ON THE LIOUVILLE FUNCTION ON RATIONAL POLYNOMIAL VALUES

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Dedicated to András Sárközy on the occasion of his 80th birthday.

Abstract. In this paper we extend a conjecture of Cassigne et al. concerning the behaviour of the Liouville function at integral polynomial values to the rational case. We solve the new conjecture for polynomials of degree at most two, and provide partial results for polynomials of degree three. We also make some remarks concerning polynomials of degree four.

1. Introduction

Liouville's function λ for positive integers n is defined as

$$\lambda(n) = (-1)^{\alpha_1+\dots+\alpha_k},$$

where $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ is the prime factorization of n. There are many results and conjectures in the literature concerning the behavior of Liouville's function on polynomial values. First we mention some results concerning the values of λ on consecutive integers. One can regard a classical conjecture of Chowla [4] as the starting point. It says (see Conjecture 1.3 in [11] for this formulation) that for any choice of $\varepsilon_i \in \{-1, 1\}$ $(i = 1, \dots, k)$, the natural density of the set

$$\{n : \lambda(n + i) = \varepsilon_i \ (i = 1, \dots, k)\}$$

exists and equals to $1/2^k$. This conjecture is wide open. Hildebrand [7] proved that for $k = 3$, the above set is infinite in each of the eight possible choices of the ε_i, while recently, Matomäki, Radziwiłł and Tao [11] showed that all these sets have positive natural lower densities.

Concerning general polynomials, first we recall another conjecture of Chowla [4], which states the following.

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Conjecture 1. *Let $f(x) \in \mathbb{Z}[x]$ not of the form $ag^2(x)$ ($a \in \mathbb{Z}$, $g(x) \in \mathbb{Z}[x]$). Then we have*

$$\sum_{n \leq x} \lambda(f(n)) = o(x).$$

This conjecture seems to be extremely hard; it has been proved only for linear polynomials. Cassaigne, Ferenczi, Mauduit, Rivat and Sárközy [3] suggested the following conjecture.

Conjecture 2. *Let $f(x)$ be a polynomial having integer coefficients with positive leading coefficient which is not of the form $ag^2(x)$ ($a \in \mathbb{Z}$, $g(x) \in \mathbb{Z}[x]$). Then the sequence $\lambda(f(n))$ changes sign infinitely often.*

One can easily check that Conjecture 2 is a simple consequence of Conjecture 1. However, Conjecture 2 is still rather difficult. For $\deg(f) = 1$ it follows from Dirichlet's theorem on primes in arithmetic progressions, but for $\deg(f) \geq 2$ Conjecture 2 is still open. In [3], in case of $\deg(f) = 2$, the conjecture is proved under certain assumptions. Further, Borwein, Choi and Ganguli [2], still in case of $\deg(f) = 2$, could give an integer A_0 explicitly in terms of $f(x)$, such that if the sequence $\lambda(f(n))$ changes sign after $n = A_0$, then it changes sign infinitely often. Using their result, they could prove Conjecture 2 for new families of quadratic polynomials. Recently, Teräväinen [17] could prove the conjecture for degree three polynomial of the form $f(x) = x(x^2 - Bx + C)$ with $B \geq 0$. (It is noted in [17] that the condition $B \geq 0$ probably can be relaxed.) Beside these, Conjecture 2 has been proved for polynomials factoring into linear or certain type of quadratic factors. Namely, the authors in [3] could handle the case where $f(x)$ is of the form $f(x) = (ax+b_1)\cdots(ax+b_k)$ with $a, b_1, \ldots, b_k$ integers, $a > 0$ and $b_1 \equiv \cdots \equiv b_k \pmod a$. Extending this result, Teräväinen [17] could completely settle the case where $f(x)$ is a product of arbitrary linear factors. Further, he could also solve Conjecture 2 in the case where $f(x)$ is the product of quadratic factors of the form $(x + A_i)^2 + 1$ ($i = 1, \ldots, k$), where $A_1, \ldots, A_k$ are distinct integers. Apart from these results, Conjecture 2 is open. We note that Ganguli and Jankauskas [6] proved related results about sign changes of quadratic polynomials taken at rational values.

