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Inverse problem of the calculus of variations on Lie groups

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Abstract

This article studies the inverse problem of the calculus of variations for the special case of the geodesic flow associated to the canonical symmetric bi-invariant connection of a Lie group. Necessary background on the differential geometric structure of the tangent bundle of a manifold as well as the Fröhlicher–Nijenhuis theory of derivations is introduced briefly. The first obstructions to the inverse problem are considered in general and then as they appear in the special case of the Lie group connection. Thereafter, higher order obstructions are studied in a way that is impossible in general. As a result a new algebraic condition on the variational multiplier is derived, that involves the Nijenhuis torsion of the Jacobi endomorphism. The Euclidean group of the plane is considered as a working example of the theory and it is shown that the geodesic system is variational by applying the Cartan–Kähler theorem. The same system is then reconsidered locally and a closed form solution for the variational multiplier is obtained. Finally some more examples are considered that point up the strengths and weaknesses of the theory.

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Keywords: Lie group; Canonical connection; Euler–Lagrange equation; Variational principle

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1. Introduction

On a Lie group G there are three kinds of natural linear connection: the plus, minus and zero connections introduced first by E. Cartan and J.A. Schouten [3]. The plus and minus connections arise from the fact that any two tangent spaces to G may be "connected" by means of a unique left or right translation, respectively. A simple way to define these connections is to give their values on left-invariant vector-fields X, Y by

$$\nabla^+_X Y = [X, Y], \qquad \nabla^-_X Y = 0,$$

and extend them to arbitrary vector fields by making them tensorial in the X argument and satisfy the Leibniz rule in the Y argument. It is easy to show that the curvature tensors of the minus and plus connections are zero, indeed that is exactly what the structure equations of the Lie algebra say, but in general, the corresponding torsion tensors are non-vanishing. In this article we shall be interested in the zero connection whose parallel transport rule is more complicated than the other two. Nonetheless its value on left invariant vector fields is given simply by

$$\nabla^0_X Y = \frac{1}{2}[X, Y].$$

Clearly ∇^0 has zero torsion but its curvature tensor is not zero in general; however, we shall prove that its curvature tensor is parallel. Thus G is in a sense a symmetric space with respect to ∇^0. All three of the canonical connections have the same geodesics.

This paper is concerned with investigating the inverse problem of Lagrangian dynamics for the geodesic flow associated to ∇^0 which henceforth shall be denoted simply by ∇ and referred to as the canonical symmetric, linear connection. Thus we wish to be able to decide if there exists a Lagrangian function E defined on an open subset of the tangent bundle TG whose Euler–Lagrange equations coincide with the geodesics of ∇ and to a lesser extent to describe all possible such Lagrangians, or more realistically, their Hessians. An easy computation shows that the curvature of ∇ is

$$R(X, Y)Z = \frac{1}{4}\big[Z, [X, Y]\big], \tag{1}$$

where X, Y and Z are arbitrary vector fields. Other important properties of ∇ that we shall discuss in Section 1 are that every left or right invariant vector field is an affine collineation and dually every left or right invariant one-form gives a first integral of the geodesics when thought of as a linear function on TG. Furthermore its Ricci tensor R_{ij} is symmetric and since the curvature tensor is parallel, R_{ij} gives a partial, that is possibly degenerate, quadratic Lagrangian function. Indeed R_{ij} is essentially just the left or right translation of the Killing form and so R_{ij} is non-degenerate precisely when G is semi-simple. In that case if G is compact it is a Riemannian type II symmetric space or more generally, if non-compact, a pseudo-Riemannian symmetric space [10]. It should be noted, however, that it is possible for an indefinite metric Lagrangian to be associated to ∇ even if G is not semi-simple or flat. As regards the inverse problem where G is not semi-simple, the only results are as far as we aware are given by [14] where the inverse problem for the cases of Lie groups up through dimension three is solved by lengthy local coordinate calculations.

In this present paper we will concentrate on the higher dimensional case and in the main part (Section 5) we push the computation as far as is reasonably possible in the generic case. In Section 6 we consider the example of the Euclidean group ASO_n and compute explicitly the integrability conditions found in Section 5. We also prove that ASO_2 is variational and use it as a worked example to explain the theory. In Section 7 we approach the inverse problem form the point of view of the Helmholtz conditions and give some examples that are intended to illuminate the theory. In particular we give a concrete solution to the inverse problem for the Euclidean group of the plane ASO_2. Clearly the Lie group problem becomes very much more difficult as the dimension of the group increases. Therefore it is profitable to have both the theoretical Spencer approach and coordinate formulations of Section 7 available. For example the case of the higher dimensional Euclidean groups is altogether more complicated with the involution tests running into dimensions in the hundreds. The coordinate version is also very difficult, not least because there is a semi-simple part. The inverse problem for Lie groups of dimension two and three has been addressed in [14]. In dimension four for indecomposable Lie algebras all algebras are solvable and the same is almost true in dimension five where only the Lie algebra of the special affine group is not solvable.

