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On the linearizability of 3-webs

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Abstract

In this article, which is an announcement of a paper submitted to *Publicationes Mathematicae, Debrecen*, we give a modern and intrinsic presentation of the linearizability theory of 3-webs on the real or complex plane, using the Nagy's view-point and the integrability theory of over-determined partial differential systems. These techniques allow us to give explicitly the linearizability conditions for 3-webs in terms of the curvature tensor of the Chern connection and its covariant derivatives. We give explicit examples of non-linearizable webs. We improve also a result of Bol on the Gronwall conjecture.

Key words: Webs, Affine structures, Gronwall conjecture.

Introduction

As it is well known, if X and Y are two independent vector fields on the plane, by the inverse functions theorem, there exist flow box coordinates (x, y) such that $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$. In particular, up to a local diffeomorphism, the two families of integral curves are straight lines.

It is an old problem, to find conditions under which this situation holds for three independent vector fields on the plane. This question is fundamental in the *theory of nomograms*, a subject related to the graphical numerical solutions of non-linear equations, and to the optimal representation of functional

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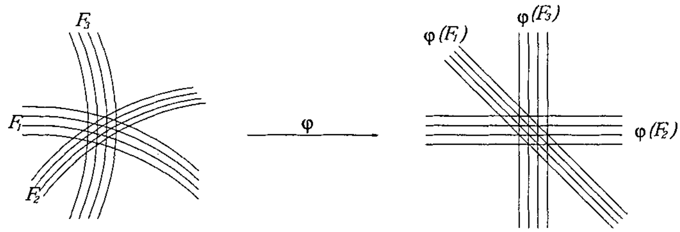
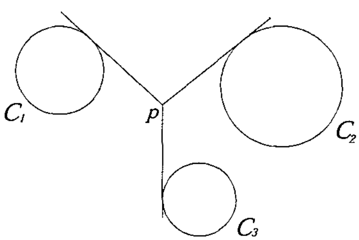
relations by level-lines and level-surfaces. In the language of web's theory, that problem translates to the one:

Linearizability problem: *Characterize the 3-webs on real or complex 2-dimensional manifolds which are equivalent, up to a local diffeomorphism, to linear webs, i.e. webs such that is corresponding foliations are straight lines in a convenient coordinate system.*

Much stronger than linearizability is the notion of parallelizability. A 3-web is called *parallelizable* when it is equivalent to three families of *parallel* lines.



The Graf-Sauer's Theorem (see for example [2], page 24) gives an elegant characterization of parallelizable webs: a linear web is parallelizable if and only if its leaves are tangent lines to an algebraic curve of degree 3. It follows in particular that there exist linear non-parallelizable 3-webs: for example the web whose leaves are tangent to three circles C_1, C_2, C_3 with mutually non-intersecting interiors (of course, only one tangent to each circle is taken from any point):



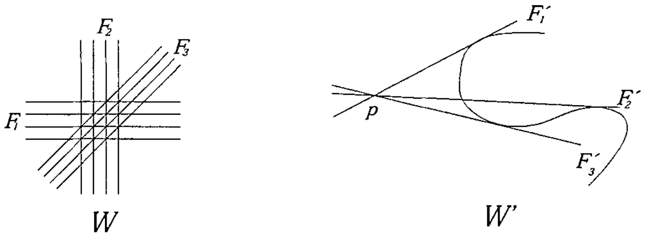
This linear web is not parallelizable because the union $C_1 \cup C_2 \cup C_3$ is not a curve of degree 3.

Parallelizable webs are characterized in terms of the curvature of its Chern connection: a web is parallelizable if and only if the curvature vanishes. However, the problem of characterizing linear webs is still open.