The purpose of this paper is to extend the problem to the rational values of polynomials in $\mathbb{Q}[x]$. For this, first note that λ naturally extends to the set of positive rationals in the following way. For positive integers n, k let

$$\lambda\left(\frac{n}{k}\right) = \frac{\lambda(n)}{\lambda(k)}.$$

Clearly, this extension is well-defined (i.e. it is independent of the representation of the rational number n/k). Note that for any positive rationals p, q we obviously have $\lambda(pq) = \lambda(p)\lambda(q)$. The above conjectures suggest that the λ values of polynomial sequences behave "randomly" or "uniformly". So concerning the values of a polynomial taken at rational points, the following conjecture seems to be natural.

Conjecture 3. *Let $f(x)$ be a polynomial with rational coefficients, having positive leading coefficient. Suppose that $f(x)$ is not of the form $ag^2(x)$ with some $a \in \mathbb{Q}$ and $g(x) \in \mathbb{Q}[x]$. For $\varepsilon \in \{-1, 1\}$ write*

$$H_\varepsilon(f) = \{x \in \mathbb{Q} \ : \ f(x) > 0 \text{ and } \lambda(f(x)) = \varepsilon\}.$$

Then there exists a real number x_0 such that H_ε is dense in (x_0, ∞), for both $\varepsilon = \pm 1$.

In this paper we prove Conjecture 3 for $\deg(f) \leq 2$, and give some partial results for $\deg(f) = 3$. In particular, we show that for cubic polynomials Conjecture 2 implies Conjecture 3. We also make some remarks concerning quartic polynomials. Our main tools will be the theory of ternary quadratic forms and quadratic twists of elliptic curves.

For polynomials of degree ≥ 5, Conjecture 3 is possibly very difficult. The reason is that in this case the equation

$$f(x) = \ell y^2 \quad (f \in \mathbb{Q}[x],\ \ell \in \mathbb{Q}), \tag{1}$$

in general defines a curve of genus at least two over $\mathbb{Q}$. Hence by a classical result of Faltings [5], equation (1) has only finitely many rational solutions x, y. Thus to prove Conjecture 3 for $\deg(f) \geq 5$, one need to control infinitely many curves (i.e. consider infinitely many values of ℓ) in (1). In contrast, if $\deg(f) \leq 4$, then the genus of (1) is at most one. Thus with careful choices of ℓ, one has to control only a few equations of the type (1) in this case. (We note that there is a similar phenomenon behind Conjecture 2 and the related results.)

2. Main results

Our first statement proves Conjecture 3 for polynomials of degree at most 2.

Theorem 2.1. *Let $f(x) \in \mathbb{Q}[x]$ with* $\deg(f) \leq 2$ *and positive leading coefficient. Then Conjecture 3 is valid for $f(x)$.*

For $\deg(f) \geq 3$ we have only partial results. We shall stick to the case $\deg(f) = 3$, though we could handle certain families of polynomials of degree four, as well. However, as these results would be far from being complete, we prefer not to give them. We shall make a remark about them after the proofs of the results concerning the cubic case.

Our first theorem for degree three polynomials shows that Conjecture 2 implies Conjecture 3 in this case.

Theorem 2.2. *Suppose that Conjecture 2 is valid for cubic polynomials. Then Conjecture 3 is also valid for cubic polynomials.*

The next theorem for cubic polynomials says that one of the sets H_ε occurring in Conjecture 3 is dense in some interval (x_0, ∞).

Theorem 2.3. *Let $f(x) \in \mathbb{Q}[x]$ be a cubic polynomial with positive leading coefficient. Then there exists a real number x_0 such that one of $H_1(f)$ and $H_{-1}(f)$ is dense in (x_0, ∞).*

Our third statement in the cubic case gives a simple tool, by the help of which one can (most probably) easily establish the validity of Conjecture 3 for any fixed cubic polynomial. For its formulation, for a negative rational number r put $\lambda(r) := \lambda(-r)$.

