Unique reconstruction of bounded sets in discrete tomography

Lajos Hajdu [1,2]

Institute of Mathematics
University of Debrecen
and the Number Theory Research Group
of the Hungarian Academy of Sciences
Debrecen, Hungary

Abstract

We consider the problem of recognizing arbitrary finite subsets of $\mathbb{Z}^2$ by X-rays corresponding to a small set of directions S. We show that if we fix any rectangle A in $\mathbb{Z}^2$ then there exists a so called valid set S of four directions (at least when A is not too "small") depending only on the size of A such that any two subsets of A can be distinguished, using these directions only. By our approach this result can easily be extended to any lattice, in any dimension.

Keywords: Discrete tomography, unique reconstruction, generating functions, polynomials in two variables.

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[2] Email: `hajdul@math.klte.hu`

1 Introduction

An important problem in discrete tomography is the question of uniqueness, i.e. to decide when the X-rays corresponding to a set S of directions allow determining an unknown set T of points completely. In the case when T is assumed to be a finite and (in the digital sense) convex subset of $\mathbb{Z}^2$, it is known by the work of Gardner and Gritzmann (see [2] and [3]) that there are sets of directions S with $|S| = 4$ which guarantee the determination of T. Moreover, these sets S are totally independent of T, and they are completely described. For example, one can take the set of directions $S = \{(2,1), (3,2), (1,1), (2,3)\}$ for this purpose. We also mention that Barucci, Del Lungo, Nivat and Pinzani [1] obtained several related (though mostly negative) results in the more general case when T is assumed to be a so-called hv-convex polyomino.

The approach of the above cited papers is geometric, and does not seem to be easily extendable to the general case, i.e. when not necessarily convex type sets are considered. Recently, Hajdu and Tijdeman worked out a purely algebraic approach for discrete tomographical problems, based on generating functions and divisibility properties of polynomials (see [4]). Using their approach they provided a new type algorithm for solving discrete tomographical problems [5], and also showed that their results can be extended to the case of emission tomography with absorption [6]. In the present paper we demonstrate that this approach can also be useful for investigating the question of unique reconstruction.

We consider the problem of recognizing an arbitrary set T of points in $\mathbb{Z}^2$. It is well-known that in full generality this problem cannot be handled, as for any fixed set S of directions it is easy to construct a finite set T which cannot be uniquely determined by the X-rays in the directions of S only (see e.g. Theorem 4.3.1 of [3]). On the other hand, for any given set H it is easy to find some degenerate direction(s) which allow(s) unique reconstruction of any subset of H. (E.g. one can take a direction with so large or so small slope (with respect to H) such that each line with this slope has at most one point with H in common.) Excluding these cases, we prove that if A is any rectangle in $\mathbb{Z}^2$ then there exists a so called valid set S of four directions (at least when A is not too "small") depending only on the size of A such that any two subsets of A can be distinguished, using these directions only. In fact we prove some more general and more precise statements, and we obtain this result as a simple consequence. We also mention that by our approach this result can easily be extended to any lattice, in any dimension.

2 Notation and results

We adopt some notation from [3] and [4]. We will be interested in subsets of $\mathbb{Z}^2$, or in other words, in functions mapping $\mathbb{Z}^2$ to $\{0,1\}$. However, for theoretical reasons it is useful to consider functions $f:\ \mathbb{Z}^2 \to \mathbb{Z}$ instead.