Another old problem related to linearizable webs is the following conjecture:

Gronwall Conjecture (1912) [9]: *If a non-parallelizable 3-web $\mathcal{W}$ in the (real or complex) plane is linearizable, then, up to a projective transformation,* *there is a unique diffeomorphism which maps $\mathcal{W}$ into a linear 3-web.*

Note that the uniqueness property does not hold for hexagonal 3-webs. Indeed, by the Graf-Sauer Theorem, a linear web $\mathcal{W}$ given by three families of parallel lines is equivalent to the web $\mathcal{W}'$ defined by the tangent lines of an algebraic curve of degree 3. But $\mathcal{W}$ and $\mathcal{W}'$ are not projectively equivalent since for any foliation of $\mathcal{W}$ all leaves pass through the same projective point (infinity) and that is not the case for $\mathcal{W}'$:



This subject was studied by the Russian school, in particular by Akivis and Goldberg ([1], [6]). The most significant results are due to Bol ([3], [4]). In [3] he suggests how to find a criterion of linearizability, although he is unable to carry out the computations through; this really requires the use of computers. On the other hand he shows that the number of the classes of diffeomorphisms, modulo projective transformations, which linearize a linearizable but non parallelizable 3-web is finite and less that 17. Note that Bol's proof consists in associating to a real 3-web two *complex* vector fields which play an essential role; so this proof cannot be interpreted in the complex case. In [4], Bol defines several classes of non projectively equivalent linear 3-webs and shows that the webs in the same classes are diffeomorphically equivalent. He expresses the opinion that one can prove that the Gronwall conjecture is false with the help of these examples. Until now, however, the Gronwall conjecture remains open.

In this paper, we give a modern presentation of the linearizability theory and we improve the result of the first Bol's paper. Our tools are the infinitesimal formalism of the web theory given by Nagy [10], and the formal integrability theory of the over-determined systems in the Spencer version (see, for example, [5], [7]). These allow us to present all the results in an intrinsic and comprehensive way in terms of the curvature of the Chern connection. In our proof the web is real or complex, so the results hold for real and complex webs.

In order to formulate our results, we introduce the following definition. (To simplify the notations, the tangent and cotangent vector bundles will be denoted by T and T^*.) Let ∇ be the Chern connection. A section L of $S^2T^* \otimes T$ on M is called *pre-linearization*, if the connection ∇^L defined by

$$\nabla^L_X Y = \nabla_X Y + L(X, Y)$$

preserves the web, that is the three families of leaves are auto-parallel curves for ∇^L. If M is 2-dimensional, then the set of the pre-linearizations is a 3-dimensional sub-bundle E of $S^2T^* \otimes T$. A pre-linearization L is called *linearization* if the connection ∇^L is flat. Of course, a 3-web is linearizable at $p \in M$ if and only if there exists a germ of a linearization at p. This approach to the linearizability problem was proposed by Akivis and employed by Goldberg in [6].

Our central idea is to introduce a projective invariant, noted s, of the pre-linearizations bundle, which we call the *base of the linearization*. Two linearizations L and $\hat{L}$ are projectively equivalent if and only if they have the same base. We show that the obstructions to the linearizability of webs can be expressed in terms of polynomials in s, whose coefficients depend on the curvature of the Chern connection and its derivatives. Our main result is the following:

Theorem *Let $\mathcal{W}$ be a differentiable 3-web on a 2-dimensional real or complex manifold M with non vanishing Chern curvature at $p \in M$. Then, there exists an algebraic sub-manifold $\mathcal{A}$ of E on a neighborhood of p, whose degree is at most 15, expressed in terms of the curvature of the Chern connection and its covariant derivatives until the 6^{th}-order, such that the linearizations of $\mathcal{W}$ are sections of E taking values in $\mathcal{A}$. Moreover, for any $L_0 \in \mathcal{A}$, there exists a neighborhood U of L_0 in $\mathcal{A}$ such that any $L \in U$ can be prolonged to a germ of a linearization. In particular:*

1) The web is linearizable if and only if $\mathcal{A} \neq \emptyset$,
2) There exists at most 15 different classes of linearizations.

1 Linear and parallel webs

Let M be a $2r$-differentiable manifold, with $r \geq 1$.