Theorem 2.4. *Let $f(x) \in \mathbb{Q}[x]$ be a cubic polynomial with positive leading coefficient. If the roots of f are distinct, then suppose further that for some non-zero rational numbers a_1, a_2 with $\lambda(a_1 a_2) = -1$, both elliptic curves*

$$f(x) = a_i y^2 \quad (i = 1, 2) \tag{2}$$

are of positive rank over $\mathbb{Q}$. Then Conjecture 3 is valid for $f(x)$.

Remark 1. In fact, in the case where the roots of f are not distinct, Theorem 2.4 is a simple consequence of Theorem 2.1 (see the proof). In other words, for such an f, Conjecture 3 immediately follows from Theorem 2.1. The main interest of Theorem 2.4 lies in the case where f has distinct roots.

Remark 2. According to a conjecture of Silverman (see e.g. Conjecture 1 in [13]), for any $f(x)$ as in Theorem 2.4, there exist infinitely many primes p such that the elliptic curve $f(x) = py^2$ is of positive rank. Clearly, this conjecture implies Conjecture 3 for cubic polynomials by Theorem 2.4. Indeed, write $f(x) = ax^3 + bx^2 + cx + d$, choose such a prime p and put

$$f^*(X) = aX^3 + bpX^2 + cp^2X + dp^3.$$

By Remark 1 we may assume that the roots of $f(x)$ are distinct. Now by Silverman's conjecture again, we can find a prime q such that the elliptic curve $f^*(X) = qY^2$ has positive rank. However, then by the substituion $(X, Y) = (px, p^2y)$ we see that the elliptic curve $f(x) =$

pqy^2 is also of positive rank. Hence assuming Silverman's conjecture, by Theorem 2.4 we obtain Conjecture 3 for cubic polynomials, indeed.

Our last statement in the cubic case shows that for a particular, infinite set of cubic polynomials, Conjecture 3 is valid. In fact, this statement (and in particular, its proof) is to demonstrate that Theorem 2.4 can be efficiently used, indeed.

Theorem 2.5. *For any positive rational N, Conjecture 3 is valid for the polynomial $f_N(x) = x^3 - N^2x$.*

Remark 3. As it is well-known, the polynomials $f_N(x)$, or the elliptic curves $f_N(x) = y^2$ are closely related to the famous set of congruent numbers. (For details and related results about them, see e.g. the survey paper [18] and the references therein.) That is why we choose this family for demonstration. In fact, one can easily get alike results for similar families of elliptic curves. Namely, let $f(x)$ be as in Theorem 2.4 with distinct roots, such that both elliptic curves in (2) are of positive ranks with some a_1, a_2 with $\lambda(a_1a_2) = -1$. Write $f(x) = ax^3 + bx^2 + cx + d$ and put

$$f_M(x) = ax^3 + bMx^2 + cM^2x + dM^3 \quad (M \in \mathbb{Q} \setminus \{0\}).$$

Then Conjecture 3 holds for all polynomials $f_M(x)$. This statement can be proved similarly to Theorem 2.5.

3. Proofs

We start with the proof of Theorem 2.1. For this, we shall need a famous theorem of Legendre concerning the representability of zero by ternary quadratic forms.

Lemma 3.1. *Let A, B, C be non-zero, square-free, pairwise coprime integers, not all of the same sign. Then the equation*

$$AX^2 + BY^2 + CZ^2 = 0 \tag{3}$$

has a non-trivial solution in integers X, Y, Z if and only if the congruences

$$t^2 \equiv -BC \pmod{A}, \quad t^2 \equiv -AC \pmod{B}, \quad t^2 \equiv -AB \pmod{C}$$

are all solvable.

Further, if (3) has a non-trivial solution, then it has infinitely many coprime solution (X, Y, Z) which can be parametrized by finitely many binary quadratic forms.

Proof. The first part of the statement is a classical theorem of Legendre [9]. The second part of the statement follows e.g. from Theorem 4 on p. 47 of Mordell [12]. $\square$

Remark 4. We shall need and explicitly give the parametrizations for the solutions (X,Y,Z) of (3) concretely in our case (in the proof of Theorem 2.1), that is why we do not give more general details about them at this point.