Let $a,b \in \mathbb{Z}$ with $\gcd(a,b)=1$ and $a \geq 0$, with the further assumption that $b=1$ if $a=0$. We call (a,b) a direction. By lines with direction (a,b) we mean lines of the form $ay=bx+t$ ($t \in \mathbb{Z}$) in the $[x,y]$ plane. Let m and n be positive integers, and let $A=\{(i,j)\in\mathbb{Z}^2\ :\ 0\leq i<m, 0\leq j<n\}$. If $f:\ A\to\mathbb{Z}$ is a function, then we write $|f|=\max\limits_{(i,j)\in A}\{|f(i,j)|\}$. Moreover, the line sum of f along the line $ay=bx+t$ is defined as $\sum\limits_{aj=bi+t,\ (i,j)\in A} f(i,j)$. If T is a subset of A and f is the characteristic function of T in A, then the above line sums (as t varies) are called the X-rays of T belonging to the direction (a,b). Let S be a finite set of directions. The functions $f,g:\ A\to\mathbb{Z}$ are called tomographically equivalent with respect to S if f and g have the same line sums along all lines corresponding to the directions in S. A set $S=\{(a_k,b_k)\}_{k=1}^d$ of directions is called valid for A, if $\sum\limits_{k=1}^{d} a_k < m$ and $\sum\limits_{k=1}^{d} |b_k| < n$. This concept is important because if S is not valid for A, then there are no (distinct) tomographically equivalent functions $A\to\mathbb{Z}$ (with respect to S) at all, and the problem of uniqueness is trivial. (This statement easily follows from our Lemma 3.1.)

The following theorem shows that in case of any size (up to small exceptions), there is a valid set of four directions such that there are no "small" non-trivial functions $f:\ A\to\mathbb{Z}$ having only zero line sums. For simplicity, from this point on we will assume that $m\geq n$ holds; clearly, this can be done without loss of generality.

Theorem 2.1 *Let m and n be integers with $m\geq n\geq 5$ and $m\neq 6$, and let $A=\{(i,j)\in\mathbb{Z}^2\ :\ 0\leq i<m, 0\leq j<n\}$. Put $d=5$ if $(m,n)\in\{(8,6),(8,8),(10,6),(12,6)\}$, and $d=4$ otherwise. Then there exists a valid set S for A consisting of d directions depending only on n if $n\geq 15$, and on m and n otherwise, such that if the function $f:\ A\to\mathbb{Z}$ has zero line sums along the lines corresponding to the directions in S and $|f|\leq 1$, then f is identically zero.*

Using Theorem 2.1, we can prove the following statement.

Theorem 2.2 *Let m, n, A and d be as in Theorem 2.1. Then there exists a valid set S for A consisting of d directions such that if the functions $f,g:\ A\to\{0,1\}$ are tomographically equivalent, then $f=g$. Further, the directions* *in S depend only on n if $n \geq 15$, and on m and n otherwise.*

Theorem 2.2 immediately implies the following

Corollary 2.3 *Let m, n, A and d be as in Theorem 2.1. Then there exists a valid set S for A consisting of d directions such that the X-rays in the directions of S determine any subset of A completely. Moreover, the directions in S depend only on n if $n \geq 15$, and on m and n otherwise.*

Now we show that three (or fewer) directions are never sufficient to distinguish all subsets of a given rectangle in $\mathbb{Z}^2$. In fact we prove a more precise statement, including the "small" values of m and n, as well.

Theorem 2.4 *Let m and n be positive integers with $m \geq n$ and $A = \{(i,j) \in \mathbb{Z}^2 : 0 \leq i < m, 0 \leq j < n\}$. Further, let S be a valid set for A consisting of d directions with*

$$d < \begin{cases} \infty, & \textit{if } n \leq 4 \textit{ or } m = 6, \\ 5, & \textit{if } (m,n) \in \{(8,6),(8,8),(10,6),(12,6)\}, \\ 4, & \textit{otherwise.} \end{cases}$$

Then there is a function $f : A \to \mathbb{Z}$ having zero line sums corresponding to the directions in S, such that $|f| = 1$.

From Theorem 2.4 we derive the following result.

Theorem 2.5 *Let A, m, n and S be as in Theorem 2.4. Then there exist distinct functions $f_1, f_2 : A \to \{0,1\}$ which are tomographically equivalent with respect to S.*

As a simple consequence of Theorem 2.5 we get the following

Corollary 2.6 *Let A, m, n and S be as in Theorem 2.4. Then there exist two distinct subsets of A having the same X-rays in the directions of S.*

3 Proofs

We need two lemmas to prove our results. To formulate them, we introduce some further notation borrowed from [4].

