Representing integers as linear combinations of power products

Lajos Hajdu and Rob Tijdeman

Abstract. Let P be a finite set of at least two prime numbers, and A the set of positive integers that are products of powers of primes from P. Let $F(k)$ denote the smallest positive integer which cannot be presented as sum of less than k terms of A. In a recent paper Nathanson asked to determine the properties of the function $F(k)$, in particular to estimate its growth rate. In this paper we derive several results on $F(k)$ and on the related function $F_{\pm}(k)$ which denotes the smallest positive integer which cannot be presented as sum of less than k terms of $A \cup (-A)$.

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1. Introduction

Let P be a nonempty finite set of at least two prime numbers, and A the set of positive integers that are products of powers of primes from P. Put $A_{\pm} = A \cup (-A)$. Then there does not exist an integer k such that every positive integer can be represented as a sum of at most k elements of $A_{\pm}$. This follows e.g. from Theorem 1 of Jarden and Narkiewicz [6], cf. [4, 1]. At a conference in Debrecen in 2010 Nathanson announced the following stronger result (see also [7]):

For every positive integer k there exist infinitely many integers n such that k is the smallest value of l for which n can be written as

$$n = a_1 + a_2 + \cdots + a_l \ (a_1, a_2, \ldots, a_l \in A_{\pm}).$$

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Let $F(k)$ be the smallest positive integer which cannot be presented as a sum of less than k terms of A and $F_{\pm}(k)$ the smallest positive integer which cannot be presented as a sum of less than k terms of $A_{\pm}$. Problem 2 of [7] Nathanson is to give estimates for $F(k)$. (The notation in [7] is different from ours.) Problem 1 is the corresponding question for $F_{\pm}(k)$ in case A consists of the pure powers of 2 and of 3.

In [5] two of the authors considered Problem 1 in the more general setting of powers of any finite set of positive integers. They gave lower and upper bounds for $F(k)$ and $F_{\pm}(k)$. In the present paper we consider Problem 2. We give lower and upper bounds for $F(k)$ and $F_{\pm}(k)$ for A as defined above.

We show that there exists an effectively computable number c depending only on P, an effectively computable number C depending only on ε and an effectively computable constant $C_{\pm}$ such that $k^{ck} < F(k) < C(kt)^{(1+\varepsilon)kt}$ and $k^{ck} < F_{\pm}(k) < \exp((kt)^{C_{\pm}})$. The method of proof is an adaptation of that in [5], but in the case of the lower bound an additional argument is needed. For the upper bound we need an extended version of a theorem of Ádám, Hajdu and Luca [1] in which a result of Erdős, Pomerance and Schmutz [2] plays an important part. We state the result of Erdős, Pomerance and Schmutz and its refinement in Section 2 and our generalization of the result of Ádám, Hajdu and Luca in Section 3. In Section 4 we derive the lower and upper bounds for $F(k)$ and $F_{\pm}(k)$. In Section 5 we apply the Qualitative Subspace Theorem to prove that for some number c^* depending only on P, k and ε the inequality $F_{\pm}(k) \leq (kt)^{(1+\varepsilon)kt}$ holds for $k > c^*$.

2. An extension of a theorem of Erdős, Pomerance and Schmutz

Let $\lambda(m)$ be the Carmichael function of the positive integer m, that is the least positive integer for which

$$b^{\lambda(m)} \equiv 1 \pmod{m}$$

for all $b \in \mathbb{Z}$ with $\gcd(b, m) = 1$. Theorem 1 of [2] gives the following information on small values of the Carmichael function.

For any increasing sequence $(n_i)_{i=1}^{\infty}$ of positive integers, and any positive constant $C_1 < 1/\log 2$, one has

$$\lambda(n_i) > (\log n_i)^{C_1 \log\log\log n_i}$$

for i sufficiently large. On the other hand, there exist a strictly increasing sequence $(n_i)_{i=1}^{\infty}$ of positive integers and a positive constant C_2, such that, for every i,

$$\lambda(n_i) < (\log n_i)^{C_2 \log\log\log n_i}.$$

This nice result does not give any information on the size of n_i. For our purposes the following quantitative version will be needed.

