SOME DIOPHANTINE PROPERTIES OF THE SEQUENCE OF S-UNITS

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Abstract. Let S be a finite set of rational primes, and let s_n denote the increasing sequence of the positive integers having all their prime factors in S. In this paper we develop a method to explicitly give the gaps in the sequence s_n. In other words, for any term s_n we can find both s_{n-1} and s_{n+1}, at least in principle, without enumerating all terms of the sequence. In the case when S contains two fixed primes, we even give an efficient algorithm to find these terms explicitly. Further, we apply our results to prove some diophantine properties of the sequence s_n.

1. Introduction

Integers having no prime factors outside a fixed set of primes play important role and are heavily investigated in several parts of number theory. For example, they play special role in diophantine number theory; see e.g. the classical survey paper of Evertse, Győry, Stewart and Tijdeman [1] or Chapter 1 of the book of Shorey and Tijdeman [7] and the references given there.

Further, the sequence formed of such integers is also of interest. To be precise, fix primes $\mathfrak{p}_1 < \cdots < \mathfrak{p}_t$, and write s_n for the sequence of integers

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composed of these primes, arranged in an increasing order. Tijdeman [8] and [9] provided sharp upper and lower bounds for the gaps between consecutive terms of the sequence, respectively. These bounds have the nice property that they are "almost" equal. Namely, Tijdeman proved that

$$\frac{s_n}{(\log s_n)^{c_1}} < s_{n+1} - s_n < \frac{s_n}{(\log s_n)^{c_2}} \tag{1.1}$$

hold with some effectively computable absolute constants c_1 and c_2 for all index n which is large enough. In the proofs of both the lower and the upper bound in (1.1) the approximation properties of the tuple $(\log \mathfrak{p}_1, \ldots, \log \mathfrak{p}_t)$ play a crucial role. These are mainly used through Baker's theory, but in establishing the upper bound also the continued fractions of $\log \mathfrak{p}_i / \log \mathfrak{p}_j$ play a vital role.

In this paper we develop a method to explicitly give the gaps in the sequence s_n. In other words, for any term s_n we can find both s_{n-1} and s_{n+1}, at least in principle, without enumerating all terms of the sequence. Again, here the approximation properties of the tuple $(\log \mathfrak{p}_1, \ldots, \log \mathfrak{p}_t)$ are decisive. In the case when there are two fixed primes, we even give an efficient algorithm to find these terms explicitly. This is done by the careful analysis of the behavior of the continued fractions of $\log \mathfrak{p}_1 / \log \mathfrak{p}_2$. Since to explain our results and methods in detail we need several notions and notation, we shall do that in the next section.

We note that our general result relies in fact on a simple geometrical observation, and our main contribution concerns the case $t = 2$. However, for the sake of completeness we prefer to include the general statement, as well.

2. Main results

Let $S = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_t\}$ be a set of t rational primes, and in the sequel suppose that $\mathfrak{p}_1 < \mathfrak{p}_2 < \cdots < \mathfrak{p}_t$. The ring of rational S-integers is denoted by $\mathbb{Z}_S$, and its unit group by $\mathbb{Z}_S^*$. Consider those S-units, which are natural numbers, and denote by (s_n) the sequence consisting of these numbers in increasing order. Clearly, any element of the sequence (s_n) can be written in the form $s_n = \mathfrak{p}_1^{c_{n,1}} \mathfrak{p}_2^{c_{n,2}} \ldots \mathfrak{p}_t^{c_{n,t}}$ with $c_{n,i} \in \mathbb{Z}_{\geq 0}$.

Consider the hyperplane $\mathcal{P} \subset \mathbb{R}^t$ defined by

$$\mathcal{P} := \{(x_1, \dots, x_t) \ : \ x_1 \log \mathfrak{p}_1 + \dots + x_t \log \mathfrak{p}_t = 0\}.$$

Then $\mathcal{P}$ is a subspace of $\mathbb{R}^t$, in particular, it clearly contains the origin. For a point $\mathbf{a} = (a_1, \dots, a_t) \in \mathbb{Z}^t_{\geq 0}$ denote by $d(\mathbf{a})$ the Euclidean distance of the point $\mathbf{a}$ from the hyperplane $\mathcal{P}$ in $\mathbb{R}^t$.

Theorem 2.1. *The following statements are true:*

(i) *For all* $\boldsymbol{a} = (a_1, \dots, a_t), \boldsymbol{b} = (b_1, \dots, b_t) \in \mathbb{Z}^t_{\geq 0}$ *we have*

$$\mathfrak{p}_1^{a_1} \dots \mathfrak{p}_t^{a_t} < \mathfrak{p}_1^{b_1} \dots \mathfrak{p}_t^{b_t} \iff d(\boldsymbol{a}) < d(\boldsymbol{b}).$$

In particular, $d(\boldsymbol{a}) = d(\boldsymbol{b})$ *if and only if* $\boldsymbol{a} = \boldsymbol{b}$.

(ii) *Let* $r \in \mathbb{R}_{>0}$, *and write*

$$c(r) := \frac{\log r}{\sqrt{\log^2 \mathfrak{p}_1 + \dots + \log^2 \mathfrak{p}_t}}.$$

Then the smallest s_n *for which* $s_n > r$ *is that* $s_n = \mathfrak{p}_1^{a_1} \dots \mathfrak{p}_t^{a_t}$ *with* $\boldsymbol{a} = (a_1, \dots, a_t) \in \mathbb{Z}^t_{\geq 0}$ *for which for every* $\boldsymbol{b} = (b_1, \dots, b_t) \in \mathbb{Z}^t_{\geq 0}$ *with* $d(\boldsymbol{b}) > c(r)$ *we have*

$$c(r) < d(\boldsymbol{a}) < d(\boldsymbol{b}).$$

Similarly, the largest s_n *for which* $s_n < r$ *is that* $s_n = \mathfrak{p}_1^{a_1} \dots \mathfrak{p}_t^{a_t}$ *with* $\boldsymbol{a} = (a_1, \dots, a_t) \in \mathbb{Z}^t_{\geq 0}$ *for which for every* $\boldsymbol{b} = (b_1, \dots, b_t) \in \mathbb{Z}^t_{\geq 0}$ *with* $d(\boldsymbol{b}) < c(r)$ *we have*

$$c(r) > d(\boldsymbol{a}) > d(\boldsymbol{b}).$$

Further, in both cases $\boldsymbol{a}$ *can be effectively determined.*

Remark. The proof of Theorem 2.1 is based upon some properties of a certain special multidimensional diophantine approximation. For the theory of multidimensional diophantine approximations of different types see the excellent survey paper of Moshcevitin [4], and the references given there.

In the special case $t = 2$ we can formulate much more precise results. In order to do so, we need to introduce some further notation. From now on let $S = \{\mathfrak{p}, \mathfrak{q}\}$ be a set of two rational primes with $\mathfrak{p} < \mathfrak{q}$. Now the sequence

(s_n) may be written in the form $s_n = \mathfrak{p}^{c_n}\mathfrak{q}^{d_n}$ with $c_n, d_n \in \mathbb{Z}_{\geq 0}$. We define the companion sequence (f_n) of (s_n) by

$$f_n := \frac{d_{n+1} - d_n}{c_n - c_{n+1}}. \tag{2.1}$$

Later we shall prove that the elements of the sequence (f_n) are always well defined (i.e. $c_n - c_{n+1} \neq 0$), they are always in lowest terms (i.e. $\gcd(d_{n+1} - d_n, c_n - c_{n+1}) = 1$), and $f_n \geq 0$, with equality precisely for values of n for which $s_n < \mathfrak{q}$.

In the statement of our results below we use notions related to the continued fractions of real numbers. Here we use these notions without any reference, however the concepts and results connected to continued fractions which are needed in the paper, are summarized in Section 3.

Given a concrete element of the sequence (s_n), the following theorem gives a simple algorithm how to determine the next element in the sequence.

