CORRIGENDUM TO THE PAPER "ON A CONJECTURE OF SCHÄFFER CONCERNING THE EQUATION $1^k + \cdots + x^k = y^n$"

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In [1], the following statement is formulated.

Lemma 3.2. *Let x be a positive integer. Then we have*

$$\nu_3(S_k(x)) = \begin{cases} \nu_3(x(x+1)), & \text{if } k=1, \\ \nu_3(x(x+1)(2x+1)) - 1, & \text{if } k \text{ is even}, \\ 0, & \text{if } x \equiv 1 \pmod 3 \text{ and } k \geq 3 \text{ is odd}, \\ \nu_3(kx^2(x+1)^2) - 1, & \text{if } x \equiv 0,2 \pmod 3 \text{ and } k \geq 3 \text{ is odd}. \end{cases}$$

Unfortunately, the proof of this statement contains a small gap, and also the last part of the argument is not correctly presented. Now we give a correct and full proof of this statement. Note that Lemma 3.2 as well as all the other statements in [1] hold true.

Proof of Lemma 3.2. To keep the presentation as simple as possible, we only consider the case of odd k. The case of even k has been handled by Sondow and Tsukerman [2]. (This has also been noted in [1]; there [2] is reference [21].)

By the arguments in [1] we may assume that $k \geq 3$ and $x \geq 3$. We proceed by induction on x. Assume that the assertion is valid for all x' with $1 \leq x' < x$ for all positive integers k.

Since $a^k \equiv a \pmod 3$ for any integer a for odd k, we clearly have that $S_k(x) \equiv 1 \pmod 3$ whenever $x \equiv 1 \pmod 3$, yielding $\nu_3(S_k(x)) = 0$ in this case. So if $x \equiv 1 \pmod 3$, then the statement holds.

When $x \equiv 0,2 \pmod 3$, then we distinguish three cases. Assume first that x is of the form $\varepsilon 3^\alpha$ with $\varepsilon = 1,2$ and $\alpha \geq 1$. In this case the argument in [1] perfectly works. Note that we need the induction hypothesis with $(3^\alpha - 1)/2$ and $3^\alpha - 1$ for $\varepsilon = 1$ and 2, respectively.

Suppose next that x is of the form $\varepsilon 3^\alpha - 1$, with ε and α as above. (This is the case not discussed in [1].) Then, by the induction hypothesis and what we have proved previously, the statement is valid for $x+1$, that is

$$\nu_3(S_k(x+1)) = \nu_3(k) + 2\alpha - 1.$$

Thus, since we have $\log k > \nu_3(k)$,

$$\nu_3((x+1)^k) = k\alpha > \nu_3(k) + 2\alpha - 1,$$

and we obtain

$$\nu_3(S_k(x)) = \nu_3(S_k(x+1)) = \nu_3(k) + 2\alpha - 1 = \nu_3(kx^2(x+1)^2) - 1.$$

So the statement follows also in this case.

Finally, assume that x is not of any of the forms above. Then write $x = \sum_{i=1}^t \varepsilon_i 3^{\alpha_i}$ with $\varepsilon_i = 1, 2$ $(i = 1, \dots, t)$ and $\alpha_1 > \cdots > \alpha_t \geq 0$. Set $z = x - \varepsilon_1 3^{\alpha_1}$. (This is the point where we change the argument in [1]: there we dealt with the number $x - \varepsilon_t 3^{\alpha_t}$ instead, and it does not work properly.) Observe that by our assumption on x, we have $\max(\nu_3(z), \nu_3(z+1)) < \alpha_1$. Moreover,

$$S_k(x) = S_k(\varepsilon_1 3^{\alpha_1} + z) = S_k(\varepsilon_1 3^{\alpha_1}) + \sum_{i=1}^{z} \sum_{j=0}^{k} \binom{k}{j} (\varepsilon_1 3^{\alpha_1})^{k-j} i^j =$$

$$= S_k(\varepsilon_1 3^{\alpha_1}) + \sum_{j=0}^{k} \binom{k}{j} (\varepsilon_1 3^{\alpha_1})^{k-j} S_j(z)$$

hold, where $S_0(y) = y$. We have $\nu_3(S_k(\varepsilon_1 3^{\alpha_1})) = \nu_3(k) + 2\alpha_1 - 1$. Further, letting $\nu_3^{(j)} = \nu_3\left(\binom{k}{j}(\varepsilon_1 3^{\alpha_1})^{k-j} S_j(z)\right)$ for $0 \leq j \leq k$, we get

$$\nu_3^{(k)} = \nu_3(kz^2(z+1)^2) - 1,$$
$$\nu_3^{(0)} = k\alpha_1 + \nu_3(z),$$
$$\nu_3^{(1)} = \nu_3(k) + (k-1)\alpha_1 + \nu_3(z(z+1)),$$

and for $1 < j < k$,

$$\nu_3^{(j)} = \nu_3\left(\binom{k}{j}\right) + (k-j)\alpha_1 + \nu_3(z(z+1)(2z+1)) - 1, \text{ if } j \text{ is even},$$

$$\nu_3^{(j)} = \nu_3\left(\binom{k}{j}\right) + (k-j)\alpha_1 + \nu_3(jz^2(z+1)^2) - 1, \text{ if } j \text{ is odd}.$$

Recalling $\max(\nu_3(z), \nu_3(z+1)) < \alpha_1$ and noting that $s - 1 - \log s \geq 0$ for any positive integer s, we obtain

$$\nu_3^{(0)} - \nu_3^{(k)} > (k-2)\alpha_1 - \nu_3(k) + 1 \geq k - 1 - \log k \geq 0,$$
$$\nu_3^{(1)} - \nu_3^{(k)} > (k-2)\alpha_1 + 1 \geq k - 1 \geq 0.$$

Using further

$$\nu_3\left(\binom{k}{j}\right) = \nu_3\left(\binom{k}{k-j}\right) \geq \max(\nu_3(k) - \nu_3(j), \nu_3(k) - \nu_3(k-j))$$

for $1 < j < k$, we get

$$\nu_3^{(j)} - \nu_3^{(k)} > (k-j-1)\alpha_1 - \nu_3(k-j) \geq k-j-1-\log(k-j) \geq 0$$

if j is even, and

$$\nu_3^{(j)} - \nu_3^{(k)} \geq (k-j)\alpha_1 > 0$$

if j is odd. Hence

$$\nu_3^{(k)} < \nu_3^{(j)} \ (0 \leq j < k) \quad \text{and} \quad \nu_3^{(k)} < \nu_3(S_k(\varepsilon_1 3^{\alpha_1})).$$

Therefore we obtain

$$\nu_3(S_k(x)) = \nu_3^{(k)} = \nu_3(kz^2(z+1)^2) - 1.$$

As $\nu_3(x) = \nu_3(z)$ and $\nu_3(x+1) = \nu_3(z+1)$, hence the lemma follows.

Note that the argument in [1] goes along the same lines. However, because of the not appropriate choice of z (indicated before), it does not work properly, some inequalities in the proof of Lemma 3.2 in [1] fail in certain cases. □

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References

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