If (a, b) is a direction, then put

$$f_{(a,b)}(x, y) = \begin{cases} x^a y^b - 1, & \text{if} \ \ a > 0,\ b > 0, \\ x^a - y^{-b}, & \text{if} \ \ a > 0,\ b < 0, \\ x - 1, & \text{if} \ \ a = 1,\ b = 0, \\ y - 1, & \text{if} \ \ a = 0,\ b = 1. \end{cases}$$

Let m and n be positive integers and $A = \{(i, j) \in \mathbb{Z}^2 \ : \ 0 \leq i < m, 0 \leq j < n\}$. If $S = \{(a_k, b_k)\}_{k=1}^d$ is a set of directions, then write $F_S(x, y) = \prod_{k=1}^{d} f_{(a_k,b_k)}(x, y)$. Finally, for any function $f : \ A \to \mathbb{Z}$ let G_f be the generating function of f, i.e.

$$G_f(x, y) = \sum_{(i,j)\in A} f(i, j) x^i y^j.$$

The following result is due to Hajdu and Tijdeman [4].

Lemma 3.1 *Let m, n be positive integers and $A = \{(i, j) \in \mathbb{Z}^2 \ : \ 0 \leq i < m, 0 \leq j < n\}$, and let S be an arbitrary set of directions. Assume that $f : \ A \to \mathbb{Z}$ has zero line sums along the lines corresponding to the directions in S. Then $F_S(x, y)$ divides $G_f(x, y)$ over $\mathbb{Z}$.*

Proof. The statement simply follows from the proof of Theorem 1 of [4]. □

The next statement gives some insight into the structure of the set of coefficients of $F_S(x, y)$.

Lemma 3.2 *Let $S = \{(a_k, b_k)\}_{k=1}^d$ be any set of directions. Suppose that $F_S(x, y)$ has a coefficient outside the set $\{-1, 0, 1\}$. Then there exist two disjoint subsets S_1 and S_2 of S such that $|S_1| \equiv |S_2| \pmod 2$ and*

$$\sum_{(a,b)\in S_1} (a, b) = \sum_{(a,b)\in S_2} (a, b).$$

Proof. Observe that in $F_S(x, y)$ the terms of the form $x^a - y^{-b}$ corresponding to some $b < 0$ can be replaced by $y^{-b}(x^a y^b - 1)$. Making this change, our argument is easier to follow. Suppose that for the coefficient c_{ij} of $x^i y^j$ in $F_S(x, y)$ for some i and j we have $|c_{ij}| \geq 2$. This means that (i, j) can be obtained as the sum of elements of S at least in two ways. That is, for some distinct subsets I_1 and I_2 of S we have

$$(i,j) = \sum_{(a,b)\in I_1} (a,b) = \sum_{(a,b)\in I_2} (a,b).$$

Obviously, if the parities of $|I_1|$ and $|I_2|$ are different, then the contribution of these decompositions to c_{ij} is altogether zero. Hence, in addition we may assume that $|I_1| \equiv |I_2| \pmod 2$. Putting $S_1 = I_1 \setminus (I_1 \cap I_2)$ and $S_2 = I_2 \setminus (I_1 \cap I_2)$, the statement follows. □

Now we turn to the proofs of the theorems and their corollaries.

Proof of Theorem 2.1 We construct a valid set S of four directions in each case.