Theorem 2.2. *Let the sequences $(s_n), (c_n) and (d_n)$ have the same meaning as above. Suppose that we are given $s_k = \mathfrak{p}^{c_k}\mathfrak{q}^{d_k}$. Then we can compute s_{k+1} in the following way:*

- *Let $\frac{u_1}{v_1}$ be the upper convergent of $\frac{\log \mathfrak{p}}{\log \mathfrak{q}}$ with maximal denominator for which $v_1 \leq c_k$ holds.*
- *Let $\frac{u_2}{v_2}$ be the lower convergent of $\frac{\log \mathfrak{p}}{\log \mathfrak{q}}$ with maximal numerator for which $u_2 \leq d_k$ holds.*
- *Put $x := |v_1 \log \mathfrak{p} - u_1 \log \mathfrak{q}| - |v_2 \log \mathfrak{p} - u_2 \log \mathfrak{q}|$ and*

$$c_{k+1} = \begin{cases} c_k - v_1 & \text{if } x < 0, \\ c_k + v_2 & \text{if } x > 0, \end{cases} \qquad d_{k+1} = \begin{cases} d_k + u_1 & \text{if } x < 0, \\ d_k - u_2 & \text{if } x > 0. \end{cases} \tag{2.2}$$

Then we have $s_{k+1} = \mathfrak{p}^{c_{k+1}}\mathfrak{q}^{d_{k+1}}$.

Remark. In view of the method of the proof, having s_k one can explicitly give the term s_{k-1} of the sequence, similarly to the term s_{k+1}. However, since in the light of Theorem 2.2 this can be done in the obvious way, we omit the details.

In the following theorem we summarize basic properties of the companion sequence, which sequence describes how the exponents of $\mathfrak{p}$ and $\mathfrak{q}$ change when we move from s_n to s_{n+1}.

Theorem 2.3. *Let the sequences $(s_n), (c_n), (d_n)$ and (f_n) have the same meaning as above. Then we have the following properties:*

(i) *The sequence (f_n) is well-defined, i.e. $c_{n+1} \neq c_n$ for all $n \in \mathbb{N}$.*

(ii) *We have $f_n \geq 0$ for all $n \in \mathbb{N}$, with equality precisely for those values of n for which $s_n < \mathfrak{q}$.*

(iii) *All companion fractions f_n are convergents of $\frac{\log \mathfrak{p}}{\log \mathfrak{q}}$, and*
- *if f_n is an upper convergent then $c_{k+1} < c_k$ and $d_{k+1} > d_k$,*
- *if f_n is a lower convergent then $c_{k+1} > c_k$ and $d_{k+1} < d_k$.*

(iv) *Suppose that the smallest index n such that $f_n = \frac{u}{v}$ is k. Then*
- *if $\frac{u}{v}$ is an upper convergent of $\frac{\log \mathfrak{p}}{\log \mathfrak{q}}$ then we have $s_k = \mathfrak{p}^v$ and $s_{k+1} = \mathfrak{q}^u$;*
- *if $\frac{u}{v}$ is a lower convergent of $\frac{\log \mathfrak{p}}{\log \mathfrak{q}}$ then we have $s_k = \mathfrak{q}^u$ and $s_{k+1} = \mathfrak{p}^v$.*

Conversely,
- *if $s_k = \mathfrak{p}^v$ and $s_{k+1} = \mathfrak{q}^u$ then $f_k = \frac{u}{v}$ is an upper convergent of $\frac{\log \mathfrak{p}}{\log \mathfrak{q}}$ and k is the index of $\frac{u}{v}$ in the sequence (f_n);*
- *if $s_k = \mathfrak{q}^u$ and $s_{k+1} = \mathfrak{p}^v$ then $f_k = \frac{u}{v}$ is a lower convergent of $\frac{\log \mathfrak{p}}{\log \mathfrak{q}}$ and k is the index of $\frac{u}{v}$ in the sequence (f_n).*

(v) *Let $\frac{p_{i,j}}{q_{i,j}}$ be a convergent of $\frac{\log \mathfrak{p}}{\log \mathfrak{q}}$. The number of occurrences of $\frac{p_{i,j}}{q_{i,j}}$ in the sequence (f_n) is exactly $p_{i+1}q_{i+1}$, where $\frac{p_{i+1}}{q_{i+1}}$ is the principal convergent of $\frac{\log \mathfrak{p}}{\log \mathfrak{q}}$ following the principal convergent $\frac{p_i}{q_i} = \frac{p_{i,0}}{q_{i,0}}$.*

To understand well the structure of our sequence (s_n) we need to know how the corresponding companion sequence (f_n) behaves. Some of the most important arising questions are the following:

- if we know the value of f_n then which values can be taken by f_{n-1} and f_{n+1} respectively
- how many consecutive elements of the sequence f_n may have the same value $\frac{p_{i,j}}{q_{i,j}}$.

Theorems 2.4 and 2.5 give a precise answer to these questions. In one hand we prove that an intermediate convergent cannot be the value of two consecutive elements of (f_n), and that there are at most $a_{j+2} + 1$ consecutive elements of (f_n) which assume the same value $\frac{p_j}{q_j}$. Further our Theorems describe all possible patterns formed by exactly k $(1 \le k \le a_{j+2}+1)$ consecutive elements of (f_n) assuming the same value $\frac{p_j}{q_j}$, and by the preceding and the following elements. Moreover, our Lemmas in Section 6 give necessary and sufficient conditions for $c_n = \mathrm{ord}_{\mathfrak{q}}\, s_n$ and $d_n = \mathrm{ord}_{\mathfrak{p}}\, s_n$ so that s_{n-1} is the starting point of such a concrete pattern.

In the following Theorem 2.4 we answer the above question for principal convergents, and in Theorem 2.5 we do the same for intermediate convergents.

Theorem 2.4. *Let us suppose that in the sequence of companion fractions we have the following pattern:*

$$f_{n-1} \neq \frac{p_l}{q_l},\ f_n = f_{n+1} = \cdots = f_{n+k-1} = \frac{p_l}{q_l},\ f_{n+k} \neq \frac{p_l}{q_l}. \tag{2.3}$$

Then we have $1 \le k \le a_{l+1}+1$, *and for* (f_{n-1}, f_{n+k}) *we have the following possibilities:*

(i) *If* $1 \le k < a_{l+1}$ *then*

$$\begin{aligned}(f_{n-1}, f_{n+k}) \in \Bigg\{ & \left(\frac{p_{l+1}}{q_{l+1}}, \frac{p_{l+1}}{q_{l+1}}\right), \left(\frac{p_{l-1,k-1}}{q_{l-1,k-1}}, \frac{p_{l-1,k-1}}{q_{l-1,k-1}}\right), \\ & \left(\frac{p_{l-1,k-1}}{q_{l-1,k-1}}, \frac{p_{l-1,k}}{q_{l-1,k}}\right) \left(\frac{p_{l-1,k}}{q_{l-1,k}}, \frac{p_{l-1,k-1}}{q_{l-1,k-1}}\right) \left(\frac{p_{l-1,k}}{q_{l-1,k}}, \frac{p_{l-1,k}}{q_{l-1,k}}\right)\Bigg\}\end{aligned} \tag{2.4}$$

(ii) *If* $k = a_{l+1}$ *then*

$$\begin{aligned}(f_{n-1}, f_{n+k}) \in \Bigg\{ & \left(\frac{p_{l+1}}{q_{l+1}}, \frac{p_{l+1}}{q_{l+1}}\right), \left(\frac{p_{l-1,k-1}}{q_{l-1,k-1}}, \frac{p_{l-1,k-1}}{q_{l-1,k-1}}\right), \\ & \left(\frac{p_{l-1,k-1}}{q_{l-1,k-1}}, \frac{p_{l+1}}{q_{l+1}}\right) \left(\frac{p_{l+1}}{q_{l+1}}, \frac{p_{l-1,k-1}}{q_{l-1,k-1}}\right)\Bigg\}\end{aligned} \tag{2.5}$$

(iii) *If* $k = a_{l+1}+1$ *then*

$$(f_{n-1}, f_{n+k}) = \left(\frac{p_{l+1}}{q_{l+1}}, \frac{p_{l+1}}{q_{l+1}}\right). \tag{2.6}$$

Theorem 2.5. *Suppose that* $f_n = \frac{p_{l,j}}{q_{l,j}}$ *with some* $1 \le j < a_{l+2}$ *(i.e.* f_n *is an intermediate convergent). Then we have*

$$f_{n-1} = f_{n+1} = \frac{p_{l+1}}{q_{l+1}}. \tag{2.7}$$

3. Continued fractions

In this section we summarize important properties of the continued fraction expansion and the corresponding convergents of real numbers. For the general theory of continued fractions we refer to the classical books [2], [5], [6] and the references given there. If $S = \{\mathfrak{p}, \mathfrak{q}\}$, then, as we have seen, the structure of the sequence of natural S-units is strongly connected to the convergents of the real number $\frac{\log \mathfrak{p}}{\log \mathfrak{q}}$. The proofs of the properties listed below may be found in [2], [5] and [6].

