

**EXAMPLES FOR SETS S_m
NOT HAVING PROPERTY $P(T)$**

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In this file we provide examples concerning a problem of Pillai and its generalizations. We briefly describe the problems and relevant results here. For more details on the problems and results see the paper [1].

Let S_m denote a set of m consecutive integers and T some set of positive integers containing 1. We say that S_m has property $P(T)$ if there exists an $x \in S_m$ such that $\gcd(x, y) \notin T$ for all $y \in S_m, y \neq x$. It was shown in [1] that S_m does not have property $P(T)$ if and only if the set of integers in $[1, m]$ has an H -covering with an appropriate set H which depends on T . Further, for certain choices of T a theoretical bound $M(T)$ was given such that for each $m \geq M(T)$ there exists an H -covering of $[1, m]$ (and hence a set S_m not having property $P(T)$). In the ensuing examples, for the chosen T we give an H -covering of $[1, m]$ with the appropriate H and $m < M(T)$.

To give the H -coverings, we use the following notation. In each case under consideration, we indicate the set T , the set H used to make the appropriate covering, and we give the covering $f : H \rightarrow [1, m]$ by the help of ordered pairs of the form $[h, f(h)]$ ($h \in H$). For example, in the original problem of Pillai we have $T = \{1\}$, $H = \{p \mid p \text{ is a prime}\}$. For $m = 17$ we have the H -covering given by

$$[2, 1], [3, 2], [5, 6], [7, 3], [11, 1], [13, 4].$$

That is, the function $f : H \rightarrow [1, 17]$ defined by

$$f(2) = 1, f(3) = 2, f(5) = 6, f(7) = 3, f(11) = 1, f(13) = 4$$

is an H -covering of $[1, 17]$. If there is an $h \in H$ such that in the actual H -covering no pair occurs with h as the first entry, then this h is called a spared element and in this case H is not a minimal covering.

The examples are to be read as follows. When a single value of m is specified the H -covering that follows is a covering of $[1, m]$. When a range of m is specified as $m_1 \leq m \leq m_2$, then the H -covering that follows is in fact the H -covering of $[1, m_2]$. To obtain an H -covering of $[1, m]$ for any m in the given range, one should restrict to only those pairs $[h, f(h)]$ with $h + f(h) \leq m$. Finally, note that if an m -value with

$m < M(T)$ is not indicated, then for that $[1, m]$ **NO H -COVERING EXISTS!**

Example I: $T = \{1, 2\}$

In this case $M(T) = 56$ and for each m we have

$$H = \{p \mid p \text{ is an odd prime } < m\} \cup \{4\}.$$

For $m < M(T)$ we have the following H -coverings.

$25 \leq m \leq 35$:

$[3, 1], [4, 3], [5, 4], [7, 5], [11, 6], [13, 8], [17, 1], [19, 1], [23, 2], [29, 1], [31, 1].$

$36 \leq m \leq 55$:

$[3, 1], [4, 1], [5, 3], [7, 6], [11, 3], [13, 11], [17, 15], [19, 7],$
 $[23, 12], [29, 1], [31, 2], [37, 2], [41, 1], [43, 1], [47, 4], [53, 1].$

Example II: $T = \{1, 2, 3\}$

In this case $M(T) = 100$ and for each m we have

$$H = \{p \mid p \text{ is a prime with } 3 < p < m\} \cup \{4, 6, 9\}.$$

For $m < M(T)$ we have the following H -coverings.

 $m = 49$:

$$[4, 3], [5, 2], [6, 1], [7, 2], [9, 1], [11, 7], [13, 8], [17, 7], [19, 14], [23, 3], [29, 20], \\ [31, 5], [37, 1], [41, 4], [43, 6], [47, 1].$$

 $53 \leq m \leq 55$:

$$[4, 1], [5, 2], [6, 1], [7, 2], [9, 1], [11, 4], [13, 1], [17, 1], [19, 1], [23, 11], [29, 24], \\ [31, 5], [37, 1], [41, 6], [43, 8], [47, 3], [53, 1].$$

 $m = 56$:

$$[4, 1], [5, 2], [6, 5], [7, 3], [9, 8], [11, 6], [13, 4], [17, 2], [19, 15], [23, 17], [29, 18], \\ [31, 20], [37, 16], [41, 14], [43, 3], [47, 1], [53, 1].$$

 $57 \leq m \leq 77$:

$$[4, 1], [5, 1], [6, 5], [7, 6], [9, 5], [11, 8], [13, 2], [17, 5], [19, 5], [23, 12], [29, 18], \\ [31, 7], [37, 3], [41, 1], [43, 1], [47, 10], [53, 4], [59, 1], [61, 3], [67, 3], [71, 1], [73, 2].$$

 $78 \leq m \leq 99$:

$$[4, 2], [5, 3], [6, 4], [7, 2], [9, 4], [11, 3], [13, 6], [17, 7], [19, 1], [23, 12], [29, 27], \\ [31, 29], [37, 24], [41, 18], [43, 14], [47, 8], [53, 21], [59, 15], [61, 17], [67, 11], \\ [71, 16], [73, 5], [79, 10], [83, 12], [89, 10], [97, 4].$$

Example III: $T = T_p = \{p^\alpha \mid \alpha \geq 0\}$

For each m we have

$$H = \{q \mid q \text{ is a prime } < m\} \setminus \{p\}.$$

Here we need to distinguish several subcases.