Proof of Theorem 2.1. Since the case where $\deg(f)=0$ is excluded, we need to check the possibilities $\deg(f)=1,2$ only. The case $\deg(f)=1$ is in fact trivial, but we give a simple argument. Writing $f(x)=ax+b$ with $a,b\in\mathbb{Q}$, $a>0$, for $\varepsilon=\pm1$ we have

$$\begin{aligned} H_\varepsilon := \{x\in\mathbb{Q} : f(x)>0,\ \lambda(f(x))=\varepsilon\} = \\ = \left\{\frac{1}{a}y-\frac{b}{a} : y\in\mathbb{Q},\ y>0,\ \lambda(y)=\varepsilon\right\}. \end{aligned}$$

Thus to prove our claim, it is sufficient to check that the sets

$$H_\varepsilon^* = \{y\in\mathbb{Q} : y>0,\ \lambda(y)=\varepsilon\}$$

are dense in $(0,\infty)$. To see this, let α be an arbitrary positive real number, $(r_n)_{n=1}^\infty$ be a sequence of positive rationals tending to α, and set

$$t_n := \frac{p_{\pi(n^2)+1}}{n^2}\quad (n=1,2,\dots).$$

Here $\pi(x)$ is the prime counting function, so the numerator of t_n is the smallest prime exceeding n^2. Note that $\lambda(t_n)=-1$ for every $n=1,2,\dots$. Further, we trivially have $\lim\limits_{n\to\infty} t_n=1$. (This follows from standard estimates concerning $\pi(x)$; see e.g. [14].) Put

$$r_n^{(1)} = \begin{cases} r_n & \text{if } \lambda(r_n)=1, \\ t_nr_n & \text{if } \lambda(r_n)=-1 \end{cases}\quad (n=1,2,\dots)$$

and

$$r_n^{(-1)} = \begin{cases} t_nr_n & \text{if } \lambda(r_n)=1, \\ r_n & \text{if } \lambda(r_n)=-1 \end{cases}\quad (n=1,2,\dots).$$

Then we have $\lambda(r_n^{(1)})=1$ and $\lambda(r_n^{(-1)})=-1$ $(n=1,2,\dots)$, and

$$\lim_{n\to\infty} r_n^{(1)} = \lim_{n\to\infty} r_n^{(-1)} = \alpha,$$

which implies our claim in this case.

Let now $\deg(f)=2$. Observe that Conjecture 3 is valid for $f(x)$ if and only if it is valid for $Af(Bx+C)$ for any rationals A,B,C with

$A > 0$, $B > 0$. Hence without loss of generality we may assume that $f(x)$ is of the form

$$f(x) = x^2 + a \quad (a \in \mathbb{Z} \text{ square-free}).$$

Further, here a is non-zero, otherwise $f(x)$ would be of the excluded shape. Consider the equation

$$f(x) = \ell y^2 \tag{4}$$

in rationals x, y, where ℓ is a square-free integer greater than one, such that for all prime divisors q_j $(j = 1, \dots, J)$ of ℓ we have

$$q_j \equiv 1 \pmod{4|a|} \quad (j = 1, \dots, J). \tag{5}$$

Letting $x = x_1/x_2$ and $y = y_1/y_2$ with integral unknowns x_1, x_2, y_1, y_2 the above equation yields

$$X^2 + aY^2 - \ell Z^2 = 0, \tag{6}$$

where $X = x_1 y_2$, $Y = x_2 y_2$, $Z = y_1 x_2$. As a and ℓ are square-free, $\ell > 1$ and $\gcd(a, \ell) = 1$, Lemma 3.1 gives that (6) has a non-trivial solution in integers X, Y, Z if and only if

$$t^2 \equiv \ell \pmod{|a|} \quad \text{and} \quad t^2 \equiv -a \pmod{\ell} \tag{7}$$

are both solvable. Obviously, by (5), the first congruence is solvable (one can take e.g. $t = 1$). We claim that the second congruence in (7) is solvable, as well. For this, write

$$|a| = 2^\nu p_1 \dots p_I \tag{8}$$

where $\nu \in \{0, 1\}$, and p_i $(i = 1, \dots, I)$ are distinct odd primes. If $|a| \le 2$ then our claim immediately follows by (5) and by the well-known identities