Let $0 \neq \alpha \in \mathbb{R}$ be a real number and define $a_0, a_1, a_2, \dots$ in the following way: $\alpha_0 := \alpha$, $a_0 := [\alpha_0]$, $\alpha_{i+1} := \{\frac{1}{\{\alpha_i\}}\}$, $a_{i+1} := [\frac{1}{\alpha_{i+1}}]$, . The sequence (a_n) is called the continued fraction of α. In the sequel, for $0 \neq \alpha \in \mathbb{R}$ we shall denote by $[a_0, a_1, a_2 \dots]$ the continued fraction expansion of α. Put

$$p_{-2} = 0,\ p_{-1} = 1, \quad p_i = a_i p_{i-1} + p_{i-2} \quad (i \geq 0) \tag{3.1}$$

and

$$q_{-2} = 1,\ q_{-1} = 0, \quad q_i = a_i q_{i-1} + q_{i-2} \quad (i \geq 0). \tag{3.2}$$

The fractions p_i/q_i for $i \geq 0$ are called the principal convergents of α. Further, for non-negative integers i and j put

$$p_{i,j} = j p_{i+1} + p_i, \quad q_{i,j} = j q_{i+1} + q_i. \tag{3.3}$$

The fractions

$$\frac{p_{i,j}}{q_{i,j}} = \frac{j p_{i+1} + p_i}{j q_{i+1} + q_i} \quad 1 \leq j \leq a_{i+2} - 1 \tag{3.4}$$

are called the intermediate convergents of α. We mention, that in many cases it is comfortable to let in (3.4) the index j assume also the values 0 and a_{i+2}, in these cases the resulting fraction in (3.4) being a principal convergent, namely:

$$\frac{p_{i,0}}{q_{i,0}} = \frac{p_i}{q_i} \quad \text{and} \quad \frac{p_{i,a_{i+2}}}{q_{i,a_{i+2}}} = \frac{p_{i+2}}{q_{i+2}}. \tag{3.5}$$

The principal convergents and intermediate convergents together are called convergents. For the convergents of α we have the following properties:

$$\cdots < \frac{p_i}{q_i} < \cdots < \frac{p_{i,j}}{q_{i,j}} < \frac{p_{i,j+1}}{q_{i,j+1}} < \cdots < \frac{p_{i+2}}{q_{i+2}} < \ldots \qquad \text{if } i \text{ is even,} \tag{3.6}$$

$$\cdots > \frac{p_i}{q_i} > \cdots > \frac{p_{i,j}}{q_{i,j}} > \frac{p_{i,j+1}}{q_{i,j+1}} > \cdots > \frac{p_{i+2}}{q_{i+2}} > \cdots > \qquad \text{if } i \text{ is odd,} \tag{3.7}$$

and $p_{i,j-1}q_{i,j} - p_{i,j}q_{i,j-1} = (-1)^j$ for $i \geq 0$ and $1 \leq j \leq a_{i+2} - 1$. In the sequel the fractions (3.6) of even indices will also be referred to as *lower convergents*, while the fractions (3.7) of odd indices as *upper convergents*. This terminology is clearly justified by the fact, that lower convergents of α are smaller then α, while upper convergents of α are larger then α.

We say that

- the rational number $\frac{p}{q}$ is *a best approximation to* α if for every rational number $\frac{b}{c}$ with denominator $c < q$ we have

$$|q\alpha - p| < |c\alpha - b| \tag{3.8}$$

- the rational number $\frac{p}{q}$ is *a best lower approximation to* α if $\frac{p}{q} < \alpha$ and for every rational number $\frac{b}{c} < \alpha$ with denominator $c < q$ we have

$$q\alpha - p < c\alpha - b \tag{3.9}$$

- the rational number $\frac{p}{q}$ is *a best upper approximation to* α if $\frac{p}{q} > \alpha$ and for every rational number $\frac{b}{c} > \alpha$ with denominator $c < q$ we have

$$p - q\alpha < b - c\alpha. \tag{3.10}$$

The first statement of the following lemma is a well-known property of principal convergents (see e.g. [2], [5], [6]), while the second and third statements are due to Kimberling [3]; see also Section 37 of [5].

Lemma 3.1. *Let $\alpha \neq 0$ be a real number, and denote by p_i/q_i for $i \geq 0$ the principal convergents of α and by $\frac{p_{i,j}}{q_{i,j}}$ for $i \geq 0$, $1 \leq j < a_{i+2}$ the intermediate convergents of α. Then the following statements are true:*

(i) *If $\frac{b}{c}$ satisfies $|c\alpha - b| < |q_i\alpha - p_i|$ then $c \geq q_{i+1}$*

(ii) *The best lower approximates to α are the lower convergents to α, i.e. the fractions $\frac{p_{i,j}}{q_{i,j}}$ for even i and $0 \le j < a_{i+2}$.*

(iii) *The best upper approximates to α are the upper convergents to α, i.e. the fractions $\frac{p_{i,j}}{q_{i,j}}$ for odd i and $0 \le j < a_{i+2}$.*

Remark. The first statement of Lemma 3.1 implies as a simple corollary that the best approximates to α are the principal convergents of α. In the last two statements of Lemma 3.1 among the best lower and upper approximations $\frac{p_{i,j}}{q_{i,j}}$ for $0 \le j < a_{i+2}$ we can find the principal convergents, i.e. the fractions with $j = 0$, and the intermediate convergents, i.e. the fractions with $1 \le j < a_{i+2}$.

The next lemma is a classical result for continued fractions again (see e.g. [2], [5], [6]).

Lemma 3.2. *Suppose that $\frac{p}{q} \ne 0$ is a convergent to a positive real number α. Then $\frac{q}{p}$ is a convergent to $\frac{1}{\alpha}$. The parity of the index of $\frac{q}{p}$ among the convergents of $\frac{1}{\alpha}$ is opposite to the parity of the index of $\frac{p}{q}$ among the convergents of α.*

4. Applications

In this section we give some diophantine applications of our results.

Theorem 4.1. *There exist infinitely many indices k such that the terms $s_k, s_{k+1}, s_{k+2}, s_{k+3}$ form a geometric progression.*

Proof of Theorem 4.1. In fact we prove more. First note that since $\alpha := \frac{\log \mathfrak{p}}{\log \mathfrak{q}}$ is transcendental, the continued fraction expansion of α contains infinitely many terms > 1, so there are either infinitely many odd values of n with $a_{n+1} > 1$ or there are either infinitely many even values of n with $a_{n+1} > 1$.

First suppose that there are infinitely many odd values of n with $a_{n+1} > 1$ and take a fixed odd index n such that $a_{n+1} > 1$. Observe that then we have

$$q_{n+1} = a_{n+1} q_n + q_{n-1} \ge 2q_n + q_{n-1},$$

and

$$p_{n+1} = a_{n+1} p_n + p_{n-1} \ge 2p_n + p_{n-1}.$$

Choose integers A and B subject to the following restrictions:

$$3q_n \le A < 3q_n + q_{n-1}, \quad 0 \le B < p_{n-1}. \tag{4.1}$$

We claim that with any of the above choices for A and B, writing $s_k = \mathfrak{p}^A\mathfrak{q}^B$ we have $f_k = \frac{p_n}{q_n}$, and the terms $s_k, s_{k+1}, s_{k+2}, s_{k+3}$ form a geometric progression. To check these assertions, observe that both

$$A < q_n + q_{n+1} \le \min\{q_{n+2}, q_{n,1}\} \quad \text{and} \quad B + 2p_n < p_{n+1}$$

holds. Hence by Theorem 2.2 we clearly get that

$$s_k = \mathfrak{p}^A\mathfrak{q}^B,\ s_{k+1} = \mathfrak{p}^{A-q_n}\mathfrak{q}^{B+p_n},\ s_{k+2} = \mathfrak{p}^{A-2q_n}\mathfrak{q}^{B+2p_n},\ s_{k+3} = \mathfrak{p}^{A-3q_n}\mathfrak{q}^{B+3p_n}$$

is a desired geometric progression. Since by our assumption there are infinitely many indices n having the desired property, the statement follows.