Case III/1: $p = 2$

In this case $M(T) = 384$. For $m < M(T)$ we have the following H -coverings.

$m = 86$:

$$\begin{aligned} & [3, 2], [5, 3], [7, 4], [11, 9], [13, 1], [17, 3], [19, 17], [23, 11], [29, 16], \\ & [31, 30], [37, 15], [41, 10], [43, 6], [47, 22], [53, 19], [59, 24], [61, 21], \\ & [67, 3], [71, 5], [73, 12], [79, 7], [83, 1]. \end{aligned}$$

$m = 87$:

$$\begin{aligned} & [3, 1], [5, 1], [7, 3], [11, 9], [13, 5], [17, 14], [19, 16], [23, 4], [29, 10], \\ & [31, 16], [37, 23], [41, 33], [43, 29], [47, 30], [53, 32], [59, 4], [61, 1], \\ & [67, 2], [71, 15], [73, 12], [79, 8], [83, 1]. \end{aligned}$$

$88 \leq m \leq 125$:

$$\begin{aligned} & [3, 1], [5, 4], [7, 1], [11, 9], [13, 12], [17, 11], [19, 3], [23, 10], [29, 18], \\ & [31, 1], [37, 35], [41, 27], [43, 23], [47, 1], [53, 30], [59, 6], [61, 26], \\ & [67, 21], [71, 17], [73, 7], [79, 2], [83, 5], [89, 4], [97, 4], [101, 6], \\ & [103, 7], [107, 4], [109, 13], [113, 10]. \end{aligned}$$

$126 \leq m \leq 141$:

$$\begin{aligned} & [3, 2], [5, 3], [7, 5], [11, 2], [13, 3], [17, 17], [19, 11], [23, 22], [29, 2], \\ & [31, 4], [37, 2], [41, 23], [43, 27], [47, 37], [53, 9], [59, 52], [61, 6], \\ & [67, 2], [71, 1], [73, 36], [79, 21], [83, 44], [89, 10], [97, 32], [101, 25], \\ & [103, 27], [107, 14], [109, 15], [113, 7], [127, 5], [131, 4], [137, 2], [139, 2]. \end{aligned}$$

$142 \leq m \leq 169$:

[3, 1], [5, 1], [7, 5], [11, 6], [13, 9], [17, 10], [19, 4], [23, 15], [29, 3],
 [31, 29], [37, 30], [41, 16], [43, 16], [47, 45], [53, 24], [59, 10], [61, 53],
 [67, 65], [71, 63], [73, 20], [79, 62], [83, 84], [89, 73], [97, 50], [101, 43],
 [103, 5], [107, 18], [109, 14], [113, 7], [127, 8], [131, 12], [137, 2],
 [139, 1], [149, 1], [151, 4], [157, 1], [163, 1], [167, 1].

 $170 \leq m \leq 215$:

[3, 1], [5, 4], [7, 3], [11, 6], [13, 3], [17, 9], [19, 18], [23, 2], [29, 7],
 [31, 1], [37, 16], [41, 16], [43, 6], [47, 15], [53, 25], [59, 51], [61, 41],
 [67, 19], [71, 31], [73, 23], [79, 47], [83, 12], [89, 4], [97, 79], [101, 66],
 [103, 11], [107, 30], [109, 33], [113, 35], [127, 20], [131, 27], [137, 21],
 [139, 2], [149, 12], [151, 4], [157, 8], [163, 5], [167, 10], [173, 10],
 [179, 9], [181, 10], [191, 4], [193, 4], [197, 3], [199, 8], [211, 1].

 $216 \leq m \leq 296$:

[3, 2], [5, 3], [7, 5], [11, 10], [13, 1], [17, 17], [19, 17], [23, 22], [29, 10],
 [31, 6], [37, 10], [41, 24], [43, 25], [47, 47], [53, 15], [59, 11], [61, 29],
 [67, 5], [71, 64], [73, 42], [79, 31], [83, 49], [89, 67], [97, 30], [101, 15],
 [103, 69], [107, 100], [109, 81], [113, 52], [127, 93], [131, 46], [137, 88],
 [139, 60], [149, 82], [151, 11], [157, 57], [163, 71], [167, 4], [173, 38],
 [179, 2], [181, 7], [191, 19], [193, 9], [197, 8], [199, 16], [211, 5],
 [223, 14], [227, 17], [229, 17], [233, 16], [239, 17], [241, 18], [251, 11],
 [257, 8], [263, 4], [269, 7], [271, 8], [277, 9], [281, 10], [283, 11], [293, 2].