Definition 1.1 *A $3-$web on M is a triple of foliations $\{F_1, F_2, F_3\}$ such that the tangent spaces to the leaves of any two different foliations are complementary subspaces of T.*

We will call the leaves of the foliations $\{F_1, F_2, F_3\}$ horizontal, vertical and transversal. Likewise, we call their tangent spaces *horizontal, vertical and transversal* and denote them by T^h, T^v and T^t. We will use the Nagy's formalism (cf. [10]). In particular h (resp. v) is the horizontal (resp. vertical) projection, j is the associated product structure, ∇ is the Chern connection.

An *adapted basis* at $p \in M$ is a basis $\{e_i, e_{i+r}\}_{i=1,\dots,r}$ of T_pM such that $e_i \in T_p^h$ and $e_{i+r} = j(e_i)$.

From now on, we suppose that dim $M = 2$. Note that in this case the torsion of the Chern connection vanishes, and in an adapted basis the curvature is given by only one scalar function R^1_{121} defined by:

$$R(e_1, e_2)e_1 = R^1_{121}e_1.$$

By the inverse functions theorem, we can find local coordinates (x, y) at a neighborhood of $p \in M$ such that $\mathcal{W}$ can be written as

$$x = cte, \quad y = cte, \quad f(x, y) = cte. \tag{1}$$

Definition 1.2 *A 3−web on a 2-dimensional affine space is called* linear *(resp.* parallel*) if the leaves of the three foliations are straight lines (resp. parallel straight lines). A 3-web on M is called* linearizable *(resp.* parallelizable*) at $p \in M$ if it is equivalent to a linear (resp. parallel) 3-web modulo a local diffeomorphisms.*

Definition 1.3 *Let $\mathcal{W}$ be a 3-web on a 2-manifold, and ∇ the Chern connection. A symmetrical (1,2)-tensor field L is called* pre-linearization *if the connection ∇^L defined as*

$$\nabla^L_X Y = \nabla_X Y + L(X, Y)$$

preserves the web, that is the leaves are auto-parallel curves with respect to ∇^L. The set of pre-linearizations is a 3-dimensional sub-bundle E of $S^2T^* \otimes T$. A pre-linearization is a linearization *if the connection ∇^L is flat. Two pre-linearizations L and $\hat{L}$ are* projectively equivalent *if the connections ∇^L and $\nabla^{\hat{L}}$ are projectively related, that is there exists $\omega \in \Lambda^1(M)$ such that*

$$\nabla^L_X Y = \nabla^{L'}_X Y + \omega(X)Y + \omega(Y)X.$$

Proposition 1.4 *A tensor field L in $S^2T^* \otimes T$ is a* pre-linearization *if and only if*

1. $vL(hX, hY) = 0$,
2. $hL(vX, vY) = 0$,
3. $L(hX, hY) + jL(jhX, jhY) - hL(jhX, hY)$
   $- hL(hX, jhY) - jvL(jhX, hY) - jvL(hX, jhY) = 0.$

Moreover, L is a linearization *if in addition*

$$\nabla_X L(Y, Z) - \nabla_Y L(X, Z) + L(X, L(Y, Z)) - L(Y, L(X, Z)) + R(X, Y)Z = 0$$

holds, for any $X, Y, Z \in T$, where R denotes the curvature of the Chern connection.

Let $L \in E$ be a pre-linearization. We introduce the tensors $x,\ y,\ z : T^h \oplus T^h \to T^h$ defined by

$$\begin{cases} x\,(hX,\, hY) = L\,(hX,\, hY) \\ y\,(hX,\, hY) = jL\,(jhX,\, jhY) \\ z\,(hX,\, hY) = hL\,(hX\,, jhY) \end{cases} \tag{2}$$

The following elementary proposition is the key to our main theorem.

Proposition 1.5 *Let M be a 2-manifold, $\mathcal{W}$ a 3-web on M and $\{e_1, e_2\}$ a frame at $p \in M$ adapted to the web. If L is a pre-linearization at p, set*

$$s := 2L^1_{12} - L^2_{22}$$

where L^k_{ij} are the components of L: $L(e_i, e_j) = L^k_{ij} e_k$. Then two pre-linearizations L and $\hat{L}$ are projectively equivalent if and only if $s = \hat{s}$.