Lemma 1 ([5], **Theorem 1).** *There exist positive constants C_3, C_4 such that for every large integer i there is an integer m with*

$$\log m \in [\log i, (\log i)^{C_3}] \text{ and } \lambda(m) < (\log m)^{C_4 \log\log\log m}.$$

3. An extension of a theorem of Ádám, Hajdu and Luca

Let k be a positive integer. Put

$$H_{P,k} = \{n \in \mathbb{Z} : n = \sum_{i=1}^{l} a_i \text{ with } l \leq k\}$$

where $a_i \in A$ $(i = 1, 2, \ldots, k)$. For $H \subseteq \mathbb{Z}$ and $m \in \mathbb{Z}, m \geq 2$, we write $\sharp H$ for the cardinality of the set H and

$$H(\text{mod } m) = \{i : 0 \leq i < m, h \equiv i \pmod m \text{ for some } h \in H\}.$$

The next theorem is a generalization of a result from [1].

Theorem 1. *Let C_3, P and k be given as above. There is a constant C_5 such that for every sufficiently large integer i there exists an integer m with $\log m \in [\log i, (\log i)^{C_3}]$ and*

$$\sharp H_{P,k} \pmod m < (\log m)^{C_5 kt \log\log\log m}.$$

In the proof of Theorem 1 the following lemma is used.

Lemma 2. ([1], Lemma 1). *Let $m = q_1^{\alpha_1} \cdots q_z^{\alpha_z}$ where $q_1, \ldots, q_z$ are distinct primes and $\alpha_1, \ldots, \alpha_z$ positive integers, and let $b \in \mathbb{Z}$. Then*

$$\sharp\{b^u \pmod m : u \geq 0\} \leq \lambda(m) + \max_{1 \leq j \leq z} \alpha_j.$$

Proof of Theorem 1. Let i be a large integer. Choose m according to Lemma 1. Write m as a product of powers of distinct primes as in Lemma 2. Lemma 2 implies that

$$\sharp\{h \pmod m : h \in H_{P,k}\} \leq \left(\lambda(m) + \max_{1 \leq j \leq z} \alpha_j + 1\right)^{kt}.$$

On the other hand, with the constant C_4 from Lemma 1,

$$\lambda(m) + \max_{1 \leq j \leq z} \alpha_j < (\log m)^{C_4 \log\log\log m} + \frac{\log m}{\log 2}.$$

The combination of both inequalities yields the theorem. □

4. Effective results on combinations of power products

Suppose we want to express the positive integer n as a finite sum of elements of A. For this we apply the greedy algorithm. If we subtract the largest element of A not exceeding n from n, we are left with a rest which is less than $n/(\log n)^{c_1}$ for some number $c_1 > 0$ depending only on the two smallest elements of P according to [8]. We can iterate subtracting the largest element of A not exceeding the rest from the rest and as long as the rest exceeds $\exp(\sqrt{\log n})$ reduce the rest each time by a factor at least $(\log n)^{c_1/2}$. If the rest is smaller than $\exp(\sqrt{\log n})$ we can reduce the rest each step by a factor larger than some constant $c_2 > 1$, with c_2 depends only on the smallest prime from P. Thus we find that the sum of

$$k \leq \frac{2\log n}{c_1 \log\log n} + \frac{\sqrt{\log n}}{\log c_2}$$

elements of A suffices to represent n. This implies the lower bound k^{ck} for $F(k)$ in Theorem 2(i) below. Of course, $F(k) \leq F_{\pm}(k)$ for all k.

For an upper bound for $F(k)$ we study the number of representations of positive integers up to n as $\sum_{j=1}^{l} a_j$ with $a_j \in A, l \leq k$. Since the number of elements of $A \cup \{0\}$ not exceeding n is at most $(C_6 \log n)^t$, the number of represented integers is at most $(C_6 \log n)^{kt}$. If this number is less than n, then we are sure that some positive integer $\leq n$ is not represented. This is the case if

$$kt < \frac{\log n}{\log\log n + \log C_6}.$$

Suppose $n > (kt)^{(1+\varepsilon)kt}$. Then it follows from the monotonicity of the function $\log x/(\log\log x + C_6)$ for large x that

$$\frac{\log n}{\log\log n + C_6} > \frac{(1+\varepsilon)kt\log kt}{\log(kt) + \log((1+\varepsilon)\log(kt)) + C_6} > kt$$

for kt sufficiently large. By choosing C_7 suitably for the smaller values of kt, it suffices for all values of kt that $n \geq C_7(kt)^{(1+\varepsilon)kt}$. Thus

$$F(k) \leq C_7(kt)^{(1+\varepsilon)kt}.$$

Next we consider representations by sums of elements from $A_{\pm}$. We write $H^*_{P,k} = \{n \in \mathbb{Z} : n = \sum_{j=1}^{l} a_j \text{ with } a_j \in A_{\pm}, l \leq k\}$. Choose the smallest positive integer $i > 10$ such that $j > (\log j)^{C_5 kt \log\log\log j}$ for $j \geq i$. Then $i < 2(\log i)^{C_5 kt \log\log\log i}$. It follows that