Now we also have to deal with the case when there are only finitely many odd values of n with $a_{n+1} > 1$. However, in this case there are infinitely many even values of n with $a_{n+1} > 1$ and choosing any such n a similar construction is possible as above, just we have to choose A and B subject to the restrictions

$$0 \le A < q_{n-1}, \quad 3p_n \le B < 3p_n + p_{n-1},$$

and the desired geometric progression shall be

$$s_k = \mathfrak{p}^A\mathfrak{q}^B,\ s_{k+1} = \mathfrak{p}^{A+q_n}\mathfrak{q}^{B-p_n},\ s_{k+2} = \mathfrak{p}^{A+2q_n}\mathfrak{q}^{B-2p_n},\ s_{k+3} = \mathfrak{p}^{A+3q_n}\mathfrak{q}^{B-3p_n}.$$

□

Remark. Assuming that α is not badly approximable, the sequence (s_n) contains arbitrary long geometric progressions. Indeed, in this case one can find infinitely many indices n with $a_n > K$ for any K, and applying the argument in the proof of Theorem 4.1, our claim follows.

For $n \ge 1$ write

$$S_n = \prod_{i=1}^{n} s_i.$$

Theorem 4.2. *There exist infinitely many indices n such that S_n is a perfect square.*

Proof of Theorem 4.2. We give a construction similar to the one from the proof of Theorem 4.1. Take an index k such that $a_{n+1} > 1$ and n is odd, and let

$$A = \begin{cases} 3q_n, & \text{if } q_n \text{ is odd,} \\ 3q_n + 1, & \text{if } q_n \text{ is even,} \end{cases}$$

and

$$B = \begin{cases} 0, & \text{if } p_n \text{ is odd,} \\ 1, & \text{if } p_n \text{ is even.} \end{cases}$$

We mention, that here depending on our parity conditions we chose one of the smallest possible values for both A and B, which fulfil (4.1).

Now put $s_{k+1} := \mathfrak{p}^A\mathfrak{q}^B$ and observe that

$$\mathfrak{p}^A\mathfrak{q}^B,\ \mathfrak{p}^{A-q_n}\mathfrak{q}^{B+p_n},\ \mathfrak{p}^{A-2q_n}\mathfrak{q}^{B+2p_n},\ \mathfrak{p}^{A-3q_n}\mathfrak{q}^{B+3p_n}$$

are four consecutive terms in the sequence (s_n). Write

$$S_k = \mathfrak{p}^U\mathfrak{q}^V.$$

Then

$$S_{k+1} = \mathfrak{p}^{U+A}\mathfrak{q}^{V+B},\ S_{k+2} = \mathfrak{p}^{U+2A-q_n}\mathfrak{q}^{V+2B+p_n},\ S_{k+3} = \mathfrak{p}^{U+3A-3q_n}\mathfrak{q}^{V+3B+3p_n},$$

and observe that by the choices of A, B and $\gcd(p_n, q_n) = 1$ we have that in one of the pairs

$$(U, V),\ (U+A, V+B),\ (U+2A-q_n, V+2B+p_n),\ (U+3A-3q_n, V+3B+3p_n)$$

both entries are even. If we have infinitely many possibilities to choose an odd n with $a_{n+1} > 1$ our statement follows. If there are only finitely many such indices n, then we have infinitely many even values of n such that $a_{n+1} > 1$ and we use a similar construction to conclude the proof of our Theorem 4.2. □

5. Proofs of Theorems 2.1, 2.2 and 2.3

Proof of Theorem 2.1. (i) By elementary linear algebra for any $\mathbf{y} \in \mathbb{R}^t$ we have

$$d(\mathbf{y}) = y_1\frac{\log \mathfrak{p}_1}{u} + \cdots + y_t\frac{\log \mathfrak{p}_t}{u},$$

where $u = \sqrt{\log^2 \mathfrak{p}_1 + \cdots + \log^2 \mathfrak{p}_t}$. Thus for $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_{\geq 0}^t$ we have $d(\mathbf{a}) < d(\mathbf{b})$ if and only if

$$a_1 \log \mathfrak{p}_1 + \cdots + a_t \log \mathfrak{p}_t < b_1 \log \mathfrak{p}_1 + \cdots + b_t \log \mathfrak{p}_t$$

that is, if and only if

$$\mathfrak{p}_1^{a_1} \dots \mathfrak{p}_t^{a_t} < \mathfrak{p}_1^{b_1} \dots \mathfrak{p}_t^{b_t}.$$

This concludes the proof of part (i) of the theorem.
(ii) We prove the assertion only for the case when we are searching for the smallest s_n larger than r. The other case can be proved in the same way.

Let $r \in \mathbb{R}_{>0}$ and let s_k be the smallest term of the strictly increasing sequence (s_n) with $s_k > r$. Clearly, we have $s_k = \mathfrak{p}_1^{a_1} \dots \mathfrak{p}_t^{a_t}$ with some $\mathbf{a} = (a_1, \dots, a_t) \in \mathbb{Z}_{\geq 0}^t$. By assumption we have

$$a_1 \frac{\log \mathfrak{p}_1}{u} + \cdots + a_t \frac{\log \mathfrak{p}_t}{u} > \frac{\log r}{u},$$

where $u = \sqrt{\log^2 \mathfrak{p}_1 + \cdots + \log^2 \mathfrak{p}_t}$. This means that $d(\mathbf{a}) > \frac{\log r}{u}$ holds. Let $\mathbf{b} = (b_1, \dots, b_t) \in \mathbb{Z}_{\geq 0}^t$ arbitrary with $\mathbf{b} \neq \mathbf{a}$ and $d(\mathbf{b}) > \frac{\log r}{u}$, and suppose that $d(\mathbf{b}) \leq d(\mathbf{a})$. As $d(\mathbf{a}) = d(\mathbf{b})$ is impossible, we in fact have $d(\mathbf{b}) < d(\mathbf{a})$. However, then by

$$d(\mathbf{a}) > d(\mathbf{b}) > \frac{\log r}{u}$$

we get

$$\mathfrak{p}_1^{a_1} \dots \mathfrak{p}_t^{a_t} > \mathfrak{p}_1^{b_1} \dots \mathfrak{p}_t^{b_t} > r,$$

contradicting the minimality of s_n with $s_n > r$, as above. Hence for any $\mathbf{b} \neq \mathbf{a}$ with $d(\mathbf{b}) > \frac{\log r}{u}$ we necessarily have

$$d(\mathbf{b}) > d(\mathbf{a}) > r.$$

Finally, note that taking $c_1 \in \mathbb{N}$ the smallest exponent for which $p_1^{c_1} > r$, we have $d(\mathbf{c}) > \frac{\log r}{u}$ with $\mathbf{c} = (c_1, 0, \dots, 0)$. Thus by $d(\mathbf{c}) > d(\mathbf{a}) > \frac{\log r}{u}$, $\mathbf{a}$ is an integral point in a bounded region of $\mathbb{R}^t$, so it can be found effectively. This concludes the proof of part (ii) of the statement. $\square$

Proof of parts (i)-(iii) Theorem 2.3. To prove (i) suppose indirectly that for some $n \in \mathbb{N}$ we have $c_n = c_{n+1}$. Then by $s_{n+1} > s_n$ we must have $d_{n+1} > d_n$. However, then by $\mathfrak{p} < \mathfrak{q}$ we get

$$s_n = \mathfrak{p}^{c_n} \mathfrak{q}^{d_n} < \mathfrak{p}^{c_n+1} \mathfrak{q}^{d_n} < \mathfrak{p}^{c_n} \mathfrak{q}^{d_n+1} \leq s_{n+1},$$

which contradicts the fact that s_n and s_{n+1} are consecutive terms in the strictly increasing sequence (s_n).