 $297 \leq m \leq 383$:

[3, 2], [5, 4], [7, 2], [11, 4], [13, 10], [17, 12], [19, 16], [23, 13], [29, 4],
 [31, 9], [37, 1], [41, 37], [43, 9], [47, 21], [53, 12], [59, 8], [61, 28], [67, 7],
 [71, 61], [73, 44], [79, 17], [83, 62], [89, 18], [97, 60], [101, 96], [103, 90],
 [107, 76], [109, 87], [113, 27], [127, 45], [131, 106], [137, 85], [139, 123],
 [149, 2], [151, 66], [157, 149], [163, 108], [167, 43], [173, 134], [179, 134],
 [181, 57], [191, 124], [193, 123], [197, 55], [199, 42], [211, 110], [223, 102],
 [227, 31], [229, 3], [233, 22], [239, 88], [241, 87], [251, 25], [257, 80], [263, 6],
 [269, 23], [271, 2], [277, 3], [281, 10], [283, 3], [293, 2], [307, 33], [311, 32], [313, 33],
 [317, 38], [331, 26], [337, 23], [347, 16], [349, 21], [353, 19], [359, 16], [367, 9], [373, 9].

Case III/2: $p = 3$

In this case $M(T) = 705$. For $m < M(T)$ we have the following H -coverings.

 $m = 77$:

[2, 1], [5, 2], [7, 2], [11, 4], [13, 1], [17, 3], [19, 18], [23, 5], [29, 10],
 [31, 5], [37, 1], [41, 34], [43, 3], [47, 3], [53, 24], [59, 1], [61, 3],
 [67, 8], [71, 6], [73, 3].

 $m = 81$:

[2, 1], [5, 4], [7, 4], [11, 6], [13, 3], [17, 2], [19, 1], [23, 7], [29, 8],
 [31, 26], [37, 15], [41, 40], [43, 38], [47, 1], [53, 3], [59, 22], [61, 1],
 [67, 12], [71, 10], [73, 5], [79, 1].

 $82 \leq m \leq 87$:

[2, 1], [5, 2], [7, 2], [11, 4], [13, 11], [17, 6], [19, 18], [23, 14], [29, 8],
 [31, 3], [37, 1], [41, 5], [43, 36], [47, 7], [53, 28], [59, 5], [61, 20],
 [67, 1], [71, 10], [73, 5], [79, 1], [83, 1].

 $88 \leq m \leq 93$:

[2, 1], [5, 1], [7, 6], [11, 8], [13, 2], [17, 10], [19, 12], [23, 22], [29, 24],
 [31, 18], [37, 21], [41, 38], [43, 42], [47, 40], [53, 32], [59, 1], [61, 3],
 [67, 3], [71, 1], [73, 14], [79, 4], [83, 1], [89, 3].

 $94 \leq m \leq 99$:

[2, 1], [5, 1], [7, 6], [11, 8], [13, 2], [17, 10], [19, 12], [23, 1], [29, 14],
 [31, 22], [37, 32], [41, 38], [43, 15], [47, 42], [53, 40], [59, 1], [61, 3],
 [67, 1], [71, 23], [73, 18], [79, 3], [83, 4], [89, 3], [97, 1].

 $100 \leq m \leq 117$:

[2, 1], [5, 1], [7, 1], [11, 7], [13, 2], [17, 14], [19, 13], [23, 19], [29, 10],
 [31, 12], [37, 20], [41, 34], [43, 1], [47, 5], [53, 5], [59, 1], [61, 38],
 [67, 30], [71, 1], [73, 24], [79, 21], [83, 4], [89, 1], [97, 1], [101, 1],
 [103, 1], [107, 3], [109, 3], [113, 1].

$118 \leq m \leq 133$:

[2, 1], [5, 1], [7, 4], [11, 6], [13, 12], [17, 10], [19, 5], [23, 22], [29, 5],
 [31, 20], [37, 30], [41, 2], [43, 34], [47, 1], [53, 52], [59, 58], [61, 54],
 [67, 3], [71, 40], [73, 42], [79, 1], [83, 39], [89, 35], [97, 1], [101, 8],
 [103, 14], [107, 1], [109, 1], [113, 5], [127, 1], [131, 1].

 $134 \leq m \leq 326$:

[2, 1], [5, 4], [7, 1], [11, 5], [13, 7], [17, 11], [19, 13], [23, 7], [29, 13],
 [31, 9], [37, 12], [41, 6], [43, 26], [47, 42], [53, 10], [59, 52], [61, 5],
 [67, 1], [71, 56], [73, 58], [79, 1], [83, 48], [89, 1], [97, 41], [101, 39],
 [103, 2], [107, 11], [109, 1], [113, 18], [127, 1], [131, 1], [137, 5],
 [139, 13], [149, 7], [151, 15], [157, 15], [163, 15], [167, 13], [173, 9],
 [179, 7], [181, 15], [191, 9], [193, 13], [197, 13], [199, 13], [211, 9],
 [223, 17], [227, 15], [229, 19], [233, 17], [239, 17], [241, 21], [251, 19],
 [257, 15], [263, 13], [269, 9], [271, 11], [277, 9], [281, 9], [283, 9],
 [293, 3], [307, 5], [311, 7], [313, 7], [317, 5].