In this paper the Frölicher–Nijenhuis theory of derivations of the exterior algebra of differential forms $\Lambda(M)$ on a smooth manifold M is used extensively. The reader may consult [5] or [9] for further details but we mention here that there are two basic types of derivations from which all others are obtained. The first kind of derivation, "of type i_*", is purely algebraic and involves an interior product of a vector field by a form or more generally a vector-valued form by a form. If L belongs to the space of vector fields $\mathfrak{X}(M)$ on M and ω is a k-form then $i_L\omega$ is the standard interior product. If, however, L is a vector-valued l-form then $i_L\omega$ is a $k+l-1$ vector-valued form. The second kind of derivation of $\Lambda(M)$, "of type d_*", involves the exterior derivative d. If again $L \in \mathfrak{X}(M)$ and ω is a k-form then $d_L\omega$ is the Lie derivative of ω by L. If, however, L is a vector-valued l-form then $d_L\omega$ is a $k+l$-vector-valued form. The action of $i_L\omega$ and $d_L\omega$ are determined by their effect on forms of degree one and zero, respectively and the fact that they act as derivations. If L is a vector-valued form or endomorphism field and f is a function then $d_L f$ is simply the value of L applied to df. In fact d_L is the graded commutator of the derivations i_L and d. Another useful fact is that, in the case where L is non-singular, $d_L \cdot d_L$ vanishes if and only if the Nijenhuis torsion of L is zero. Indeed, if K and L are vector-valued forms of degree k and l it is possible to show that the commutator $[d_K, d_L]$ is again a derivation of type d_* and therefore there exists a vector-valued form $[K, L]$ of degree $k+l-1$ such that

$$[d_K, d_L] = d_{[K,L]}.$$

The other formalism which is employed is the Spencer–Goldschmidt theory of over-determined systems of partial differential equations. The reader may find more details in [2] for example. In order to make the paper intelligible we have found it useful to summarize some of the theory developed in [9]. In particular the word "spray" below is synonymous with "second order vector field" although the only systems that we shall ever consider are the geodesics of a linear connection for which some authors reserve the term "spray". Finally the summation convention on repeated indices applies unless the contrary is stated.

2. Properties of the canonical connection

Let E_α denote a basis for the left invariant vector fields on G. Then the structure constants $C^\gamma_{\alpha\beta}$ of the Lie algebra $\mathfrak{g}$ are defined by

$$[E_\alpha, E_\beta] = C^\gamma_{\alpha\beta} E_\gamma. \tag{2}$$

Following the conventions of [10] a left invariant vector field associated to an element X in T_eG is denoted by $\tilde{X}$; that is, $\tilde{X}(x) = L_{x*}X$, where x and e denote a typical and identity group elements, respectively, and L_x denotes left translation. Likewise using R_x for right translation the right invariant vector field induced by X is denoted by $\tilde{X}^{R(x)}$ so that $\tilde{X}^{R(x)} = R_{x_*}X$.

Lemma 2.1. *In the definition of* ∇ *one can equally use right invariant vector fields instead of left invariant vector fields.*

Proof. Let E_i be a basis for the Lie algebra of G, that is T_eG. According to the conventions introduced above and letting x denote a generic element of G, there must exist a matrix of functions $f_{ij}(x)$ such that

$$\tilde{E}_i^{R(x)} = f_{ik}(x)\tilde{E}_k. \tag{3}$$

If we calculate the quantity $\nabla_{\tilde{E}_i^{R(x)}}\tilde{E}_j^{R(x)} - \frac{1}{2}[\tilde{E}_i^{R(x)}, \tilde{E}_j^{R(x)}]$ using the definition of ∇ and the Koszul axioms, we find that it is zero if and only if

$$f_{ik}(\tilde{E}_k f_{jl})\tilde{E}_l + f_{jl}(\tilde{E}_l f_{ik})\tilde{E}_k = 0, \tag{4}$$

where the point x in the group has been suppressed. If we interchange k and l in the second term above we find that the latter condition is equivalent to

$$f_{ik}(\tilde{E}_k f_{jl}) + f_{jk}(\tilde{E}_k f_{il}) = 0. \tag{5}$$

Starting from the condition above that relates left and right invariant fields and using the fact that the left invariant $\tilde{E}_l$ and right invariant vector fields $\tilde{E}_i^{R(x)}$ commute we find that

$$\tilde{E}_l f_{ik} + f_{im}C_{lm}^k = 0. \tag{6}$$

However, because of the skew-symmetry in C_{lm}^k the latter condition implies (5) and hence in the definition of ∇ we can equally use right invariant vector fields. □