Definition 1.6 *The tensor s defined by $s = 2z - y$ is called the* base *of L.*

Proposition 1.5 shows us that two pre-linearizations are projectively equivalent if and only if they have the same base.

Since E is a rank-3 vector bundle, it can be parametrized by the tensors x, y, z. However, taking into account the central role of the base tensor and some symmetries of the problem, we will parameterize E by the tensors $\{t,\ z,\ s\}$, where $t = \frac{1}{2}(x + y - 2z)$.

2 The system of linearizations

In order to fix the notations, we suppose that M is complex, but all the considerations and proofs can be translated immediately to the real case. If $B \to M$ is a vector bundle on M, then $Sec(B)$ will denote the sheaf of sections of B and $J_k(B)$ the vector bundle of $k-$jets of sections of B.

To solve to the linearizability problem for a given web $\mathcal{W}$, one should study the local integrability of the first order regular partial differential operator $P_1 : E \longrightarrow \Lambda^2 T^* \otimes T$ defined, for any $L \in E$, by

$$\begin{aligned}(P_1 L)(X, Y, Z) = (\nabla_X L)(Y, Z) - (\nabla_Y L)(X, Z) \\ + R(X, Y)Z + L(X, L(Y, Z)) - L(Y, L(X, Z))\end{aligned} \tag{3}$$

where $X,\ Y,\ Z \in T$.

We will use the theory of the formal integrability of Spencer (cf. for example [5]). The notations are those of [7], where an accessible survey of the theory is also given. If P is a quasi-linear differential operator of order k, $R_{k,p}(P)$ will denote the bundle of the formal solutions of order k at p, and $\sigma_{k+\ell}(P)$ the symbol of the ℓ-th order prolongation $p_\ell(P)$ of P. We also write $g_{k+\ell} = \mathrm{Ker}\sigma_{k+\ell}$ and $K = \mathrm{Coker}\,\sigma_{k+1}$. K is the space of obstructions.

If Q is a tensor field in $(\otimes^q T^{h^*}) \otimes T^h$, we denote by Q_1 and Q_2 the tensor fields in $(\otimes^{q+1} T^{h^*}) \otimes T^h$ defined by

$$\begin{aligned} Q_1\,(hX, hX_1, ..., hX_q) &= (\nabla_{hX} Q)\,(hX_1, ..., hX_q) \\ Q_2\,(hX, hX_1, ..., hX_q) &= (\nabla_{jhX} Q)\,(hX_1, ..., hX_q) \end{aligned} \tag{4}$$

By recursion, we define the successive covariant derivatives, with the convention that $Q_{i_1,i_2} = (Q_{i_2})_{i_1}$. Note that

$$Q_{12} - Q_{21} = (q-1)\,\mathcal{R}\,t \tag{5}$$

for $Q \in \mathcal{S}ec(\otimes^q T^{h^*}) \otimes T^h$ where $\mathcal{R}$ is the tensor $\mathcal{R} : T^h \oplus T^h \oplus T^h \to T^h$ defined by

$$\mathcal{R}(hX,\, hY)\,hZ = R(jhX,\, hY)\,hZ \tag{6}$$

and R is the curvature of the Chern connection. Then the linearization system $P_1(L) = 0$ can be written as

$$P_1 L = \begin{cases} t_1 = st + t^2 \\ t_2 = \frac{1}{3}s_1 - \frac{2}{3}s_2 + zt - \frac{1}{3}\mathcal{R} \\ z_1 = \frac{2}{3}s_1 - \frac{1}{3}s_2 + zt + \frac{1}{3}\mathcal{R} \\ z_2 = -zs + z^2 \end{cases} \tag{7}$$

where z^2 means the $(1-3)$ tensor defined by $z^2\,(hX, hY, hZ) = z\,(z\,(hX,\, hY),\, hZ)$. Similarly, we define the product zt , the power z^3 (which is a $(1-4)$ tensor), etc...