 $327 \leq m \leq 704$:

[2, 2], [5, 3], [7, 7], [11, 5], [13, 8], [17, 16], [19, 13], [23, 19], [29, 23],
 [31, 25], [37, 6], [41, 15], [43, 42], [47, 27], [53, 39], [59, 50], [61, 45],
 [67, 8], [71, 8], [73, 61], [79, 41], [83, 69], [89, 18], [97, 11], [101, 37],
 [103, 18], [107, 10], [109, 57], [113, 95], [127, 83], [131, 85], [137, 75],
 [139, 73], [149, 1], [151, 9], [157, 47], [163, 39], [167, 155], [173, 14],
 [179, 141], [181, 4], [191, 131], [193, 2], [197, 129], [199, 2], [211, 8],
 [223, 144], [227, 142], [229, 146], [233, 144], [239, 142], [241, 148], [251, 140],
 [257, 59], [263, 4], [269, 8], [271, 55], [277, 118], [281, 29], [283, 14],
 [293, 31], [307, 17], [311, 94], [313, 12], [317, 2], [331, 78], [337, 78],
 [347, 70], [349, 70], [353, 68], [359, 76], [367, 72], [373, 74], [379, 70],
 [383, 68], [389, 68], [397, 64], [401, 64], [409, 62], [419, 62], [421, 64],
 [431, 60], [433, 62], [439, 62], [443, 74], [449, 70], [457, 70], [461, 68],
 [463, 68], [467, 68], [479, 58], [487, 62], [491, 60], [499, 58], [503, 56],
 [509, 56], [521, 54], [523, 56], [541, 44], [547, 40], [557, 48], [563, 44],
 [569, 46], [571, 54], [577, 50], [587, 52], [593, 56], [599, 58], [601, 66],
 [607, 62], [613, 64], [617, 64], [619, 70], [631, 64], [641, 60], [643, 62].

Case III/3: $p = 5$

In this case $M(T) = 431$. For $m < M(T)$ we have the following H -coverings.

 $47 \leq m \leq 49$:

[2, 1], [3, 1], [7, 4], [11, 9], [13, 12], [17, 2], [19, 5], [23, 3], [29, 1],
[31, 14], [37, 8], [41, 6], [43, 1], [47, 1].

 $50 \leq m \leq 51$:

[2, 1], [3, 2], [7, 1], [11, 1], [13, 3], [17, 6], [19, 9], [23, 1], [29, 1],
[31, 18], [37, 10], [41, 4], [43, 3], [47, 1].

 $52 \leq m \leq 55$:

[2, 1], [3, 1], [7, 2], [11, 3], [13, 6], [17, 7], [19, 12], [23, 3], [29, 18],
[31, 20], [37, 1], [41, 1], [43, 8], [47, 1], [53, 1].

 $56 \leq m \leq 65$:

[2, 1], [3, 1], [7, 1], [11, 4], [13, 12], [17, 3], [19, 18], [23, 1], [29, 1],
[31, 1], [37, 5], [41, 14], [43, 1], [47, 6], [53, 2], [59, 1], [61, 1].

 $66 \leq m \leq 430$:

[2, 1], [3, 1], [7, 6], [11, 3], [13, 5], [17, 8], [19, 12], [23, 3], [29, 1],
[31, 1], [37, 1], [41, 24], [43, 23], [47, 9], [53, 1], [59, 1], [61, 2],
[67, 1], [71, 3], [73, 5], [79, 5], [83, 3], [89, 3], [97, 1], [101, 7],
[103, 11], [107, 9], [109, 11], [113, 15], [127, 7], [131, 7], [137, 3],
[139, 19], [149, 13], [151, 19], [157, 19], [163, 17], [167, 15], [173, 19],
[179, 15], [181, 17], [191, 15], [193, 29], [197, 31], [199, 37], [211, 37],
[223, 31], [227, 37], [229, 47], [233, 49], [239, 45], [241, 47], [251, 39],
[257, 37], [263, 43], [269, 39], [271, 41], [277, 41], [281, 45], [283, 49],
[293, 57], [307, 53], [311, 51], [313, 61], [317, 63], [331, 55], [337, 53],
[347, 49], [349, 53], [353, 67], [359, 63], [367, 61].

Case III/4: $p = 7$

In this case $M(T) = 425$. For $m < M(T)$ we have the following H -coverings.

 $43 \leq m \leq 49$:

[2, 1], [3, 1], [5, 1], [11, 8], [13, 12], [17, 1], [19, 1], [23, 1], [29, 14],
[31, 1], [37, 2], [41, 1], [43, 1], [47, 1].

 $50 \leq m \leq 57$:

[2, 1], [3, 2], [5, 1], [11, 1], [13, 4], [17, 5], [19, 5], [23, 5], [29, 10],
[31, 18], [37, 3], [41, 1], [43, 9], [47, 1], [53, 1].