Corollary 2.2. ∇ *is right invariant and hence bi-invariant.*

Clearly the curvature tensor (on left invariant vector fields) is given by (1). Furthermore, G is a symmetric space in the sense that R is a parallel tensor field. Indeed suppose that W, X, Y and Z are left-invariant vector fields. Then from (1) and (2) we have that

$$\begin{aligned}
4\nabla_W R(X,Y)Z &= 1/2\big[W,[Z,[X,Y]]\big] - 4R(\nabla_W X, Y)Z - 4R(X, \nabla_W Y)Z - 4R(X,Y)\nabla_W Z \\
&= 1/2\big[W,[Z,[X,Y]]\big] - \big[Z,[\nabla_W X, Y]\big] - \big[Z,[X,\nabla_W Y]\big] - \big[\nabla_W Z,[X,Y]\big] \\
&= 1/2\big[W,[Z,[X,Y]]\big] - 1/2\big[Z,[[W,X],Y]\big] \\
&\quad - 1/2\big[Z,[X,[W,Y]]\big] - 1/2\big[[W,Z],[X,Y]\big] \\
&= 1/2\big(\big[Z,[W,[X,Y]]\big] - \big[Z,[[W,X],Y]\big] - \big[Z,[X,[W,Y]]\big]\big) \\
&= 0.
\end{aligned}$$

It follows from (3) that ∇ is flat if and only if the Lie algebra $\mathfrak{g}$ of G is nilpotent of order two. Clearly left and right invariant vector fields are auto-parallel. Hence the geodesics of ∇ are translates either to the right or left of one-parameter subgroups of G, that is of the form $x(\exp(tX))$ or $(\exp(tX))x$, where

X and x are in $\mathfrak{g}$ and G, respectively. The Ricci tensor $R_{\alpha\beta}$ of ∇ is symmetric and bi-invariant. In fact, in the basis $\{E_\alpha\}$ of left invariant vector fields the Ricci tensor $R_{\alpha\beta}$ is given by

$$R_{\alpha\beta} = \frac{1}{4}\, C^{\lambda}_{\beta\mu} C^{\mu}_{\alpha\lambda} \tag{7}$$

from which the symmetry of $R_{\alpha\beta}$ becomes apparent. Indeed, $R_{\alpha\beta}$ is obtained by translating to the left or right one quarter of the Killing form. Since $R^{\delta}_{\alpha\beta\gamma}$ is a parallel tensor field and $R_{\alpha\beta}$ is symmetric, it follows that Ricci gives rise to a quadratic Lagrangian which may, however, not be regular. If G is semi-simple then the Killing form provides a bi-invariant metric whose Levi-Civita connection is ∇. For this reason, in what follows it will usually be assumed that G is not semi-simple.

We remind the reader that a one-form $\omega_\alpha\, dx^\alpha$ on G gives rise to a function $\omega_\alpha y^\alpha$ on TG, that is linear in the fibres, where (x^α) is a system of local coordinates on G and (x^α, y^α) the induced system on TG. Conversely any such linear function gives rise to a one-form and this construction is a general feature of second order systems and in no way depends on the group structure on G.

Proposition 2.3. *Any left or right invariant one-form on G gives rise to a linear first integral on TG.*

Proof. Let ω be a one-form on G that is left-invariant and let X and Y be left-invariant vector fields. The function $\langle Y, \omega\rangle$ is left invariant and so is constant. Hence

$$X\langle Y, \omega\rangle = \langle \nabla_X Y, \omega\rangle + \langle Y, \nabla_X \omega\rangle = 0. \tag{8}$$

Now interchange X and Y and add the resulting equations and use the definition of ∇. One finds that

$$\langle X, \nabla_Y \omega\rangle + \langle Y, \nabla_X \omega\rangle = 0. \tag{9}$$

Since the last condition is tensorial in X and Y it follows that ω satisfies Killing's equation and hence gives a first integral on TG. $\square$

Alternatively just notice that since a left or right invariant vector field is a geodesic field, a left or right invariant one- form, respectively, must be constant along the geodesic.

Proposition 2.4. *Consider the following conditions for a one-form ω on G:*

(i) *ω is left-invariant and closed.*
(ii) *ω is right-invariant and closed.*
(iii) *ω is bi-invariant.*
(iv) *ω is parallel.*

Then we have the following implications: (iii) *implies* (i); (iii) *implies* (ii); *each of* (i), (ii) *or* (iii) *implies* (iv).