Now we turn to the proof of (ii). Indirectly, suppose that for some $n \in \mathbb{N}$ we have $f_n < 0$. Since $s_{n+1} > s_n$ clearly $c_n > c_{n+1}$ and $d_n > d_{n+1}$ cannot hold simultaneously. So the only way that f_n could be negative is that $c_n < c_{n+1}$ and $d_n < d_{n+1}$. However, then

$$s_n = \mathfrak{p}^{c_n}\mathfrak{q}^{d_n} < \mathfrak{p}^{c_n+1}\mathfrak{q}^{d_n} < \mathfrak{p}^{c_n+1}\mathfrak{q}^{d_n+1} \le s_{n+1},$$

a contradiction again. This shows that $f_n \ge 0$.

Now suppose that $f_n = 0$ and $s_n > \mathfrak{q}$. Then we have $d_{n+1} = d_n$, and by $s_{n+1} > s_n$ clearly $c_n < c_{n+1}$. Let c be that positive integer for which $\mathfrak{p}^{c-1} < \mathfrak{q} < \mathfrak{p}^c$. If $d_n \ge 1$ then

$$s_n = \mathfrak{p}^{c_n}\mathfrak{q}^{d_n} < \mathfrak{p}^{c_n+c}\mathfrak{q}^{d_n-1} < \mathfrak{p}^{c_n+1}\mathfrak{q}^{d_n} \le s_{n+1},$$

and if $d_n = 0$ then

$$s_n = p^{c_n} < p^{c_n-c+1}\mathfrak{q} < p^{c_n+1} \le s_{n+1},$$

and by $s_n > \mathfrak{q}$ we see that $c_n - c + 1 \ge 0$. So we find a contradiction again, proving part (ii) of the theorem.

For the proof of (iii) suppose that $f_n = \frac{d_{n+1}-d_n}{c_n-c_{n+1}}$ is a companion fraction such that $c_n > c_{n+1}$ and $d_{n+1} > d_n$. We clearly have

$$\begin{aligned} &0 < s_{n+1} - s_n < s_k - s_n \\ &\qquad\qquad \text{for every } k > n+1 \text{ with } c_k < c_n \text{ and } d_k > d_n. \end{aligned} \tag{5.2}$$

Using $s_i = \mathfrak{p}^{c_i}\mathfrak{q}^{d_i}$ we get

$$0 < (d_{n+1} - d_n) - (c_n - c_{n+1})\frac{\log \mathfrak{p}}{\log \mathfrak{q}} < (d_k - d_n) - (c_n - c_k)\frac{\log \mathfrak{p}}{\log \mathfrak{q}}. \tag{5.3}$$

Now if k runs through all the possible values with $k > n+1$, $c_k < c_n$ and $d_k > d_n$ then $(c_n - c_k, d_k - d_n)$ in (5.3) runs through all pairs (c, d) of positive integers with $0 < c < c_n - c_{n+1}$, $d > 0$ and $\frac{d}{c} > \frac{\log \mathfrak{p}}{\log \mathfrak{q}}$. Indeed, for any (c, d) with the above properties we have $s_k := \mathfrak{p}^{c_n-c} \cdot \mathfrak{q}^{d_n+d} > s_n \cdot \mathfrak{p}^{-c} \cdot \mathfrak{q}^d > s_n$. Thus for this particular s_k we have $d_k - d_n = d$ and $c_n - c_k = c$ in (5.3). Hence by (5.3) we see that $f_n = \frac{d_{n+1}-d_n}{c_n-c_{n+1}}$ is a best upper approximate to $\frac{\log \mathfrak{p}}{\log \mathfrak{q}}$, which in turn means by Lemma 3.1 that f_n is an upper convergent to $\frac{\log \mathfrak{p}}{\log \mathfrak{q}}$.

Now suppose that $f_n = \frac{d_{n+1}-d_n}{c_n-c_{n+1}}$ is a companion fraction such that $c_n < c_{n+1}$ and $d_{n+1} < d_n$. We clearly have

$$0 < s_{n+1} - s_n < s_k - s_n \quad \text{for every } k > n+1 \text{ with } c_k > c_n \text{ and } d_k < d_n. \tag{5.4}$$

Similarly as above, we can deduce

$$0 < (c_{n+1} - c_n) - (d_n - d_{n+1})\frac{\log \mathfrak{q}}{\log \mathfrak{p}} < (c_k - c_n) - (d_n - d_k)\frac{\log \mathfrak{q}}{\log \mathfrak{p}}. \tag{5.5}$$

Now a similar reasoning as above shows that $\frac{c_{n+1}-c_n}{d_n-d_{n+1}}$ is an upper convergent to $\frac{\log \mathfrak{q}}{\log \mathfrak{p}}$, which by Lemma 3.1 proves that $f_n = \frac{d_{n+1}-d_n}{c_n-c_{n+1}}$ is a lower convergent to $\frac{\log \mathfrak{p}}{\log \mathfrak{q}}$. □

Proof of Theorem 2.2. Write $s_k = \mathfrak{p}^{c_k}\mathfrak{q}^{d_k}$. Our goal is to determine $s_{k+1} = \mathfrak{p}^{c_{k+1}}\mathfrak{q}^{d_{k+1}}$. By (iii) of Theorem 2.3 we know that $f_k = \frac{u}{v} = \frac{d_{k+1}-d_k}{c_k-c_{k+1}}$ is a convergent of $\frac{\log \mathfrak{p}}{\log \mathfrak{q}}$.

If f_n is an upper convergent, then it must be the upper convergent with maximal denominator for which $v \le c_k$ holds. Indeed, if this is not true, we have the following cases.

- If $v > c_k$ then we must have $c_k < c_{k+1}$ and since now f_n is an upper convergent, this contradicts (iii) of Theorem 2.3.
- If $v \le c_k$ but v is not maximal among the denominators of the upper convergents having this property, then we have another upper convergent $\frac{u'}{v'}$, with $0 < v < v' \le c_k$. Thus $\frac{u'}{v'}$ is a best upper approximate to $\frac{\log \mathfrak{p}}{\log \mathfrak{q}}$ so by (iii) of Lemma 3.1, putting $c := c_k - v'$ and $d := d_k + u'$ we obtain

$$0 < (d - d_k) - (c_k - c)\frac{\log \mathfrak{p}}{\log \mathfrak{q}} < (d_{k+1} - d_k) - (c_k - c_{k+1})\frac{\log \mathfrak{p}}{\log \mathfrak{q}}.$$

  This is equivalent to

$$s_k < \mathfrak{p}^c\mathfrak{q}^d < s_{k+1},$$

  which contradicts the assumption that s_k and s_{k+1} are consecutive elements of (s_n).

Similarly, using (ii) of Lemma 3.2 we can prove that if f_n is a lower convergent, then it must be the lower convergent with maximal numerator for which $u \le d_k$ holds.

Now it is clear, that we have only those two possibilities for c_{k+1} and d_{k+1} which are stated in (2.2). We only need to prove that the decision among the two possibilities can be made based on the sign of

$$x := |v_1 \log \mathfrak{p} - u_1 \log \mathfrak{q}| - |v_2 \log \mathfrak{p} - u_2 \log \mathfrak{q}| .$$

In order to show this, let us first suppose indirectly, that $x < 0$ but $c_{k+1} = c_k + v_2$ and $d_{k+1} = d_k - u_2$. Since $\frac{u_1}{v_1}$ is an upper convergent and $\frac{u_2}{v_2}$ is a lower convergent we clearly have $v_1 \log \mathfrak{p} - u_1 \log \mathfrak{q} < 0$ and $v_2 \log \mathfrak{p} - u_2 \log \mathfrak{q} > 0$. This shows that $x < 0$ can be rewritten in the form

$$0 < -v_1 \log \mathfrak{p} + u_1 \log \mathfrak{q} < v_2 \log \mathfrak{p} - u_2 \log \mathfrak{q},$$

which can be reformulated as

$$1 < \mathfrak{p}^{-v_1} \mathfrak{q}^{u_1} < \mathfrak{p}^{v_2} \mathfrak{q}^{-u_2}.$$

Multiplying this by $s_k = \mathfrak{p}^{c_k} \mathfrak{q}^{d_k}$, and using that $c_{k+1} = c_k + v_2$ and $d_{k+1} = d_k - u_2$, we get

$$s_k = \mathfrak{p}^{c_k} \mathfrak{q}^{d_k} < \mathfrak{p}^{c_k - v_1} \mathfrak{q}^{d_k + u_1} < \mathfrak{p}^{c_k + v_2} \mathfrak{q}^{d_k - u_2} = s_{k+1},$$

which contradicts the fact that s_k and s_{k+1} are consecutive elements of the sequence (s_n). This means that if $x < 0$ then $c_{k+1} = c_k - v_1$ and $d_{k+1} = d_k + u_1$ must be valid.

