbb{Z}^n$ or in other words in the nD *digital plane*.

Definition. *Let p and q be two points in $\mathbb{Z}^n$. Let $\mathrm{Pr}_i(p)$ denote the* i*th coordinate of the point p. Let m be an integer such that $0 \leq m \leq n$. p and q are m-neighbours, if:*

- $|\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)| \leq 1$ *for* $1 \leq i \leq n$, *and*
- $\sum_{i=1}^{n} |\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)| \leq m$.

Definition. *The infinite sequence $B = \{b(i) : i \in \mathbb{N}$ and $b(i) \in \{1, 2, \ldots, n\}\}$ is called a generalized nD-neighbourhood sequence. If for some $l \in \mathbb{N}$, $b(i) = b(i + l)$ holds for every $i \in \mathbb{N}$, then B is called periodic, with a period l, or simply l-periodic. In this case we will use the abbreviation $B = \{b(1), \ldots, b(l)\}$.*

Remark. *This concept of the generalized nD-neighbourhood sequences is a generalization of the notion of neighbourhood sequences introduced in* [2]*. In* [2]*, and later in* [1] *and* [3] *only periodic sequences were investigated.*

Definition. *Let p and q be two points in $\mathbb{Z}^n$ and $B = \{b(i) : i \in \mathbb{N}\}$ a generalized nD-neighbourhood sequence. The point sequence $\Pi(p, q; B)$ – which has the form $p = p_0, p_1, \ldots, p_m = q$, where p_{i-1} and p_i are $b(i)$-neighbours for $1 \leq i \leq m$ – is called a path from p to q determined by B. The length $|\Pi(p, q; B)|$ of the path $\Pi(p, q; B)$ is m.*

Definition. *Let p and q be two points in $\mathbb{Z}^n$ and B a generalized nD-neighbourhood sequence. The distance of p and q is defined as the length of the minimal path $\Pi^*(p, q; B)$.*

$$d(p, q; B) = |\Pi^*(p, q; B)|.$$

If we define the distance in the above way, then we cannot have a metric in $\mathbb{Z}^n$ for every nD-neighbourhood sequence. It can be illustrated with a very simple counterexample. Let $B = \{2, 1\}$, $n = 2$, $p = (0, 0)$, $q = (1, 1)$ and $r = (2, 2)$. Now $d(p, q; B) = 1$, $d(q, r; B) = 1$, but $d(p, r; B) = 3$.

There is a result in [2] about how we can decide whether the distance function related to a periodic neighbourhood sequence is a metric on the n-dimensional digital plane, or not.

Notation. *Let p and q be two points in $\mathbb{Z}^n$, and $B = \{b(i) : i \in \mathbb{N}\}$ a generalized nD-neighbourhood sequence. If B is periodic then let l denote any of its periods, otherwise put $l = \infty$. Let*

$$x = (x(1), x(2), \ldots, x(n)),$$

where x *is the nonincreasing ordering of* $|\operatorname{Pr}_i(p) - \operatorname{Pr}_i(q)|$*, that is,* $x(i) \geq x(j)$ *if* $i < j$*. Put*

$$a_i = \sum_{j=1}^{n-i+1} x(j),$$

$$b_i(j) = \begin{cases} b(j), & \text{if } b(j) < n-i+2, \\ n-i+1, & \text{otherwise}, \end{cases}$$

$$f_i(j) = \begin{cases} \sum_{k=1}^{j} b_i(k), & \text{if } 1 \leq j \leq l, \\ 0, & \text{if } j = 0. \end{cases}$$

The following result (cf. [2]) provides an algorithm for the calculation of the distance $d(p, q; B)$ defined above.

Theorem (see [2]). *Let* p *and* q *be two points in* $\mathbb{Z}^n$*, and* $B = \{b(i) : i \in \mathbb{N}\}$ *a periodic* nD*-neighbourhood sequence with period* l*. Using the above notations, for* $i = 1, \ldots, n$ *put*

$$g_i(j) = f_i(l) - f_i(j-1) - 1, \quad 1 \leq j \leq l.$$

We can determine the length of the minimal path (distance) between p *and* q *determined by* B *in the following way:*

$$d(p, q; B) = \max_{i=1}^{n} d_i(p, q),$$

$$\text{where } d_i(p, q) = \sum_{j=1}^{l} \left\lfloor \frac{a_i + g_i(j)}{f_i(l)} \right\rfloor.$$

The set of the neighbourhood sequences forms a lattice with several properties, so we need some definitions and remarks from lattice theory, as well.