3 The compatibility system P_2

Considering s as a parameter, the system (7) can be seen as a Frobenius system in the unknown functions t and z. By the permutation formula (5), the integrability condition is

$$\begin{cases} z_{12} - z_{21} = \mathcal{R}z \\ t_{12} - t_{21} = \mathcal{R}t \end{cases} \tag{8}$$

where the parameter s has to verify

$$s_{12} - s_{21} = \mathcal{R}s \tag{9}$$

An easy computation shows that the equations (8) can be expressed in terms of s:

$$P_2 = \begin{cases} s_{22} = 2s_{21} - ss_2 + 2ss_1 + \mathcal{R}s + \mathcal{R}_2 \\ s_{11} = 2s_{21} - 2ss_2 + ss_1 + \mathcal{R}s + \mathcal{R}_1 \end{cases} \tag{10}$$

The linearizability of the web is equivalent to the local solvability of the system P_2.

Since the bundle T^h is 1-dimensional, all the tensorial bundles $\otimes^p_q T^h$ are also 1-dimensional and therefore, fixing a basis e_1 of T^h, we can identify the tensors Q on these bundles with their components. In order to simplify the notations, we may consider s, s_i, s_{ij} etc as complex functions, noticing however that the indices are considered as indices of covariant derivative, for which the permutation formula (5) holds. With this abuse of notations, P_2 can be seen as a map

$$p_0(P_2) = J_2(\mathbb{C}) \to \mathbb{C}^2$$

It is easy to prove that every 2^{nd}–order solution of P_2 at p can be lifted into a 3^{rd}–order solution. Unfortunately, P_2 is not 2-acyclic. Indeed, the sequence

$$0 \longrightarrow g_{\ell+1}(P_2) \longrightarrow g_\ell(P_2) \otimes T^* \overset{\delta_\ell(P_2)}{\longrightarrow} g_{\ell-1}(P_2) \otimes \Lambda^2 T^* \longrightarrow 0$$

is not exact $\forall l \geq 2$. For $\ell = 3$ we have $\operatorname{rank} \delta_3 = \dim g_3 \otimes T^* - \dim g_4 = 0 < \dim g_2 \otimes \Lambda^2 T^* = 1$, so $\delta_3(P_2)$ is not onto and the corresponding Spencer cohomology group $H^2_2(P_2)$ does not vanish. Thus there exist obstructions in the prolongation of the system, and because of that the whole study is complicated.

4 The first obstructions

Let us denote by P_3 the first prolongation of P_2. We have

Lemma 4.1 *P_3 is involutive. Moreover, any 3^{rd}–order solution of P_3 can be lifted into a 4^{th}–order solution if and only if $\varphi = 0$, where*

$$\begin{aligned} \varphi := &- 24\mathcal{R}s_{21} - (24\mathcal{R}s + 12\mathcal{R}_1 - 6\mathcal{R}_2)s_1 + (24\mathcal{R}s + 6\mathcal{R}_1 - 8\mathcal{R}_2)s_2 \\ &+ 3\mathcal{R}s^3 + (-4\mathcal{R}_2 - 3\mathcal{R}_{22} + \mathcal{R}_{21} + 2\mathcal{R}_{12} - 13\mathcal{R}^2 - 3\mathcal{R}_{11})s \\ &+ 2\mathcal{R}_{122} - \mathcal{R}_{221} - \mathcal{R}_{112} - 5\mathcal{R}\mathcal{R}_1 - 2\mathcal{R}_{121} - 11\mathcal{R}\mathcal{R}_2 \end{aligned} \tag{11}$$

REMARK - If $\mathcal{R} = 0$, then $\varphi = 0$, so every 3^{rd}–order solution of P_3 can be lifted into a 4^{th}–order solution. Since P_3 is also involutive, it follows that P_3 is formally integrable. So we have:

Corollary 4.2 *Let $\mathcal{W}$ be a parallelizable 3-web on $\mathbb{C}^2$, and $p \in \mathbb{C}^2$. Then, for every $L_0 \in E_p$ there exists a germ L of a linearization which extends L_0.*

According to the Theorem of Graf-Sauer, for a parallelizable web, there exist projectively non-equivalent linearizations. Indeed, if we take L_0 and $L_0{}' \in E_p$ such that $s_p \neq s'_p$, and extend them to germs of linearizations, we obtain two projectively non-equivalent linearizations.