First assume that $n \geq 15$. Put $S = \{(1,2),(1,3),(1,k),(1,5-k)\}$, where $k = [(n-1)/2]$. Observe that S is valid for A. Further, we have

$$F_S(x,y) = (xy^2-1)(xy^3-1)(xy^k-1)(x-y^{k-5}) =$$

$$= x^4y^{k+5} - x^3(y^{2k}+y^{k+3}+y^{k+2}+y^5) + x^2(y^{2k-2}+y^{2k-3}+2y^k+y^3+y^2)$$
$$-x(y^{2k-5}+y^{k-2}+y^{k-3}+1)+y^{k-5}.$$

Let now $f:\ A \to \mathbb{Z}$ be any function having zero line sums along the lines corresponding to the directions in S. Then by Lemma 3.1 we get that $F_S(x,y)$ divides $G_f(x,y)$ over $\mathbb{Z}$. Observe that $n-2 \leq \deg_y(F_S(x,y)) \leq n-1$. Hence there exist polynomials $P_0, P_1 \in \mathbb{Z}[x]$ such that

$$G_f(x,y) = (yP_1(x)+P_0(x))F_S(x,y). \tag{1}$$

As we intend to show that either f is identically zero or $|f| \geq 2$, without loss of generality we may assume that at least one of $P_0(0)$ and $P_1(0)$ is non-zero. Then a simple calculation yields that at least one of the absolute values of the coefficients of x^2y^k and x^2y^{k+1} on the right hand side of (1) is ≥ 2. (Note that to verify this assertion it is sufficient to consider the coefficients of P_0 and P_1 corresponding to $1, x, x^2$.) Hence the theorem is proved in this case.

Assume next that $n \leq 14 < m$. Observe that by switching the roles of m and n in the first part of the proof, we get the same conclusion. Hence the theorem follows also in this case. (Note that now the directions in S depend also on m.)

In the remaining cases with $m \leq 14$ it is easy to find appropriate valid sets S yielding unique reconstruction. We summarize the results of our computations in Table 1. In the table we indicate all the pairs (m,n) with $14 \geq m \geq n \geq 5$ and $m \neq 6$, and the corresponding valid sets S. Note

Table 1
Appropriate valid sets for $14 \geq m \geq n \geq 5$ with $m \neq 6$

(m,n)	a valid set S
$(5,5)$	$(1,-1),(1,0),(1,1),(1,2)$
$(7,5),(7,6)$	$(1,-1),(1,1),(2,-1),(2,1)$
$(7,7),(8,7)$	$(0,1),(1,-2),(2,-1),(3,-2)$
$(8,5)$	$(0,1),(1,1),(2,-1),(3,1)$
$(8,6)$	$(0,1),(1,-1),(1,1),(2,1),(3,-1)$
$(8,8)$	$(1,-1),(1,1),(1,3),(2,-1),(2,1)$
$(9,n)$	$(1,1),(2,-1),(2,1),(3,-1)$
$(10,5)$	$(1,-1),(1,1),(2,1),(4,1)$
$(10,6)$	$(0,1),(1,-1),(1,1),(2,1),(5,-1)$
$(10,7)$	$(0,1),(1,-2),(3,2),(4,1)$
$(10,8)$	$(1,-2),(1,2),(2,-1),(4,-1)$
$(10,9)$	$(0,1),(2,-3),(2,3),(4,1)$
$(10,10)$	$(1,-3),(1,3),(2,-1),(4,-1)$
$(11,n)$	$(1,-1),(2,1),(3,-1),(4,1)$
$(12,5)$	$(1,-1),(1,1),(3,1),(5,1)$
$(12,6)$	$(0,1),(1,-1),(1,1),(2,1),(7,-1)$
$(12,7)$	$(1,-2),(1,1),(3,2),(5,1)$
$(12,8)$	$(1,-2),(1,2),(3,-1),(5,-1)$
$(12,9)$	$(1,1),(2,-3),(2,3),(5,1)$
$(12,10)$	$(1,-3),(2,-1),(2,3),(5,-1)$
$(12,11)$	$(1,-4),(1,1),(3,4),(5,1)$
$(12,12)$	$(1,-4),(1,4),(3,-1),(5,-1)$
$(13,n)$	$(1,-1),(2,1),(3,1),(6,1)$
$(14,5)$	$(1,-1),(1,1),(4,1),(6,1)$
$(14,6)$	$(2,-1),(2,1),(4,-1),(4,1)$
$(14,7)$	$(1,-2),(2,1),(3,2),(6,1)$
$(14,8)$	$(1,-2),(2,-1),(3,2),(6,-1)$
$(14,9)$	$(0,1),(2,-3),(4,3),(6,1)$
$(14,10)$	$(1,-3),(2,3),(3,-1),(6,-1)$
$(14,11)$	$(1,-4),(2,1),(3,4),(6,1)$
$(14,12)$	$(1,-4),(1,4),(4,-1),(6,-1)$
$(14,13)$	$(1,-5),(2,1),(3,5),(6,1)$
$(14,14)$	$(1,-5),(1,5),(4,-1),(6,-1)$