Definition. *Let* $(P, \leq)$ *be a partially ordered set. An element* $a \in P$ *is the least upper bound (greatest lower bound) of a subset* $S \subseteq P$ *if for all* $x \in S$*,* $a \geq x$ $(a \leq x)$*, and* $b \geq a$ $(b \leq a)$ *for every upper bound (lower bound)* b *of* S*. Moreover, if every pair of elements* $\{(x, y) : x, y \in P\}$ *has a least upper bound* $x \vee y$ *and a greatest lower bound* $x \wedge y$ *then* $(P, \leq)$ *is called a lattice.*

Definition. *The lattice* $(P, \leq)$ *is distributive if for all* $x, y, z \in P$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Remark. *Clearly,* $(P, \leq)$ *is distributive if and only if*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Definition. *The lattice* $(P, \leq)$ *is complete if its every subset* $S \subseteq P$ *has a least upper bound* $\bigvee S$ *and a greatest lower bound* $\bigwedge S$*.*

Remark. *It is well known that* $(P, \leq)$ *is complete if its every subset has a least upper bound.*

Definition. *Let* $(P, \leq)$ *be a complete lattice and* $S \subseteq P$*. The set* $S^c = \{x \in P : x \leq \bigvee S\}$ *is called the closure of* S*.*

3. NEIGHBOURHOOD SEQUENCES IN FINITE DIMENSION

In this chapter we examine whether a relation exists between the distance functions generated by two neighbourhood sequences. This question was investigated in [1] first (for periodic 2D neighbourhood sequences), and we recall an example from that paper to describe the complexity of this problem. Let $B_1 = \{1, 1, 2\}$, $B_2 = \{1, 1, 1, 2, 2, 2\}$. Let $o = (0, 0)$, $p = (3, 1)$ and $q = (6, 3)$. In this case $d(o, p; B_1) = 3 < 4 = d(o, p; B_2)$, but $d(o, q; B_1) = 7 > 6 = d(o, q; B_2)$. So the distances generated by B_1 and B_2 cannot be compared.

The authors in [1] showed that a nice ordering relation can be introduced for periodic neighbourhood sequences in 2D with the help of the functions $f_i(j)$ (see the Notation section in Chapter 2). Fazekas proved a similar statement in 3D (see [3]). We extend these results to nD with arbitrary $n \in \mathbb{N}$, to generalized nD-neighbourhood sequences. We do not use periodic sequences any more, so our result is new for $n = 2$ and 3, as well.

We would like to compute the distance of two points with respect to a generalized nD-neighbourhood sequence. To do this, first we need the following lemma.

Lemma. *Let p and q be two points in $\mathbb{Z}^n$ with $\sum_{i=1}^{n} |\operatorname{Pr}_i(p) - \operatorname{Pr}_i(q)| = l$. Let $A = \{a(i) : i \in \mathbb{N}\}$ and $B = \{b(i) : i \in \mathbb{N}\}$ be two generalized nD-neighbourhood sequences, with $a(i) = b(i)$ for $i \le l$. Then $d(p, q; A) = d(p, q; B)$.*

Proof. First, it is clear that $d(p, q; A) \le l$. Let $d(p, q; A) = h$, and let $p = p_0, p_1, \dots, p_h = q$ be a path from p to q determined by A in $\mathbb{Z}^n$. However, by $h \le l$ and $a(i) = b(i)$ for $1 \le i \le l$, we obtain that p_{i-1} and p_i are $b(i)$-neighbours for $i = 1, \dots, h$, hence $d(p, q; B) \le h = d(p, q; A)$. The opposite inequality can be proved in a similar way, and the lemma follows. □

Theorem. *Using the above introduced notation, for any generalized nD-neighbourhood sequences $B_1 = \{b^{(1)}(i) : i \in \mathbb{N}\}$ and $B_2 = \{b^{(2)}(i) : i \in \mathbb{N}\}$*

$$d(p, q; B_1) \le d(p, q; B_2), \qquad \text{for all } p, q \in \mathbb{Z}^n$$

if and only if

$$f_k^{(1)}(i) \ge f_k^{(2)}(i), \qquad \text{for all } i \in \mathbb{N},\ k \in \{1, \dots, n\},$$

where $f_k^{(1)}(i)$ and $f_k^{(2)}(i)$ correspond to B_1 and B_2, respectively.

Proof. The theorem can be proved similarly to the 2D and 3D case (c.f. [2,3]). For length limitations we omit this proof. □

Now we go on with examining the structure of the set of the generalized nD-neighbourhood sequences. It is interesting to analyse some of its subsets, as well.

Definition. *Let S_n, S'_n, and $S'_n(l)$ be the sets of generalized, periodic, and l-periodic ($l \in \mathbb{N}$) nD-neighbourhood sequences, respectively. For any $B_1, B_2 \in S_n$ the relation $\sqsupseteq^*$ is defined in the following way:*

$$B_1 \sqsupseteq^* B_2 \qquad \Leftrightarrow \qquad f_k^{(1)}(i) \ge f_k^{(2)}(i)$$

for all $i \in \mathbb{N}$ and $k \in \{1, \dots, n\}$.

Remark. *According to the previous theorem the relation $\sqsupseteq^*$ is a partial ordering in S_n, and also all in its subsets. Moreover, this relation $\sqsupseteq^*$ in 2D and 3D, is identical to those introduced by Das [1] and Fazekas [3], respectively.*

In most cases the sets S_n, S'_n, and $S'_n(l)$ do not form a nice structure under $\sqsupseteq^*$. The following statements provide some information about the structural properties of the investigated sets. As we are limited in length here, we usually omit the proof of the statements.