We can prove similarly that if $x > 0$ we have $c_{k+1} = c_k + v_2$ and $d_{k+1} = d_k - u_2$. $\square$

Now to continue the proof of Theorem 2.3 we need the following lemma:

Lemma 5.1. *Suppose that $s_n = \mathfrak{p}^{c_n} \mathfrak{q}^{d_n}$. Then we have the following:*

(i) $f_n = \frac{p_{2i,j}}{q_{2i,j}}$ *with* $0 \le j < a_{2i+2}$ *if and only if*

$$\begin{cases} p_{2i,j} \le d_n < p_{2i,j+1}, \\ 0 \le c_n < q_{2i+1}. \end{cases} \tag{5.6}$$

(ii) $f_n = \frac{p_{2i+1,j}}{q_{2i+1,j}}$ *with* $0 \le j < a_{2i+3}$ *if and only if*

$$\begin{cases} 0 \le d_n < p_{2i+2}, \\ q_{2i+1,j} \le c_n < q_{2i+1,j+1}. \end{cases} \tag{5.7}$$

Proof. This is just a simple consequence of our Theorem 2.2. □

We are ready to finish the proof of Theorem 2.3:

Proof of (iv) and (v) of Theorem 2.3. Let $f_k = \frac{p_{2i,j}}{q_{2i,j}}$ be a lower convergent, where $0 \le j < a_{2i+2}$ (it may be both a principal and an intermediate convergent). Then by Lemma 5.1 we clearly have (5.6). Thus the number of elements of the companion sequence which are equal to $\frac{p_{2i,j}}{q_{2i,j}}$ is $(p_{2i,j+1} - p_{2i,j})q_{2i+1} = p_{2i+1}q_{2i+1}$, and this proves (v) for lower convergents. Similarly, if $f_k = \frac{p_{2i+1,j}}{q_{2i+1,j}}$ is an upper convergent, where $0 \le j < a_{2i+3}$, then by Lemma 5.1 we get (5.7), so the number of elements of the companion sequence which are equal to $\frac{p_{2i+1,j}}{q_{2i+1,j}}$ is $p_{2i+2}(q_{2i+1,j+1} - q_{2i+1,j}) = p_{2i+2}q_{2i+2}$, and this proves (v) for upper convergents, so we have completed the proof of (v).

If $f_k = \frac{p_{2i,j}}{q_{2i,j}}$ is a lower convergent, then by (5.6) it is clear that the smallest possible value for s_k is $\mathfrak{q}^{p_{2i,j}}$ and if $f_k = \frac{p_{2i+1,j}}{q_{2i+1,j}}$, then by (5.7) it is clear that the smallest possible value for s_k is $\mathfrak{p}^{q_{2i+1,j}}$. The converse statements are also trivial consequences of (5.6) and (5.7), so this concludes the proof of (iv). □

6. Proof of Theorems 2.4 and 2.5

In order to prove Theorem 2.4 and 2.5 we need to separate the cases where l is odd and l is even. Here we only prove the case when l is odd and we mention that the other case can be proved in the very same way. During the proofs we shall use (3.3) several times without further reference.

For the rest of this section put $l := 2i + 1$.

First we prove Theorem 2.5, since its proof is much simpler.

Proof of Theorem 2.5. Lemma 5.1 shows that if $1 \le j < a_{2i+3}$ then $f_n = \frac{p_{2i+1,j}}{q_{2i+1,j}}$ is equivalent to

$$\begin{cases} 0 \le d_n < p_{2i+2} \\ q_{2i+1,j} \le c_n < q_{2i+1,j+1}. \end{cases} \tag{6.8}$$

Further, $f_n = \frac{p_{2i+1,j}}{q_{2i+1,j}}$ also yields $c_{n+1} = c_n - q_{2i+1,j}$ and $d_{n+1} = d_n + p_{2i+1,j}$. These, together with (6.8) show that we have

$$\begin{cases} p_{2i+1,j} \le d_{n+1} < p_{2i+2} + p_{2i+1,j} \\ \quad 0 \le c_{n+1} < q_{2i+1,j+1} - q_{2i+1,j}, \end{cases} \tag{6.9}$$

this latter being equivalent to

$$\begin{cases} jp_{2i+2} + p_{2i+1} \le d_{n+1} < (j+1)p_{2i+2} + p_{2i+1} \\ \qquad 0 \le c_{n+1} < q_{2i+2}. \end{cases} \tag{6.10}$$

Now using $1 \le j < a_{2i+3}$ (6.10) has the consequence

$$\begin{cases} p_{2i+2} \le d_{n+1} < p_{2i+3} + p_{2i+2} \\ \quad 0 \le c_{n+1} < q_{2i+3}, \end{cases} \tag{6.11}$$

which proves

$$f_{n+1} = \frac{p_{2i+2}}{q_{2i+2}}. \tag{6.12}$$

Now we prove the statement $f_{n-1} = \frac{p_{2i+2}}{q_{2i+2}}$. Suppose indirectly that

$$f_{n-1} \ne \frac{p_{2i+2}}{q_{2i+2}}. \tag{6.13}$$

This is equivalent to the negation of the following condition:

$$\begin{cases} p_{2i+2} \le d_n + p_{2i+2} < p_{2i+3} + p_{2i+2} \\ \quad 0 \le c_n - q_{2i+2} < q_{2i+3}. \end{cases} \tag{6.14}$$

However, the negation of (6.14) is

$$d_n \notin [0, p_{2i+3}[\tag{6.15}$$

or

$$c_n \notin [q_{2i+2}, q_{2i+3} + q_{2i+2}[. \tag{6.16}$$

However, using $q_{2i+3} = a_{2i+3}q_{2i+2} + q_{2i+1}$ and $1 \le j < a_{2i+3}$ it is easily seen that both (6.15) and (6.16) contradict (6.8). Thus the indirect assumption is false, and we have

$$f_{n-1} = \frac{p_{2i+2}}{q_{2i+2}}. \tag{6.17}$$

Now (6.12) and (6.17) is just what we had to prove. □

The proof of Theorem 2.4 is more complicated, so we split it into several lemmas. However, these lemmas may be interesting themselves, too. Recall that $l := 2i + 1$.

Lemma 6.1. *Suppose that* $s_n = \mathfrak{p}^{c_n}\mathfrak{q}^{d_n}$. *Then*

$$f_n = f_{n+1} = \cdots = f_{n+k-1} = \frac{p_{2i+1}}{q_{2i+1}} \tag{6.18}$$

is equivalent to

$$\begin{cases} \quad 0 \leq & d_n < p_{2i+2} - (k-1)p_{2i+1} \\ kq_{2i+1} \leq & c_n < q_{2i+2} + q_{2i+1}. \end{cases} \tag{6.19}$$

Proof. Put $s_j = \mathfrak{q}^{c_j}\mathfrak{p}^{d_j}$ for $j \in \mathbb{N}$. By (6.18) we have $c_{n+l} = c_n - lq_{2i+1}$ and $d_{n+l} = d_n + lp_{2i+1}$ for $l = 0, \dots, k-1$. Thus, by Lemma 5.1, more precisely by (5.7) we have

$$\begin{cases} \quad 0 \leq d_n + lp_{2i+1} < p_{2i+2} & \text{for } l = 0, \dots, k-1 \\ p_{2i+1} \leq c_n - lq_{2i+1} < q_{2i+2} + q_{2i+1} & \text{for } l = 0, \dots, k-1. \end{cases}$$

In fact this is a system of $2k$ inequalities, k of them containing c_n, and the other k containing d_n. It is easy to see that the solution of this is just (6.19). $\square$