$$\left(\frac{-1}{q}\right) = (-1)^{\frac{q-1}{2}} \quad \text{and} \quad \left(\frac{2}{q}\right) = (-1)^{\frac{q^2-1}{8}} \tag{9}$$

of the Legendre symbol valid for any odd prime q. So we may assume that $|a| > 2$. Then we have $I \ge 1$ in (8). Further, (9) implies that

$$\left(\frac{-1}{q_j}\right) = 1 \quad (j = 1, \dots, J), \tag{10}$$

and if a is even, then also that

$$\left(\frac{2}{q_j}\right) = 1 \quad (j = 1, \dots, J). \tag{11}$$

Note that if a is odd, then we do not need to know the values of $\left(\frac{2}{q_j}\right)$. These assertions, using the quadratic reciprocity law of the Legendre symbol by (5) show that the second congruence of (7) is also valid. Indeed, for any $1 \le i \le I$, $1 \le j \le J$ we have

$$\left(\frac{p_i}{q_j}\right) = (-1)^{\frac{(p_i-1)(q_j-1)}{4}} \left(\frac{q_j}{p_i}\right) = 1,$$

and this by (10) and (11) clearly implies our claim.

Altogether, we see that if $\ell > 1$ is square-free with prime divisors satisfying (5), then equation (6) has a non-trivial solution. Lemma 3.1 also gives that then (6) has infinitely many solutions, which can be parametrized. As we shall need precise details about this parametrization, we give it explicitly in our case.

Fix two integers $\ell = \ell_1, \ell_2$ of the above shape, with $\lambda(\ell_1) = 1$ and $\lambda(\ell_2) = -1$, respectively. We distinguish two (similar) cases. Assume first that $a > 0$. Let ℓ be any of ℓ_1, ℓ_2, and let (X_0, Y_0, Z_0) be a non-trivial integer solution of (6) with non-negative entries. Observe that $Z_0 \neq 0$, and also that one of X_0, Y_0 is not zero. Following the argument on p. 47 of Mordell, with rational parameters r, p, q set

$$X = rX_0 + p, \quad Y = rY_0 + q, \quad Z = rZ_0.$$

We want X, Y, Z to form a solution to (6). Thus after substitution, we get

$$r = -\frac{1}{2}\frac{p^2 + aq^2}{pX_0 + aqY_0}$$

and

$$X = -\frac{1}{2}\frac{-p^2X_0 - 2apqY_0 + aq^2X_0}{pX_0 + aqY_0}, \; Y = -\frac{1}{2}\frac{p^2Y_0 - 2pqX_0 - aq^2Y_0}{pX_0 + aqY_0},$$

$$Z = -\frac{1}{2}\frac{(p^2 + aq^2)Z_0}{pX_0 + aqY_0}.$$

What is of utmost importance for us, is that the denominators of X, Y, Z are the same, and assuming from this point on that p, q are integers, the numerators of X, Y, Z form an integral solution to (6). (This can also be checked directly.) This yields

$$x = \frac{x_1}{x_2} = \frac{X}{Y} = \frac{-p^2X_0 - 2apqY_0 + aq^2X_0}{p^2Y_0 - 2pqX_0 - aq^2Y_0}. \tag{12}$$

Recall that for all these values of x, we have $\lambda(f(x)) = \lambda(\ell_i)$ (with $i \in \{1, 2\}$ arbitrary, but fixed). We only need to check that the rationals x given by (12) form a dense set in some interval (x_0, ∞). Letting $s = p/q$, we obtain

$$x = \frac{-X_0s^2 - 2aY_0s + aX_0}{Y_0s^2 - 2X_0s - aY_0}.$$

Clearly, it is sufficient to show that for any real number δ large enough, we can find some real number s such that $x = x(s) = \delta$. The latter equality can be rewritten as

$$(X_0 + \delta Y_0)s^2 + (2aY_0 - 2\delta X_0)s - \delta a Y_0 - aX_0 = 0.$$

Since the discriminant (and one of the roots) of the left hand side is positive, our claim follows in this case.