Theorem. *$(S_2, \sqsupseteq^*)$ is a complete distributive lattice.*

Unfortunately, this theorem does not hold in higher dimensions.

Proposition. *$(S_n, \sqsupseteq^*)$ is not a lattice for $n \geq 3$.*

Proof. We can find a counterexample to prove this statement. □

Similar unkind properties of $\sqsupseteq^*$ also occur in case of some special sets of periodic sequences. In the following we present these "negative" results. The "negative" statements can be proved by finding a counterexample in that special case.

Proposition. *$(S'_n, \sqsupseteq^*)$ is not a lattice for $n \geq 2$.*

As we mentioned above, $S'_2(l)$ is a distributive lattice for every $l \in \mathbb{N}$ (see [1]). This statement cannot be generalized to nD with $n \geq 3$.

Proposition. *$(S'_n(l), \sqsupseteq^*)$ is not a lattice for any $l \geq 2$, $n \geq 3$.*

These results show that we cannot obtain a nice structure neither in S_n, nor in various subsets of it with respect to the relation $\sqsupseteq^*$. A new ordering relation can be introduced with close connection to $\sqsupseteq^*$. Moreover, S_n and its investigated subsets will form much nicer structures under this new relation.

Definition. *For any $B_1 = \{b^{(1)}(i) : i \in \mathbb{N}\}$, $B_2 = \{b^{(2)}(i) : i \in \mathbb{N}\} \in S_n$ we define the relation $\sqsupseteq$ in the following way:*

$$B_1 \sqsupseteq B_2 \quad \Leftrightarrow \quad b^{(1)}(i) \geq b^{(2)}(i), \quad \text{for every } i \in \mathbb{N}.$$

Remark. *$\sqsupseteq^*$ is a proper refinement of $\sqsupseteq$ in S_n, S'_n, and $S'_n(l)$.*

Now we examine S_n, S'_n, and $S'_n(l)$ with respect to $\sqsupseteq$. The structures we get are much nicer than in the case of $\sqsupseteq^*$.

Proposition. *$(S_n, \sqsupseteq)$ is a complete distributive lattice with greatest lower bound $\bigwedge S_n = \{1\}$ and least upper bound $\bigvee S_n = \{n\}$.*

Proof. The necessary conditions can be checked easily. □

Proposition. *$(S'_n, \sqsupseteq)$ is a distributive lattice with greatest lower bound $\bigwedge S'_n = \{1\}$ and least upper bound $\bigvee S'_n = \{n\}$.*

However, the ordering relation $\sqsupseteq$ behaves worse in S'_n than in S_n. This can be shown by the following "negative" result, which can be proved by finding a counterexample.

Proposition. *For* $n \geq 2$, $(S'_n, \sqsupseteq)$ *is not a complete lattice.*

Proposition. $(S'_n(l), \sqsupseteq)$ *is a distributive lattice for every* $n, l \in \mathbb{N}$.

Proof. As for any $A_1, A_2 \in S'_n(l)$ the sequences $A_1 \vee A_2$ and $A_1 \wedge A_2$ defined in S_n are also in $S'_n(l)$, the statement holds immediately. □

4. FUTURE TREND

It seems to be interesting to analyse another subset of S_n, namely the set of the at most l-periodic generalized neighbourhood sequences $S'_n(l_{\geq})$. This analyses was performed in 2D and 3D by Das and Fazekas, respectively.

Our aim is to make the same analyses in the ∞-dimensional digital plane $\mathbb{Z}^{\infty}$. Moreover, the symbol ∞ can be used as a member of a generalized neighbourhood sequence. The same ordering relations can be introduced, and similar structural statements can be obtained. The ∞-dimensional sets, in some sense, are the closure of the union of the finite dimensional ones. It can be also interesting to examine not only $\mathbb{Z}^n$ but other kind of grids used in digital image processing, or make a deeper analyses of the defined ordering relations.

References

1. P.P. Das, *Lattice of octagonal distances in digital geometry*, Pattern Recognition Letters **11** (1990), 663-667.
2. P.P. Das, P.P. Chakrabarti, B.N. Chatterji, *Distance functions in digital geometry*, Information Sciences **42** (1987), 113-136.
3. A. Fazekas, *Lattice of distances based on 3D-neighbourhood sequences*, Acta Mathematica Academiae Pedagogicae Nyiregyháziensis **15** (1999), 55-60.
4. A. Rosenfeld, J.L. Pfaltz, *Distance functions on digital pictures*, Pattern Recognition **1** (1968), 33-61.

Kossuth Lajos University,
Institute of Mathematics and Informatics,
H-4010 Debrecen P.O.Box 12.

E-mail address: fattila@math.klte.hu
hajdua@math.klte.hu
hajdul@math.klte.hu