Lemma 6.2. *Suppose that* $s_n = \mathfrak{p}^{c_n}\mathfrak{q}^{d_n}$ *and* $1 \leq k \leq a_{2i+2} + 1$. *Then*

$$f_{n-1} = \frac{p_{2i+2}}{q_{2i+2}},\ f_n = f_{n+1} = \cdots = f_{n+k-1} = \frac{p_{2i+1}}{q_{2i+1}},\ f_{n+k} = \frac{p_{2i+2}}{q_{2i+2}} \tag{6.20}$$

is equivalent to

$$\begin{cases} \max(0, p_{2i+2} - kp_{2i+1}) \leq & d_n < p_{2i+2} - (k-1)p_{2i+1} \\ \quad \max(kq_{2i+1}, q_{2i+2}) \leq & c_n < q_{2i+2} + q_{2i+1}. \end{cases} \tag{6.21}$$

Proof. Using Lemma 5.1 and Lemma 6.1 it is easily seen that (6.20) is equivalent to

$$\begin{cases} p_{2i+2} \leq & d_n + p_{2i+2} < p_{2i+2} + p_{2i+3} \\ 0 \leq & c_n - q_{2i+2} < q_{2i+3} \\ 0 \leq & d_n < p_{2i+2} - (k-1)p_{2i+1} \\ kq_{2i+1} \leq & c_n < q_{2i+2} + q_{2i+1} \\ p_{2i+2} \leq & d_n + kp_{2i+1} < p_{2i+2} + p_{2i+3} \\ 0 \leq & c_n - kq_{2i+1} < q_{2i+3} \end{cases} \tag{6.22}$$

and this set of conditions clearly is equivalent to (6.21). □

Lemma 6.3. *Suppose that* $s_n = \mathfrak{p}^{c_n}\mathfrak{q}^{d_n}$ *and* $1 \leq k \leq a_{2i+2} + 1$*. Then*

$$f_{n-1} = \frac{p_{2i,k-1}}{q_{2i,k-1}}, \ f_n = f_{n+1} = \cdots = f_{n+k-1} = \frac{p_{2i+1}}{q_{2i+1}}, \ f_{n+k} = \frac{p_{2i,k-1}}{q_{2i,k-1}} \tag{6.23}$$

is equivalent to

$$\begin{cases} 0 \leq & d_n < p_{2i} \\ kq_{2i+1} \leq & c_n < kq_{2i+1} + q_{2i}. \end{cases} \tag{6.24}$$

Proof. Using Lemma 5.1 and Lemma 6.1 it is easily seen that (6.23) is equivalent to

$$\begin{cases} p_{2i,k-1} \leq & d_n + p_{2i,k-1} < p_{2i,k} \\ 0 \leq & c_n - q_{2i,k-1} < q_{2i+1} \\ 0 \leq & d_n < p_{2i+2} - (k-1)p_{2i+1} \\ kq_{2i+1} \leq & c_n < q_{2i+2} + q_{2i+1} \\ p_{2i,k-1} \leq & d_n + kp_{2i+1} < p_{2i,k} \\ 0 \leq & c_n - kq_{2i+1} < q_{2i+1} \end{cases} \tag{6.25}$$

and (using also (3.3)) this set of conditions is clearly equivalent to (6.24). □

Lemma 6.4. *Suppose that* $s_n = \mathfrak{p}^{c_n}\mathfrak{q}^{d_n}$ *and* $1 \leq k < a_{2i+2} + 1$*. Then*

$$f_{n-1} = \frac{p_{2i,k-1}}{q_{2i,k-1}}, \ f_n = f_{n+1} = \cdots = f_{n+k-1} = \frac{p_{2i+1}}{q_{2i+1}}, \ f_{n+k} = \frac{p_{2i,k}}{q_{2i,k}} \tag{6.26}$$

is equivalent to

$$\begin{cases} p_{2i} \le d_n < p_{2i+1} \\ kq_{2i+1} \le c_n < kq_{2i+1} + q_{2i}. \end{cases} \tag{6.27}$$

Proof. Here we have to split the proof in two cases, depending on $k < a_{2i+2}$ or $k = a_{2i+2}$.

If $k < a_{2i+2}$ then using Lemma 5.1 and Lemma 6.1 it is easily seen that (6.26) is equivalent to

$$\begin{cases} p_{2i,k-1} \le & d_n + p_{2i,k-1} < p_{2i,k} \\ 0 \le & c_n - q_{2i,k-1} < q_{2i+1} \\ 0 \le & d_n < p_{2i+2} - (k-1)p_{2i+1} \\ kq_{2i+1} \le & c_n < q_{2i+2} + q_{2i+1} \\ p_{2i,k} \le & d_n + kp_{2i+1} < p_{2i,k+1} \\ 0 \le & c_n - kq_{2i+1} < q_{2i+1} \end{cases} \tag{6.28}$$

and (using also (3.3)) this set of conditions is clearly equivalent to (6.27).

If $k = a_{2i+2}$ then the same argument applies, except that the last two conditions in (6.28) are replaced by

$$\begin{aligned} p_{2i+2} \le \; & d_n + kp_{2i+1} < p_{2i+2} + p_{2i+3} \\ 0 \le \; & c_n - kq_{2i+1} < q_{2i+3}. \end{aligned} \tag{6.29}$$

However, this set of conditions will be equivalent to the same (6.27) as in the case $k < a_{2i+2}$. □

Lemma 6.5. *Suppose that* $s_n = \mathfrak{p}^{c_n}\mathfrak{q}^{d_n}$ *and* $1 \le k < a_{2i+2} + 1$. *Then*

$$f_{n-1} = \frac{p_{2i,k}}{q_{2i,k}},\ f_n = f_{n+1} = \cdots = f_{n+k-1} = \frac{p_{2i+1}}{q_{2i+1}},\ f_{n+k} = \frac{p_{2i,k-1}}{q_{2i,k-1}} \tag{6.30}$$

is equivalent to

$$\begin{cases} 0 \le d_n < p_{2i} \\ kq_{2i+1} + q_{2i} \le c_n < (k+1)q_{2i+1}. \end{cases} \tag{6.31}$$

Proof. Here we have to split the proof in two cases, depending on $k < a_{2i+2}$ or $k = a_{2i+2}$.

If $k < a_{2i+2}$ then using Lemma 5.1 and Lemma 6.1 it is easily seen that (6.26) is equivalent to

$$\begin{cases} p_{2i,k} \leq & d_n + p_{2i,k} < p_{2i,k+1} \\ 0 \leq & c_n - q_{2i,k} < q_{2i+1} \\ 0 \leq & d_n < p_{2i+2} - (k-1)p_{2i+1} \\ kq_{2i+1} \leq & c_n < q_{2i+2} + q_{2i+1} \\ p_{2i,k-1} \leq & d_n + kp_{2i+1} < p_{2i,k} \\ 0 \leq & c_n - kq_{2i+1} < q_{2i+1} \end{cases} \tag{6.32}$$

and (using also (3.3)) this set of conditions is clearly equivalent to (6.31).

If $k = a_{2i+2}$ then the same argument applies, except that the first two conditions in (6.32) are replaced by

$$\begin{aligned} p_{2i+2} \leq & \quad d_n + p_{2i+2} < p_{2i+2} + p_{2i+3} \\ 0 \leq & \quad c_n - q_{2i+2} < q_{2i+3}. \end{aligned} \tag{6.33}$$

However, this set of conditions will be equivalent to the same (6.31) as in the case $k < a_{2i+2}$. □

Lemma 6.6. *Suppose that* $s_n = \mathfrak{p}^{c_n}\mathfrak{q}^{d_n}$ *and* $1 \leq k < a_{2i+2}$*. Then*

$$f_{n-1} = \frac{p_{2i,k}}{q_{2i,k}},\ f_n = f_{n+1} = \cdots = f_{n+k-1} = \frac{p_{2i+1}}{q_{2i+1}},\ f_{n+k} = \frac{p_{2i,k}}{q_{2i,k}} \tag{6.34}$$

is equivalent to

$$\begin{cases} p_{2i} \leq d_n < p_{2i+1} \\ kq_{2i+1} + q_{2i} \leq c_n < (k+1)q_{2i+1}. \end{cases} \tag{6.35}$$

Remark. We mention that the case $k = a_{2i+2}$ is just the case described by Lemma 6.2.