$$\log i < C_8 kt(\log\log i)(\log\log\log i)$$

for some constant C_8. Hence $\log i < C_9 kt(\log(kt))(\log\log(kt))$ for some constant C_9. According to Theorem 2 there exists an integer m with $\log i \leq \log m \leq (\log i)^{C_3}$ such that all representations in $H^*_{P,k}$ are covered by at most $(\log m)^{C_5 kt \log\log\log m}$ residue classes modulo m. By the definition of i and the inequality $i \leq m$, we see that this number of residue classes is less than m, therefore at least one positive integer $n \leq m$ has no representation of the form $\sum_{j=1}^{k} a_j$ with $a_j \in A \cup \{0\}$ for $j = 1, \ldots, k$. Hence

$$\log n \leq \log m \leq (\log i)^{C_3} < (C_9 kt(\log kt)(\log\log kt))^{C_3} < (kt)^{C_{10}}$$

for some constant C_{10}. Thus $F_\pm(k) < \exp((kt)^{C_{10}})$.

So we have proved the following result.

Theorem 2. *Let $P = \{p_1, \ldots, p_t\}$ be a finite set of primes with $t \geq 2$. Let A be the set of integers composed of numbers from P. Let k be a positive integer. Denote by $F(k)$ the smallest positive integer which cannot be represented in the form $\sum_{i=1}^{k} a_i$ with $a_i \in A \cup \{0\}$ for all i and by $F_\pm(k)$ the smallest positive integer which cannot be represented in the form $\sum_{i=1}^{k} a_i$ with $a_i \in A_\pm \cup \{0\}$ for all i. Then, for every $\varepsilon > 0$ there are a number c depending only on the two smallest elements of P, a number C depending only on ε and an absolute constant $C_\pm$ such that*
(i) $F(k) > k^{ck}$ for all $k > 1$,
(ii) $F(k) \leq C(kt)^{(1+\varepsilon)kt}$ for all $k > 1$,
(iii) $F_\pm(k) < \exp((kt)^{C_\pm})$ for all $k > 1$.

Remark 1. In Section 5 we shall use an ineffective method to show that $C_\pm = 16$ suffices.

Remark 2. Following the proof of Theorem 3(iv) of [5] it can be shown that there are infinitely many positive integers k for which $F_\pm(k) \leq \exp(C_\pm^* kt \log(kt) \log\log(kt))$ for some suitable effectively computable constant C^*. In Section 5 we derive the better upper bound $(kt)^{(1+\varepsilon)kt}$ for $F_\pm(k)$ for all but finitely many k. However, it cannot be deduced from the proof from which value of k on this bound holds.

Remark 3. Using the above methods similar bounds can be derived if P is replaced by any finite set of positive integers.

5. Application of the ineffective Subspace theorem

By applying another version of the Subspace Theorem we derive an estmate for $F_\pm(k)$ which is much better than the bound in Theorem 2(iii) and holds for all but finitely many k's.

Theorem 3. *Under the conditions of Theorem 2 for every $\varepsilon > 0$ there is a number $c_\pm^*$ depending only on P, k and ε such that*

$$F_\pm(k) \leq (kt)^{(1+\varepsilon)kt}$$

whenever $k > c_\pm^*$.

In the proof we apply the following result of Evertse. Here the p-adic value $|x|_p$ is defind as $|x|p^{-r}$ where $p^r||x$.

Lemma 3 (**[3], Corollary 1).** *Let c, d be constants with $c > 0, 0 \le d < 1$. Let S_0 be a finite set of primes and let l be a positive integer. Then there are only finitely many tuples $(x_0, x_1, \dots, x_l)$ of rational integers such that*

$$x_0 + x_1 + \cdots + x_l = 0;$$

$$x_{i_0} + x_{i_1} + \dots x_{i_s} \neq 0$$

for each proper, non-empty subset $\{i_0, i_1, \dots, i_s\}$ of $\{0, 1, \dots, l\}$;

$$\gcd(x_0, x_1, \dots, x_l) = 1;$$

$$\prod_{j=0}^{l} \left(|x_j| \prod_{p \in S_0} |x_j|_p \right) \le c \left(\max_{0 \le j \le l} |x_j| \right)^d .$$