that in case of $(m,n) \in \{(8,6),(8,8),(10,6),(12,6)\}$ we have $|S| = 5$, while in the other cases $|S| = 4$ holds.

We prove only in one instance, namely when $(m,n)=(7,6)$ that with the corresponding valid set S the statement of the theorem holds true. The proofs of the other cases are similar.

Observe that with $(m,n)=(7,6)$ and $S=\{(1,-1),(1,1),(2,-1),(2,1)\}$, we have $\deg_x(F_S(x,y))=6$ and $\deg_y(F_S(x,y))=4$. Hence if $f:\ A\to\mathbb{Z}$ has zero line sums corresponding to the directions in S, then

$$G_f(x,y)=(uy+v)F_S(x,y)=(uy+v)(x-y)(xy-1)(x^2-y)(x^2y-1)$$

holds with some integers u,v. However, a simple calculation shows that the coefficients of the terms x^3y^2 and x^3y^3 are $2v$ and $2u$, respectively. Hence either $G_f(x,y)$ is identically zero (whence f is identically zero as well), or $|f|\geq 2$, and our claim follows. $\square$

Proof of Theorem 2.2 Let $f,g:\ A\to\{0,1\}$ be two functions which are tomographically equivalent, and let S be the set of directions constructed in the proof of Theorem 2.1 for the actual choice of m and n. Put $h=f-g$ and observe that $h:\ A\to\mathbb{Z}$ has zero line sums along the lines corresponding to the directions in S, moreover, $|h|\leq 1$. Hence by Theorem 2.1 we get that h is identically zero, whence $f=g$. $\square$

Proof of Corollary 2.3 The statement is a simple reformulation of Theorem 2.2. $\square$

Proof of Theorem 2.4 Let $S=\{(a_k,b_k)\}_{k=1}^d$ be a set of valid directions with $d\leq 3$. Observe that in case of $d=1$ or 2, $F_S(x,y)$ is not identically zero and has coefficients only from $\{-1,0,1\}$. When $d=3$, by Lemma 3.2 we get again that $|F_S(x,y)|=1$. Define $f:\ A\to\mathbb{Z}$ in the following way. Let $f(i,j)$ be the coefficient of x^iy^j in $F_S(x,y)$. Then f is not identically zero, and by Lemma 3.1 has zero line sums corresponding to the directions in S. Hence the theorem follows when $d\leq 3$.

In the case when $d\geq 4$, our theorem is restricted to some "small" pairs (m,n). Suppose that (m,n) is some pair under consideration, and assume that $S=\{(a_k,b_k)\}_{k=1}^d$ is a fixed valid set of directions for A.

Assume first that $n\leq 4$. In this case by the validity of S the only possibility is given by $d=4$ and $|b_1|+|b_2|+|b_3|+|b_4|=3$. However, then Lemma 3.2 shows that $|F_S(x,y)|=1$. Defining f as above, the statement follows in this case.

We continue with $(m,n)=(6,5)$. As S is valid for A, using Lemma 3.2 we get that $d=4$ and both the a_i and the $|b_i|$ are given by either $1,1,1,1$ or $0,1,1,2$ (in some order). Moreover, if $(1,0)\in S$, then either $|F_S(x,y)|=1$ or

$|(x+1)F_S(x,y)| = 1$ holds, and defining f as before, our statement follows. So we may assume that $(1,0) \notin S$. Hence we get that

$$S = \{(0,1), (1,1), (1,-1), (2, \pm 1)\}.$$

However, for both choices of the $\pm$ sign we obtain that $|(x-1)F_S(x,y)| = 1$, and the theorem follows in this case.