Assume next that $a < 0$. Let again (X_0, Y_0, Z_0) be a non-trivial integer solution of (6) with non-negative entries. Observe that now we have $X_0 \neq 0$, and also that one of Y_0, Z_0 is not zero. Following the above argument, with rational parameters r, p, q we set

$$X = rX_0, \quad Y = rY_0 + p, \quad Z = rZ_0 + q.$$

We want X, Y, Z to form a solution to (6) again. Thus after substitution we get

$$r = -\frac{1}{2}\frac{ap^2 - \ell q^2}{apY_0 - \ell q Z_0}$$

and

$$X = -\frac{1}{2}\frac{ap^2X_0 - \ell q^2 X_0}{apY_0 - \ell q Z_0}, \quad Y = -\frac{1}{2}\frac{-ap^2Y_0 + 2\ell pqZ_0 - \ell q^2 Y_0}{apY_0 - \ell q Z_0},$$

$$Z = -\frac{1}{2}\frac{ap^2Z_0 - 2apqY_0 + \ell q^2 Z_0}{apY_0 - \ell q Z_0}.$$

Similarly as in case of $a > 0$, this yields

$$x = \frac{x_1}{x_2} = \frac{X}{Y} = \frac{ap^2X_0 - \ell q^2 X_0}{-ap^2Y_0 + 2\ell pqZ_0 - \ell q^2 Y_0}$$

now again with p, q assumed to be arbitrary integers. Recall that for all these values of x, we have again $\lambda(f(x)) = \lambda(\ell_i)$ (with $i \in \{1, 2\}$ arbitrary, but fixed). Letting $s = p/q$, we get

$$x = \frac{aX_0s^2 - \ell X_0}{-aY_0s^2 + 2\ell Z_0 s - \ell Y_0}.$$

Since for real δ large enough, $x = x(s) = \delta$ yields

$$(aX_0 + a\delta Y_0)s^2 - 2\delta\ell Z_0 s + \delta\ell Y_0 - \ell X_0 = 0$$

with positive discriminant (and a positive root) of the left hand side, our claim follows also in this case similarly as before. Hence the proof of the theorem is complete. □

To prove our results concerning cubic polynomials, we need the following lemma, which is an immediate consequence of a classical theorem due to Poincaré and Hurwitz.

Lemma 3.2. *Let $h(x) \in \mathbb{Q}[x]$ be a cubic polynomial with positive leading coefficient having distinct roots, and write α for its largest real root. Suppose that the elliptic curve*

$$E: \quad h(x) = y^2$$

is of positive rank. Then the x coordinates of the rational points on E form a dense set in the interval (α, ∞).

Proof. By a theorem due to Poincaré and Hurwitz, the rational ponts on E are dense among the real points on E with x coordinates satisfying $x \geq \alpha$. (See p. 173 of [10] and Satz 13 of [8]; c.f. also Satz 11 on p. 78 in [16].) From this the statement immediately follows. □

Now we can give the proofs of our theorems concerning cubic polynomials.

Proof of Theorem 2.2. Suppose that Conjecture 2 is valid for cubic polynomials. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree three, with positive leading coefficient. If $f(x)$ has a double root, then we may write

$$f(x) = (ax+b)g^2(x) \quad (a, b \in \mathbb{Q},\ a > 0,\ g(x) \in \mathbb{Q}[x]).$$

Hence we immediately see that by Theorem 2.1, Conjecture 3 is valid for $f(x)$. So we may assume that the roots of $f(x)$ are distinct. Further, observe that Conjecture 3 is valid for $f(x)$ if and only if it is valid for all multiples $Af(x)$ of $f(x)$ where A is any positive rational. Thus without loss of generality we may also suppose that $f(x)$ has coprime integer coefficients. Since we assumed that Conjecture 2 is valid, we can choose sequences $(u_i)_{i=1}^{\infty}$ and $(v_i)_{i=1}^{\infty}$ of positive integers such that $\lambda(f(u_i)) = 1$ and $\lambda(f(v_i)) = -1$ $(i = 1, 2, \dots)$. Hence we see that all the elliptic curves