 $58 \leq m \leq 424$:

[2, 1], [3, 1], [5, 3], [11, 9], [13, 11], [17, 2], [19, 6], [23, 3], [29, 1],
[31, 1], [37, 23], [41, 12], [43, 14], [47, 9], [53, 1], [59, 3], [61, 5],
[67, 7], [71, 9], [73, 11], [79, 11], [83, 9], [89, 7], [97, 13], [101, 13],
[103, 13], [107, 15], [109, 17], [113, 19], [127, 13], [131, 13], [137, 25],
[139, 31], [149, 33], [151, 41], [157, 37], [163, 37], [167, 45], [173, 43],
[179, 45], [181, 55], [191, 51], [193, 53], [197, 55], [199, 55], [211, 49],
[223, 41], [227, 43], [229, 47], [233, 57], [239, 55], [241, 55], [251, 49],
[257, 55], [263, 51], [269, 55], [271, 55], [277, 53], [281, 51], [283, 71],
[293, 67], [307, 59], [311, 63], [313, 67], [317, 67], [331, 59], [337, 55],
[347, 49], [349, 53], [353, 67], [359, 67].

Case III/5: $p = 11$

In this case $M(T) = 425$. For $m < M(T)$ we have the following H -coverings.

 $m = 27$:

[2, 1], [3, 2], [5, 1], [7, 3], [13, 12], [17, 1], [19, 3], [23, 4].

 $33 \leq m \leq 35$:

[2, 1], [3, 1], [5, 3], [7, 5], [13, 6], [17, 3], [19, 14], [23, 1], [29, 1], [31, 2].

 $39 \leq m \leq 49$:

[2, 1], [3, 1], [5, 1], [7, 4], [13, 12], [17, 3], [19, 5], [23, 14], [29, 1],
[31, 8], [37, 2], [41, 1], [43, 1], [47, 1].

 $50 \leq m \leq 424$:

[2, 2], [3, 1], [5, 1], [7, 1], [13, 9], [17, 5], [19, 8], [23, 23], [29, 17],
[31, 2], [37, 16], [41, 6], [43, 2], [47, 3], [53, 6], [59, 4], [61, 16],
[67, 16], [71, 18], [73, 20], [79, 26], [83, 34], [89, 30], [97, 26], [101, 24],
[103, 32], [107, 30], [109, 34], [113, 34], [127, 22], [131, 22], [137, 22],
[139, 28], [149, 24], [151, 26], [157, 28], [163, 26], [167, 28], [173, 30],
[179, 34], [181, 34], [191, 36], [193, 40], [197, 40], [199, 46], [211, 46],
[223, 50], [227, 52], [229, 58], [233, 72], [239, 76], [241, 88], [251, 82],
[257, 78], [263, 76], [269, 84], [271, 86], [277, 82], [281, 82], [283, 100],
[293, 94], [307, 88], [311, 108].

Case III/6: $p = 13$

In this case $M(T) = 425$. For $m < M(T)$ we have the following H -coverings.

 $m = 27$:

[2, 1], [3, 2], [5, 1], [7, 3], [11, 1], [17, 1], [19, 3], [23, 4].

 $31 \leq m \leq 35$:

[2, 1], [3, 1], [5, 3], [7, 5], [11, 2], [17, 14], [19, 1], [23, 6], [29, 1], [31, 1].

 $36 \leq m \leq 424$:

[2, 1], [3, 2], [5, 1], [7, 3], [11, 1], [17, 1], [19, 3], [23, 5], [29, 1],
 [31, 4], [37, 3], [41, 1], [43, 5], [47, 7], [53, 5], [59, 5], [61, 9],
 [67, 5], [71, 11], [73, 11], [79, 11], [83, 19], [89, 23], [97, 21], [101, 29],
 [103, 29], [107, 31], [109, 33], [113, 47], [127, 35], [131, 37], [137, 35],
 [139, 41], [149, 35], [151, 47], [157, 45], [163, 45], [167, 47], [173, 55],
 [179, 59], [181, 59], [191, 53], [193, 71], [197, 77], [199, 81], [211, 71],
 [223, 69], [227, 67], [229, 81], [233, 89], [239, 89], [241, 89], [251, 83],
 [257, 91], [263, 91], [269, 103], [271, 111], [277, 107], [281, 109], [283, 111],
 [293, 107], [307, 105], [311, 103], [313, 107], [317, 107].

Case III/7: $p = 17$

In this case $M(T) = 425$. For $m < M(T)$ we have the following H -coverings.

 $m = 17$:

[2, 1], [3, 1], [5, 2], [7, 1], [11, 6], [13, 1].

 $21 \leq m \leq 25$:

[2, 1], [3, 1], [5, 3], [7, 6], [11, 1], [13, 1], [19, 2], [23, 1].

 $26 \leq m \leq 27$:

[2, 1], [3, 2], [5, 1], [7, 4], [11, 1], [13, 10], [19, 3], [23, 1].

 $29 \leq m \leq 35$:

[2, 1], [3, 1], [5, 3], [7, 5], [11, 2], [13, 1], [19, 1], [23, 6], [29, 1], [31, 1].