Suppose next that $(m,n) = (6,6)$. By a similar argument as above we may assume that $d = 4$ and $(1,0) \notin S$, furthermore, by symmetry also that $(0,1) \notin S$. However, by Lemma 3.2 this leaves no space for any valid set S with $|F_S(x,y)| > 1$, and the statement follows.

In all the remaining cases the statement is restricted to $d = 4$. Let $(m,n) \in \{(8,6), (10,6), (12,6)\}$. We may assume that $|F_S(x,y)| > 1$, otherwise we are done. Then by Lemma 3.2 we get that both $a_1 + a_2 + a_3 + a_4$ and $|b_1| + |b_2| + |b_3| + |b_4|$ are even, whence by the validity of S

$$a_1 + a_2 + a_3 + a_4 \leq m - 2 \quad \text{and} \quad |b_1| + |b_2| + |b_3| + |b_4| \leq 4.$$

Suppose that say (a_1, b_1) equals to one of $(1,0)$, $(0,1)$, $(1,1)$, $(1,-1)$. Let

$$f^*(x,y) = \begin{cases} x+1, & \text{if } (a_1,b_1) = (1,0), \\ y+1, & \text{if } (a_1,b_1) = (0,1), \\ xy+1, & \text{if } (a_1,b_1) = (1,1), \\ x+y, & \text{if } (a_1,b_1) = (1,-1), \end{cases}$$

and $F^*(x,y) = f^*(x,y)F_S(x,y)$. By $|F_S(x,y)| > 1$, using a similar argument as in the proof of Lemma 3.2, one can easily see that $|F^*(x,y)| = 1$. As $\deg_x(F^*(x,y)) < m$ and $\deg_y(F^*(x,y)) < n$, this implies our statement. Hence we may assume that none of the above four directions belongs to S. However, this immediately yields a contradiction, unless $(m,n) = (12,6)$. Even in this case the only remaining possibility is given by

$$S = \{(2,-1), (2,1), (3,-1), (3,1)\},$$

when we have

$$F_S(x,y) = (x^2 - y)(x^2 y - 1)(x^3 - y)(x^3 y - 1).$$

Now one can readily verify that $|(x-1)F_S(x,y)| = 1$, which by defining f as above, implies the statement also in this case.

Finally, assume that $(m, n) = (8, 8)$. Similarly as above, if any of the directions $(1, 0)$, $(0, 1)$, $(1, 1)$, $(1, -1)$ belongs to S then we are done. In the opposite case, using that both $a_1 + a_2 + a_3 + a_4$ and $|b_1| + |b_2| + |b_3| + |b_4|$ are even (by Lemma 3.2) and are less than 8 (by the validity of S), we get that the only possibilities are given by

$$S = \{(1, -2), (1, 2), (2, -1), (2, 1)\} \quad \text{and} \quad \{(1, -1), (1, 1), (1, \pm 3), (3, \pm 1)\}.$$

However, in each case we have $|F_S(x, y)| = 1$, and defining f as before, the proof of the theorem is complete. $\square$

Proof of Theorem 2.5 Let f be the function given by Theorem 2.4, and define the functions f_1 and f_2 on A by

$$f_1(i, j) = \begin{cases} 1, & \text{if } f(i, j) = 1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_2(i, j) = \begin{cases} 1, & \text{if } f(i, j) = -1, \\ 0, & \text{otherwise.} \end{cases}$$

Then as $f_1 - f_2 = f$, the functions $f_1, f_2 : A \to \{0, 1\}$ have the same line sums corresponding to the directions in S, and the theorem follows. $\square$

Proof of Corollary 2.6 The statement is a simple reformulation of Theorem 2.5. $\square$

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