$$E_i^{(1)}: \quad f(x) = f(u_i)y^2 \quad \text{and} \quad E_i^{(2)}: \quad f(x) = f(v_i)y^2 \quad (i = 1, 2, \dots)$$

contain a rational point, namely $(u_i, 1)$ and $(v_i, 1)$, respectively. By the Specialization Theorem of Silverman (see e.g. Theorem 11.4 on p. 271 of [15]), we know that these points are torsion points only for finitely many indices i. Hence we conclude that there exists an i_0, such that both $E_{i_0}^{(1)}$ and $E_{i_0}^{(2)}$ are of positive rank. Thus, as $\lambda(f(u_{i_0})) = 1$ and $\lambda(f(v_{i_0})) = -1$, for all rational points $(x^{(1)}, y^{(1)})$ of $E_{i_0}^{(1)}$ with $y^{(1)} \neq 0$ and $(x^{(2)}, y^{(2)})$ of $E_{i_0}^{(2)}$ with $y^{(2)} \neq 0$ we have

$$\lambda(f(x^{(1)})) = 1 \quad \text{and} \quad \lambda(f(x^{(2)})) = -1,$$

respectively. Hence our claim immediately follows from Lemma 3.2. □

Proof of Theorem 2.3. The proof follows the same lines as that of Theorem 2.2. Namely, just as there, by the help of the Specialization Theorem of Silverman we can find a positive integer ℓ such that the elliptic curve

$$f(x) = \ell y^2$$

is of positive rank over $\mathbb{Q}$. Hence by Lemma 3.2 we get that either H_1 or H_{-1} is dense in some interval (x_0, ∞), according as whether $\lambda(\ell) = 1$ or $\lambda(\ell) = -1$. This proves our claim. $\square$

Proof of Theorem 2.4. Assume first that $f(x)$ has a multiple root. Then with some rationals a, b, c, d with $ac \neq 0$, we can write

$$f(x) = (ax + b)(cx + d)^2.$$

Hence the statement immediately follows from Theorem 2.1 in this case.

Assume now that f has distinct roots. Then the fact that both elliptic curves

$$E_1: \quad f(x) = a_1 y^2 \quad \text{and} \quad E_2: \ f(x) = a_2 y^2$$

are of positive rank, in view of that for all rational points (u, v) of these curves with $v \neq 0$ we have

$$\lambda(f(u)) = \lambda(a_i)$$

for $i = 1, 2$, respectively, by Lemma 3.2 implies the statement. $\square$

Proof of Theorem 2.5. As one can easily check e.g. by Magma [1], the ranks of both elliptic curves

$$X^3 - X = 5Y^2 \quad \text{and} \quad X^3 - X = 6Y^2$$

over $\mathbb{Q}$ are positive (namely, equal to one). This, using the substitutions $x = NX$ and $y = NY$ (for any positive rational N), implies that the ranks of both elliptic curves

$$x^3 - N^2 x = 5Ny^2 \quad \text{and} \quad x^3 - N^2 x = 6Ny^2$$

are also positive over $\mathbb{Q}$. This, in view of $\lambda(30N^2) = -1$ by Theorem 2.4 implies the statement. $\square$

Remark 5. As it is well-known (see e.g. Theorem 2 on p. 77 of [12]), by the help of any of its rational points, a quartic curve of the form

$$AX^4 + BX^3 + CX^2 + DX + E = Y^2 \tag{13}$$

can be transformed to a curve of the form

$$x^3 + ax + b = y^2$$

with a birational transformation. For a given quartic polynomial $f(x)$, substituting any rational value into x and writing ℓ for the square-free part of $f(x)$, we see that

$$f(x) = \ell y^2 \tag{14}$$

does have a rational point. Then we can transform (14) to a cubic equation, and we can use our Theorems 2.3 to 2.5 to get conclusions for $f(x)$ with respect to Conjecture 3. In some cases (under certain assumptions on $f(x)$) the transformation can be made explicit, and we can get explicit statements. (See e.g. [19] for the case where E in (13) is a full square in $\mathbb{Q}$.) Since this can be done only under some assumptions, and we could get only partial results towards Conjecture 3 in this case, we do not work out the details here.

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