 $36 \leq m \leq 424$:

[2, 1], [3, 1], [5, 3], [7, 5], [11, 3], [13, 6], [19, 1], [23, 1], [29, 1],
 [31, 2], [37, 5], [41, 3], [43, 7], [47, 9], [53, 7], [59, 3], [61, 5],
 [67, 5], [71, 3], [73, 13], [79, 11], [83, 9], [89, 15], [97, 17], [101, 19],
 [103, 19], [107, 25], [109, 31], [113, 31], [127, 29], [131, 33], [137, 33],
 [139, 35], [149, 27], [151, 31], [157, 29], [163, 29], [167, 33], [173, 51],
 [179, 51], [181, 61], [191, 55], [193, 59], [197, 63], [199, 71], [211, 65],
 [223, 61], [227, 67], [229, 67], [233, 69], [239, 75], [241, 85], [251, 79],
 [257, 75], [263, 73], [269, 73], [271, 79], [277, 77], [281, 75], [283, 77],
 [293, 87], [307, 77], [311, 75], [313, 89], [317, 97], [331, 89].

Case III/8: $p = 19$

In this case $M(T) = 425$. For $m < M(T)$ we have the following H -coverings.

 $17 \leq m \leq 19$:

[2, 1], [3, 1], [5, 2], [7, 1], [11, 6], [13, 1], [17, 1].

 $20 \leq m \leq 25$:

[2, 1], [3, 1], [5, 2], [7, 6], [11, 8], [13, 1], [17, 1], [23, 1].

 $26 \leq m \leq 27$:

[2, 1], [3, 2], [5, 2], [7, 4], [11, 10], [13, 3], [17, 6], [23, 1].

 $28 \leq m \leq 35$:

[2, 1], [3, 2], [5, 2], [7, 3], [11, 6], [13, 3], [17, 1], [23, 4], [29, 1], [31, 3].

 $36 \leq m \leq 424$:

[2, 1], [3, 1], [5, 3], [7, 5], [11, 3], [13, 6], [17, 3], [23, 1], [29, 1],
 [31, 2], [37, 29], [41, 1], [43, 1], [47, 3], [53, 3], [59, 1], [61, 1],
 [71, 1], [73, 1], [79, 13], [83, 3], [89, 1], [103, 1], [109, 11], [113, 1],
 [127, 7], [131, 1], [149, 1], [157, 19], [163, 1], [167, 3], [173, 1], [179, 7],
 [181, 1], [191, 1], [197, 13], [199, 1], [229, 1], [239, 7], [241, 1], [257, 3],
 [263, 13], [269, 1], [271, 1], [277, 13], [281, 1], [283, 1], [293, 1],
 [311, 19], [313, 1], [317, 7], [331, 5], [337, 5], [349, 1], [353, 1], [367, 5],
 [373, 13], [379, 1], [383, 1], [401, 1], [409, 7], [419, 1].

Case III/9: $p = 23$

In this case $M(T) = 425$. For $m < M(T)$ we have the following H -coverings.

 $17 \leq m \leq 25$:

[2, 1], [3, 1], [5, 2], [7, 1], [11, 6], [13, 1], [17, 1], [19, 1].

 $26 \leq m \leq 35$:

[2, 1], [3, 1], [5, 3], [7, 5], [11, 2], [13, 1], [17, 3], [19, 6], [29, 1], [31, 1].

 $36 \leq m \leq 424$:

[2, 1], [3, 1], [5, 3], [7, 5], [11, 2], [13, 6], [17, 3], [19, 14], [29, 1],
 [31, 5], [37, 29], [41, 1], [43, 1], [47, 3], [53, 3], [59, 1], [61, 1], [71, 1],
 [73, 1], [79, 1], [83, 3], [89, 3], [97, 29], [101, 1], [103, 1], [107, 13],
 [109, 7], [113, 1], [131, 1], [149, 1], [157, 19], [163, 1], [167, 3],
 [173, 1], [179, 7], [181, 1], [191, 1], [199, 11], [211, 1], [229, 1], [233, 1],
 [241, 5], [251, 3], [257, 3], [269, 7], [271, 1], [277, 23], [281, 1], [283, 7],
 [293, 1], [307, 23], [311, 1], [313, 1], [317, 7], [331, 5], [337, 5], [349, 1],
 [367, 5], [373, 13], [379, 1], [383, 1], [389, 3], [397, 19], [401, 1], [409, 1].

Case III/10: $p = 29$

In this case $M(T) = 425$. For $m < M(T)$ we have the following H -coverings.

 $17 \leq m \leq 424$:

[2, 1], [3, 2], [5, 6], [7, 3], [11, 1], [13, 4], [17, 1], [19, 3], [23, 5],
 [37, 3], [41, 1], [43, 15], [47, 1], [53, 1], [61, 3], [67, 3], [71, 1],
 [79, 9], [83, 1], [89, 1], [101, 1], [109, 3], [113, 5], [127, 3], [131, 1],
 [137, 1], [139, 9], [157, 5], [163, 9], [167, 1], [173, 17], [179, 1], [181, 3],
 [193, 15], [197, 1], [199, 3], [227, 1], [239, 1], [241, 3], [251, 1], [263, 37],
 [269, 1], [271, 3], [277, 3], [281, 1], [283, 9], [293, 1], [307, 3], [311, 1],
 [313, 15], [317, 5], [331, 3], [337, 3], [347, 1], [349, 3], [353, 1], [373, 5],
 [379, 3], [389, 1], [397, 3], [401, 17], [409, 5].