From now on we suppose that web under consideration is non-parallelizable, i.e., $\mathcal{R} \neq 0$.

In order to study the integrability of P_2 we have to introduce the compatibility condition $\varphi = 0$ into the system $P_3 = 0$. We arrive at a 2^{nd}–order quasi-linear system

$$P_\varphi := (P_2, \varphi = 0)$$

Lemma 4.3 *A 2^{nd}–order formal solution of P_φ at p, $(j_2 s)_p$ can be lifted to a 3^{rd}–order solution if and only if*

$$\begin{cases} \psi^1 := \ \ 24\mathcal{R}s_2^2 - 48\mathcal{R}s_1 s_2 + \alpha(s)s_1 + \beta(s)s_2 + \gamma(s) = 0, \\ \psi^2 := -24\mathcal{R}s_1^2 + 48\mathcal{R}s_1 s_2 + \hat{\alpha}(s)s_1 + \hat{\beta}(s)s_2 + \hat{\gamma}(s) = 0, \end{cases} \tag{12}$$

where α, β, $\hat{\alpha}$, $\hat{\beta}$ are polynomials in s of degree 2 and whose coefficients depend on $\mathcal{R}$ and its derivatives up to order 2, while γ and $\hat{\gamma}$ are polynomials in s of degree 3 whose coefficients are $\mathcal{R}$ and its derivatives up to order 4.

5 The linearization theorem

Since the obstructions ψ^1 and ψ^2 depend on s, they have to be put in the system P_φ and consider the differential system:

$$P_\psi = (P_2, \varphi = 0, \psi^1 = 0, \psi^2 = 0)$$

and its prolongation. Taking the covariant derivative of the equations $\psi^1 = 0$ and $\psi^2 = 0$ with respect to e_1 and e_2, we get 4 new second order equations: $\psi^1_1 = \psi^1_2 = \psi^2_1 = \psi^2_2 = 0$. However, using the equations $P_2 = 0$ and $\varphi = 0$ we can eliminate the second order derivatives of s. On the other hand, with the help of the equations (12), we can express the square terms s_1^2 and s_2^2 as a function of s_1, s_2 and s_1s_2. So, the system

$$P_\psi = 0, \quad \nabla P_{\psi^1} = 0, \quad \nabla P_{\psi^2} = 0$$

is equivalent to the system composed of $P_\psi = 0$ and the following four equations:

$$\mathcal{S} = \begin{cases} a^1 s_1 + b^1 s_2 + c^1 s_1 s_2 = d^1, \\ a^2 s_1 + b^2 s_2 + c^2 s_1 s_2 = d^2, \\ a^3 s_1 + b^3 s_2 + c^3 s_1 s_2 = d^3, \\ a^4 s_1 + b^4 s_2 + c^4 s_1 s_2 = d^4, \end{cases} \tag{13}$$

which are linear in s_1, s_2 and s_1s_2. The explicit computation shows that a_i^j, b_i^j, c_i^j and d_i^j are polynomials in s whose coefficients depend on $\mathcal{R}$ and its derivatives up to order 5. Their explicit expressions are computed with MAPLE.

Let us consider the system $\mathcal{S}$ as a linear system in the variables s_1, s_2 and s_1s_2. The computation shows that the determinant of the system $\mathcal{S}$ vanishes, so $\mathcal{S}$ is compatible. Moreover, the 3^{rd}-order minors of the systems (13) are non zero polynomial in s of degree 7. So there exists an open $\mathcal{U} \subset \mathbb{C}^2$ on which, for example,

$$D(s) := \begin{vmatrix} a^1 & b^1 & c^1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} \neq 0.$$

Solving on $\mathcal{U}$ the linear system (13) by the Cramer formulas, we get:

$$s_1 = F(s) = \frac{A(s)}{D(s)}, \quad s_2 = G(s) = \frac{B(s)}{D(s)}, \tag{14}$$

and

$$s_1 s_2 = H(s) := \frac{C(s)}{D(s)}, \tag{15}$$

where A, B, and C are given by the corresponding determinant. These functions are polynomial in s of degree 8, 8, and 11 respectively.