Proof of Theorem 4. Let n be an integer which is not divisible by any prime from P. Suppose $n = a_1 + a_2 + \cdots + a_l$ with $a_j \in A_\pm$ for $j = 1, 2, \dots, l$ with $l \le k$. Without loss of generality we may assume that l is minimal, hence $a_1 + a_2 + \cdots + a_l$ has no proper subsums which vanish. Moreover, we know that $\gcd(a_1, a_2, \dots, a_l) = 1$. We apply Lemma 3 with $c = 1, d = 1/2, S_0 = P$ to the equation $a_0 + a_1 + \cdots + a_l = 0$ with $a_0 = -n$. It follows that given k, P there only finitely many tuples $(n, a_1, a_2, \dots, a_l)$ with $\gcd(n, p_1, \dots, p_t) = 1$ and $l \le k$ such that $n = a_1 + a_2 + \cdots + a_l$ with $a_j \in A_\pm$ for $j = 1, 2, \dots, l$ and

$$n \le \left(\max_{0 \le j \le l} |a_j| \right)^{1/2},$$

hence

$$n^2 \le \max_{1 \le j \le l} |a_j|.$$

Let N_0 be the maximum of $|n|$ for all such tuples, where $N_0 = 0$ if there are no such tuples.

Next consider positive integers $n > N_0$ which are not divisible by any prime from P. Then, for any representation $n = a_1 + a_2 + \cdots + a_l$ with $a_j \in A_\pm$ for $j = 1, 2, \dots, l$ and $l \le k$, we have $|a_j| < n^2$ for $j = 1, 2, \dots, l$. Writing $a_j = \pm p_1^{s_1} \cdots p_t^{s_t}$ we obtain $\max_j s_j \le 3 \log n - 1$. The number of possible tuples $(a_1, \dots, a_l)$ for l is therefore at most $2^l (3 \log n)^{lt}$. Then the number of all possible tuples $(a_1, ..., a_j)$ with $j \le k$ is at most $2 \cdot 2^k (3 \log n)^{kt}$. Thus for $N > N_0$ there are at most $N_0 + 2 \cdot 2^k (3 \log N)^{kt}$ integers $n \le N$ coprime to P such that n is representable as sum of at most k integers from $A_\pm$. The number of positive integers $n \le N$ coprime to P is at least $N \prod_{p \in P} (1 - 1/p) - 2^t > 2^{-t} N - 2^t$. Hence for finding an n with $n \le N$ such that n is not representable in the desired form, it suffices that

$$2^{-t} N - 2^t > N_0 + 2 \cdot 2^k (3 \log N)^{kt}.$$

As in the proof of Theorem 2(ii) it follows that for every $\varepsilon > 0$ there is an unspecified number $c_\pm^*$ depending only on k, P and ε such that

$$F_\pm(k) \le (kt)^{(1+\varepsilon)kt}$$

whenever $k > c_\pm^*$. □

Remark 4. Theorem 4 is also an improvement of Theorem 3.4(iv) of [5] where, only for sums of perfect powers, a weaker bound is given.

References

[1] Zs. Ádám, L. Hajdu and F. Luca, *Representing integers as linear combinations of S-units*, Acta Arith. **138** (2009), 101–107.

[2] P. Erdős, C. Pomerance and E. Schmutz, *Carmichael's lambda function*, Acta Arith. **58** (1991), 365–385.

[3] J.-H. Evertse, *On sums of S-units and linear recurrences*, Compositio Math. **53** (1984), 225–244.

[4] L. Hajdu, *Arithmetic progressions in linear combinations of S-units*, Period. Math. Hungar. **54** (2007), 175–181.

[5] L. Hajdu and R. Tijdeman, *Representing integers as linear combinations of powers*, Publ. Math. Debrecen **79** (2011), 461–468.

[6] M. Jarden and W. Narkiewicz, *On sums of units*, Monatsh. Math. **150** (2007), 327–332.

[7] M. B. Nathanson, *Geometric group theory and arithmetic diameter*, Publ. Math. Debrecen **79** (2011), 563–572.

[8] R. Tijdeman, *On the maximal distance of integers composed of small primes*, Compos. Math. **28** (1974), 159–162.

Lajos Hajdu
Institute of Mathematics
University of Debrecen
H-4010 Debrecen, P.O. Box 12
Hungary
e-mail: `hajdul@math.unideb.hu`

Rob Tijdeman
Mathematical Institute
Leiden University
2300 RA Leiden, P.O. Box 9512
The Netherlands
e-mail: `tijdeman@math.leidenuniv.nl`