TABLE 1.

	$1 \leq k < a_{2i+2}$
D_1	$\emptyset$
D_2	$\begin{cases} d_n \in [0, p_{2i+1}[\\ c_n \in [kq_{2i+1}, (k+1)q_{2i+1}[\end{cases}$
D_3	$\begin{cases} d_n \in [p_{2i+2} - kp_{2i+1}, p_{2i+2} - (k-1)p_{2i+1}[\\ c_n \in [q_{2i+2}, q_{2i+1} + q_{2i+2}[\end{cases}$
D_4	$\emptyset$

Proof. Using Lemma 5.1 and Lemma 6.1 it is easily seen that (6.26) is equivalent to

$$(6.36) \qquad \begin{cases} p_{2i,k} \leq & d_n + p_{2i,k} < p_{2i,k+1} \\ 0 \leq & c_n - q_{2i,k} < q_{2i+1} \\ 0 \leq & d_n < p_{2i+2} - (k-1)p_{2i+1} \\ kq_{2i+1} \leq & c_n < q_{2i+2} + q_{2i+1} \\ p_{2i,k} \leq & d_n + kp_{2i+1} < p_{2i,k+1} \\ 0 \leq & c_n - kq_{2i+1} < q_{2i+1} \end{cases}$$

and (using also (3.3)) this set of conditions is clearly equivalent to (6.35). $\square$

Lemma 6.7. *Suppose that* $s_n = \mathfrak{p}^{c_n}\mathfrak{q}^{d_n}$ *and* $1 \leq k \leq a_{2i+2} + 1$*. Then*

$$(6.37) \qquad f_{n-1} \neq \frac{p_{2i+1}}{q_{2i+1}}, \ f_n = f_{n+1} = \cdots = f_{n+k-1} = \frac{p_{2i+1}}{q_{2i+1}}, \ f_{n+k} \neq \frac{p_{2i+1}}{q_{2i+1}}.$$

is equivalent to

$$(d_n, c_n) \in D_1 \cup D_2 \cup D_3 \cup D_4,$$

where the sets D_i *are given in Tables 1, 2 and 3.*

Proof. By Lemma 6.1 we already know that (6.18) is equivalent to (6.19), and by Lemma 5.1that $f_{n-1} = \frac{p_{2i+1}}{q_{2i+1}}$ is equivalent to

$$(6.38) \qquad \begin{cases} 0 \leq d_n - p_{2i+1} < p_{2i+2} \\ q_{2i+1} \leq c_n + q_{2i+1} < q_{2i+1} + q_{2i+2}. \end{cases}$$

TABLE 2.

	$k = a_{2i+2}$
D_1	$\begin{cases} d_n \in [p_{2i+2} - kp_{2i+1}, p_{2i+1}[\\ c_n \in [kq_{2i+1}, q_{2i+1} + q_{2i+2}[\end{cases}$
D_2	$\begin{cases} d_n \in [0, p_{2i+1}[\\ c_n \in [kq_{2i+1}, (k+1)q_{2i+1}[\end{cases}$
D_3	$\begin{cases} d_n \in [p_{2i+2} - kp_{2i+1}, p_{2i+2} - (k-1)p_{2i+1}[\\ c_n \in [q_{2i+2}, q_{2i+1} + q_{2i+2}[\end{cases}$
D_4	$\begin{cases} d_n \in [0, p_{2i+2} - (k-1)p_{2i+1}[\\ c_n \in [q_{2i+2}, (k+1)q_{2i+1}[\end{cases}$

TABLE 3.

	$k = a_{2i+2} + 1$
D_1	$\begin{cases} d_n \in [0, p_{2i+2} - (k-1)p_{2i+1}[\\ c_n \in [kq_{2i+1}, q_{2i+1} + q_{2i+2}[\end{cases}$
D_2	$\begin{cases} d_n \in [0, p_{2i+2} - (k-1)q_{2i+1}[\\ c_n \in [kq_{2i+1}, (k+1)q_{2i+1}[\end{cases}$
D_3	$\begin{cases} d_n \in [0, p_{2i+2} - (k-1)p_{2i+1}[\\ c_n \in [kq_{2i+1}, q_{2i+1} + q_{2i+2}[\end{cases}$
D_4	$\begin{cases} d_n \in [0, p_{2i+2} - (k-1)p_{2i+1}[\\ c_n \in [kq_{2i+1}, q_{2i+1} + q_{2i+2}[\end{cases}$

and that under the assumption $s_{n+k-1} = \mathfrak{p}^{c_n - kq_{2i+1}} \mathfrak{q}^{d_n + kp_{2i+1}}$ the statement $f_{n+k} = \frac{p_{2i+1}}{q_{2i+1}}$ is equivalent to

$$\begin{cases} 0 \leq d_n + kp_{2i+1} < p_{2i+2} \\ q_{2i+1} \leq c_n - kq_{2i+1} < q_{2i+1} + q_{2i+2}. \end{cases} \tag{6.39}$$

Clearly, the necessary and sufficient condition for (6.37) is **(6.19) and not (6.38) and not (6.39)**, however, this latter is equivalent to

$$d_n \in [0, p_{2i+2} - (k-1)p_{2i+1}[\tag{6.40}$$

and

$$c_n \in [kq_{2i+1}, q_{2i+1} + q_{2i+2}[\tag{6.41}$$

and

$$\begin{cases} d_n \in]-\infty, p_{2i+1}[\cup[p_{2i+1} + p_{2i+2}, \infty[\\ \textbf{or} \\ c_n \in]-\infty, 0[\cup[q_{2i+2}, \infty[\end{cases} \tag{6.42}$$

and

$$\begin{cases} d_n \in]-\infty, -kp_{2i+1}[\cup[p_{2i+2} - kp_{2i+1}, \infty[\\ \textbf{or} \\ c_n \in]-\infty, (k+1)q_{2i+1}[\cup[(k+1)q_{2i+1} + q_{2i+2}, \infty[. \end{cases} \tag{6.43}$$

The above system in fact leads to four systems of inequalities depending on which part of (6.42) and (6.43) is considered. We shall call the solution set of these systems by D_i for $i = 1, 2, 3, 4$, and the union of the solutions of these systems is the equivalent condition for (6.37). Depending on the value of k these solutions may differ, and the corresponding solutions to the different possibilities for k are just those summarized in Tables 1, 2 and 3. □

Proof of Theorem 2.4. To prove our theorem it is enough to show that the sets specified by the relations (6.21), (6.24), (6.27), (6.31) and (6.35) cover exactly the same possibilities for (c_n, d_n), as the set $D_1 \cup D_2 \cup D_3 \cup D_4$, where the sets D_i are given in Tables 1, 2 and 3.. We have to split our proof in three parts.

If $k < a_{2i+2}$ then (6.21) takes the form

$$\begin{cases} p_{2i+2} - kp_{2i+1} \le d_n < p_{2i+2} - (k-1)p_{2i+1} \\ q_{2i+2} \le c_n < q_{2i+2} + q_{2i+1}. \end{cases} \tag{6.44}$$

This is just the same as D_3. Further, in this case the sets specified in (6.24), (6.27), (6.31) and (6.35) give a pairwise disjoint union of the set D_2. Taking in account that we also have $D_1 = D_4 = \emptyset$ our proof is finished.

If $k = a_{2i+2}$ then (6.21) takes again the form (6.44). In this case sets D_i are not pairwise disjoint, however, here it is also easy to see that the union of the pairwise disjoint sets specified by (6.21), (6.24), (6.27) and (6.31) is just the set $D_1 \cup D_2 \cup D_3 \cup D_4$, which proves our theorem for $k = a_{2i+2}$.

Finally, the case $k = a_{2i+2} + 1$ is the simplest, since in this case (6.21) take the form

$$\begin{cases} 0 \le d_n < p_{2i+2} - (k-1)p_{2i+1} \\ kq_{2i+1} \le c_n < q_{2i+2} + q_{2i+1}. \end{cases} \tag{6.45}$$

Further $D_2 \subset D_1 = D_3 = D_4$ shows that $D_1 \cup D_2 \cup D_3 \cup D_4 = D_1$, which is just the set specified by (6.45)

□

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