On the other hand

(a) since, by (15), there should hold $F(s)\,G(s) = H(s)$, the solution s of the linearization system P_1 has to take his values on algebraic manifold defined by:

$$Q_1(s) := AB - CD = 0. \tag{16}$$

Q_1 is polynomial of degree 18 in s.

(b) For the system (14) the compatibility condition is given by the equation (9), i.e. $\nabla_{e_1} G - \nabla_{e_2} F = \mathcal{R}s$. Using A, B and D, we get that s has to take its values in the algebraic manifold defined by

$$Q_2(s) := AB'_s - A'_s B - \mathcal{R}sD^2 = 0, \tag{17}$$

where we noted A'_s (resp. B'_s) the derivative of A (resp. B) with respect to s. Of course Q_2 is polynomial in s; its degree is 15.

(c) F and G have to satisfy the equations of the system $P_\psi = 0$. We get therefore 5 polynomial equations in s, $Q_i = 0$, $(i = 3, ..., 7)$. Their degrees are 23, 23, 24, 17 and 17 respectively.

It follows that s has to take its values on the algebraic manifold $\mathcal{A} \subset E$, defined by the equations $Q_i = 0$, $i = 1, ..., 7$:

$$\mathcal{A} := \left\{ Q_i = 0 \mid i = 1, ..., 7 \right\}.$$

Finally, the system P_2 (and then the linearization system P_1) has a solution on a neighborhood of p if and only if the algebraic manifold $\mathcal{A}$ is not empty on a neighborhood of p. Moreover, if $\mathcal{A} \neq \emptyset$, then for any $s_0 \in \mathcal{A}$, there exists a neighborhood U of s_0 such that any $s \in U$ can be prolonged to a germ $\tilde{s}$ of a linearization. So we have the following

Theorem 5.1 *A C^∞ 3-web $\mathcal{W}$ with non vanishing curvature is linearizable if and only if there exists an open set $V \subset \mathbb{C}^2$ on which the resultant of the polynomials $Q_1(s), ..., Q_7(s)$ vanishes. Moreover, if this condition is satisfied, then for any $p \in V$ and for any pre-linearization $L_0 \in E_p$ whose the base s_0 lies in $\mathcal{A}$, there exists a unique germ L of a linearization such that $L_p = L_0$.*

Theorem 5.2 *If $\mathcal{W}$ is a C^∞ 3-web with non zero curvature, then there exist at most* 15 *classes of linearizations not projectively equivalent.*

Indeed, the minimum of the degrees of Q_i, $i = 1, ..., 7$ is 15 (the degree of Q_2), so these polynomials have at most 15 common different roots.

REMARK. The Gronwall conjecture can be expressed now in the following form: for any smooth non parallelizable 3-web on a 2-dimensional manifold

$$d^0[Rad\,(Q_1, ..., Q_7)] = 1$$

where *Rad* is the radical of the polynomials Q_i.

Examples

(1) Consider the web $\mathcal{W}$ defined by $x = cte$, $y = cte$, $f(x, y) = cte$, where $f(x, y) := (x + y)e^{-x}$. This web is not parallelizable in a neighborhood of

$(0,0)$ because $\mathcal{R}(0,0) = -1$. The computation gives $Rad(Q_1, \cdots, Q_7) = s+1$ on a neighborhood of $(0,0)$. Thus the web is linearizable in a neighborhood of $(0,0)$ and all the linearizations are projectively equivalent.

(2) Let $\mathcal{W}$ be the web defined by $x = cte$, $y = cte$, $f(x,y) = cte$, where

$$f = \log(x) + \frac{1}{2}\log\left(\frac{x^2+y^2}{x^2}\right) + \text{arctg}\left(\frac{y}{x}\right).$$

We have $\mathcal{R}_{(1,0)} = 2$, so $\mathcal{W}$ is not parallelizable at $(1,0)$. On the other hand the resultant of the polynomials Q_2, Q_6 is not zero at $(1,0)$. So this web is not linearizable at $(1,0)$.

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