Case III/11: $p = 31$

In this case $M(T) = 425$. For $m < M(T)$ we have the following H -coverings.

 $17 \leq m \leq 424$:

[2, 1], [3, 2], [5, 6], [7, 3], [11, 1], [13, 4], [17, 1], [19, 3], [23, 5],
 [37, 3], [41, 1], [43, 15], [47, 1], [53, 1], [61, 3], [67, 3], [71, 1],
 [79, 9], [83, 1], [89, 1], [101, 1], [109, 3], [113, 5], [127, 3], [131, 1],
 [137, 1], [139, 9], [157, 5], [163, 9], [167, 1], [173, 17], [179, 1], [181, 3],
 [193, 15], [197, 1], [199, 3], [227, 1], [239, 1], [241, 3], [251, 1], [263, 37],
 [269, 1], [271, 3], [277, 3], [281, 1], [283, 9], [293, 1], [307, 3], [311, 1],
 [313, 15], [317, 5], [331, 3], [337, 3], [347, 1], [349, 3], [353, 1], [373, 5],
 [379, 3], [389, 1], [397, 3], [401, 17], [409, 5].

Case III/12: $p = 37$

In this case $M(T) = 425$. For $m < M(T)$ we have the following H -coverings.

 $17 \leq m \leq 424$:

[2, 1], [3, 2], [5, 6], [7, 3], [11, 1], [13, 4], [17, 1], [19, 3], [23, 5],
 [31, 9], [41, 1], [43, 15], [47, 1], [53, 1], [61, 3], [67, 3], [71, 1],
 [79, 9], [83, 1], [89, 1], [107, 11], [109, 3], [113, 1], [127, 3], [131, 1],
 [137, 1], [139, 9], [157, 5], [163, 9], [167, 1], [173, 17], [179, 1], [181, 3],
 [193, 15], [197, 1], [199, 3], [227, 1], [239, 1], [241, 3], [251, 1], [263, 37],
 [269, 1], [271, 3], [277, 3], [281, 1], [283, 9], [293, 1], [307, 3], [311, 1],
 [313, 15], [317, 5], [331, 3], [347, 1], [349, 3], [353, 1], [373, 5], [379, 3],
 [389, 1], [397, 3], [401, 17], [409, 5].

Case III/13: $p = 41$

In this case $M(T) = 425$. For $m < M(T)$ we have the following H -coverings.

 $17 \leq m \leq 424$:

[2, 1], [3, 2], [5, 6], [7, 3], [11, 1], [13, 4], [17, 1], [19, 3], [23, 5],
 [29, 29], [31, 11], [37, 3], [47, 1], [53, 1], [61, 3], [67, 3], [71, 1],
 [79, 9], [83, 1], [89, 1], [101, 1], [107, 17], [109, 3], [113, 5], [127, 3],
 [131, 1], [137, 1], [139, 9], [157, 5], [163, 9], [167, 1], [173, 17], [179, 1],
 [181, 3], [193, 15], [197, 1], [199, 3], [239, 1], [241, 3], [251, 1], [263, 37],
 [269, 1], [271, 3], [277, 3], [281, 1], [283, 9], [293, 1], [307, 3], [311, 1],
 [313, 15], [317, 5], [331, 3], [337, 3], [353, 1], [367, 3], [373, 5], [379, 3],
 [389, 1], [397, 3], [409, 9].

Case III/14: $p = 43$

In this case $M(T) = 425$. For $m < M(T)$ we have the following H -coverings.

 $17 \leq m \leq 424$:

[2, 1], [3, 2], [5, 6], [7, 3], [11, 1], [13, 4], [17, 1], [19, 3], [23, 5],
 [29, 29], [31, 11], [37, 3], [47, 1], [53, 1], [61, 3], [67, 3], [71, 1],
 [79, 9], [83, 1], [89, 1], [101, 1], [107, 17], [109, 3], [113, 5], [127, 3],
 [131, 1], [137, 1], [139, 9], [157, 5], [163, 9], [167, 1], [173, 17], [179, 1],
 [181, 3], [193, 15], [197, 1], [199, 3], [239, 1], [241, 3], [251, 1], [263, 37],
 [269, 1], [271, 3], [277, 3], [281, 1], [283, 9], [293, 1], [307, 3], [311, 1],
 [313, 15], [317, 5], [331, 3], [337, 3], [353, 1], [367, 3], [373, 5], [379, 3],
 [389, 1], [397, 3], [409, 9].

Case III/15: $p = 47$

In this case $M(T) = 425$. For $m < M(T)$ we have the following H -coverings.

 $17 \leq m \leq 424$:

[2, 1], [3, 2], [5, 6], [7, 3], [11, 1], [13, 4], [17, 1], [19, 3], [23, 5],
 [31, 27], [37, 3], [41, 1], [43, 5], [53, 1], [61, 3], [67, 3], [71, 1],
 [79, 9], [83, 1], [89, 1], [101, 1], [107, 41], [109, 3], [113, 5], [127, 3],
 [131, 1], [137, 1], [139, 3], [157, 5], [163, 9], [167, 1], [173, 17], [179, 1],
 [181, 3], [193, 15], [197, 1], [199, 3], [227, 1], [239, 1], [251, 1], [263, 37],
 [269, 1], [271, 3], [277, 3], [281, 1], [283, 9], [293, 1], [307, 3], [311, 1],
 [313, 15], [317, 5], [331, 3], [337, 3], [347, 1], [349, 3], [353, 1], [373, 5],
 [379, 3], [389, 1], [397, 3], [401, 17], [409, 5], [421, 3].

Case III/16: $50 < p < 424$

In this case $M(T) = 425$. For $m < M(T)$ we have the following H -coverings.

 $17 \leq m \leq 49$:

[2, 1], [3, 2], [5, 1], [7, 3], [11, 1], [13, 4], [17, 1], [19, 3], [23, 5], [29, 11], [31, 11], [37, 11].

For $50 \leq m \leq 424$ consider the pairs

[2, 1], [3, 2], [5, 1], [7, 7], [11, 1], [13, 4], [17, 5], [23, 1], [29, 10],
 [31, 18], [37, 3], [41, 17], [43, 9], [47, 1], [53, 1], [59, 1], [61, 3],
 [71, 1], [83, 5], [89, 5], [101, 1], [109, 11], [113, 5], [127, 3], [131, 1],
 [139, 9], [149, 1], [167, 5], [173, 1], [179, 1], [181, 9], [197, 1], [199, 3],
 [211, 9], [227, 1], [229, 21], [233, 1], [239, 1], [241, 3], [257, 1], [263, 5],
 [269, 1], [271, 3], [277, 15], [281, 1], [283, 5], [311, 1], [313, 11], [317, 1],
 [331, 3], [337, 3], [347, 1], [349, 3], [353, 1], [359, 1], [367, 3], [379, 3],
 [383, 5], [389, 29], [397, 3], [401, 1], [409, 3].

The primes 19, 67, 73 are not involved in the above pairs. Hence the above set of pairs is an H -covering for $[1, m]$ if $p \in \{67, 73\}$. Now let $p \notin \{67, 73\}$. Removing the pair $[p, f(p)]$ from the above set of pairs the following numbers become uncovered up to 424 (we indicate p and the numbers which get uncovered only).

[53, [54, 372]], [59, [60, 178, 414]], [61, [64]], [71, [72, 214]], [83, [88]], [89, [94]],
 [101, [102]], [109, [120]], [113, [118]], [127, [130, 384]], [131, [132]], [139, [148]],
 [149, [150]], [167, [172]], [173, [174]], [179, [180]], [181, [190]], [197, [198]],
 [199, [202]], [211, [220]], [227, [228]], [229, [250]], [233, [234]], [239, [240]],
 [241, [244]], [257, [258]], [263, [268]], [269, [270]], [271, [274]], [277, [292]],
 [281, [282]], [283, [288]], [311, [312]], [313, [324]], [317, [318]], [331, [334]],
 [337, [340]], [347, [348]], [349, [352]], [353, [354]], [359, [360]], [367, [370]],
 [379, [382]], [383, [388]], [389, [418]], [397, [400]].

Hence for any prime p with $50 < p < 425$, leaving out the pair $[p, f(p)]$ from above and putting the primes 19, 67, 73 on the uncovered numbers, we get a covering. For example, if $p = 59$ then replacing the pair [59, 1] by the pairs [19, 3], [67, 44], [73, 49], we get an H -covering.

Case III/17: $424 \leq p$

In this case $M(T) = 425$. For $m < M(T)$ we have the following H -coverings.

 $17 \leq m \leq 424$:

[2, 1], [3, 2], [5, 1], [7, 3], [11, 1], [13, 4], [17, 1], [19, 3], [23, 5],
 [37, 3], [41, 1], [43, 15], [47, 1], [53, 1], [61, 3], [67, 3], [71, 1],
 [79, 9], [83, 1], [89, 1], [101, 1], [109, 3], [113, 5], [127, 3], [131, 1],
 [137, 1], [139, 9], [157, 5], [163, 9], [167, 1], [173, 17], [179, 1], [181, 3],
 [193, 15], [197, 1], [199, 3], [227, 1], [239, 1], [241, 3], [251, 1], [263, 37],
 [269, 1], [271, 3], [277, 3], [281, 1], [283, 9], [293, 1], [307, 3], [311, 1],
 [313, 15], [317, 5], [331, 3], [337, 3], [347, 1], [349, 3], [353, 1], [373, 5],
 [379, 3], [389, 1], [397, 3], [401, 17], [409, 5].

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