Proof. (i) implies (iv): The fact that ω is closed implies that for any two vector fields X and Y, in particular right invariant ones, that

$$\langle X, \nabla_Y \omega\rangle - \langle Y, \nabla_X \omega\rangle = 0. \tag{10}$$

On the other hand according to Proposition 2.3

$$\langle X, \nabla_Y \omega\rangle + \langle Y, \nabla_X \omega\rangle = 0. \tag{11}$$

Hence ω is parallel.

The proof that (ii) implies (iv) is similar to the proof that (i) implies (iv).

(iii) implies (i): A lemma of Helgason [10] states that if a one-form is bi-invariant then it is closed. Hence we are reduced to proving that (i) implies (iv), which has already been done. □

We note next that any left-invariant vector field will be a "Killing vector field" or affine collineation of ∇, that is to say an infinitesimal symmetry of ∇. Indeed if X and Y are also left-invariant one finds that the Lie derivative of ∇ along Z is given by

$$\begin{aligned}(L_Z\nabla)_X Y &= [Z, \nabla_X Y] - \nabla_{[Z,X]} Y - \nabla_X [Z, Y] \\ &= \frac{1}{2}\big\{\big[Z, [X.Y]\big] + \big[[X, Z], Y\big] + \big[X, [Y, Z]\big]\big\} \\ &= 0\end{aligned}$$

because of the Jacobi identity. The same argument applies equally to right-invariant vector fields. Hence:

Proposition 2.5. *Every left or right invariant vector field on G is an infinitesimal symmetry of ∇.*

3. Connections, covariant differentials, Lagrangians

The natural framework of the calculus of variations is the tangent bundle, where one can present the obstructions to the integrability of the Euler–Lagrange system in an intrinsic and natural way [9]. We present here the basic objects that play a role in the theory.

If M is a manifold TM denotes its tangent space and π the natural submersion. Let $J : TTM \to TTM$ be the canonical vertical endomorphism and $C \in \mathfrak{X}(TM)$ the Liouville vector-field: if (x^α) is a local coordinate system on M and (x^α, y^α) is the induced coordinate system on TM, then

$$J = dx^\alpha \otimes \frac{\partial}{\partial y^\alpha}, \qquad C = y^\alpha \frac{\partial}{\partial y^\alpha}.$$

Definition 3.1 [7]. A (non-linear) *connection* on M is a tensor field of type (1–1) Γ on TM such that $J\Gamma = J$ and $\Gamma J = -J$. The connection is called linear if $[C, \Gamma] = 0$, and Γ is $\mathcal{C}^1$ on the 0 section.

If Γ is a connection, $\Gamma^2 = id_{TTM}$ and the eigenspace corresponding to the eigenvalue -1 is the vertical space V_z. Then, at any $z \in TM$, we have the splitting

$$T_z TM = H_z \oplus V_z,$$

where H_z is the eigenspace corresponding to $+1$. The subspace H_z is called the horizontal space. In the sequel we will write

$$h := \frac{1}{2}(I + \Gamma), \qquad v := \frac{1}{2}(I - \Gamma),$$

for the *horizontal* and *vertical* projectors. Locally we have:

$$h\left(\frac{\partial}{\partial x^\alpha}\right) = \frac{\partial}{\partial x^\alpha} - \Gamma_\alpha^\beta(x, y)\frac{\partial}{\partial y^\beta}, \qquad h\left(\frac{\partial}{\partial y^\alpha}\right) = 0,$$

where Γ_α^β are the coefficients of the connection. If the connection is linear, the coefficients $\Gamma_\alpha^\beta(x, y)$ are linear in y and one would naturally write $y^\gamma \Gamma_{\alpha\gamma}^\beta$ where $\Gamma_{\alpha\gamma}^\beta$ are the classical Christoffel components.

Proposition 3.2 [7]. *If S is a spray, than $\Gamma := [J, S]$ is a connection.*

Definition 3.3. The torsion of a connection Γ is the vectorial 2-form $t := [J, h]$ and the curvature is $R := -\frac{1}{2}[h, h]$.

For every spray S, the connection $\Gamma = [J, S]$ has zero torsion. Moreover, the vertical distribution is integrable, and therefore $h[vX, vY] = 0$ and so

$$R(X, Y) = -v[hX, hY].$$

Every connection Γ on M determines an almost complex structure F on TM which interchanges the horizontal and the vertical spaces. More precisely F is the unique (1–1) tensor field such that $FJ = h$ and $Fh = -J$. In the case when $\Gamma = [J, S]$ we have

$$F = h[S, h] - J. \tag{12}$$

In Section 4 we have to do a calculation that is crucial for our analysis. It is convenient for that purpose to introduce the following definition of covariant derivative that enables the standard definition to be generalized considerably even though we shall not need that generalization here [9]. Recall first that if $z \in TM$ and $\pi(z) = x$, there is a natural isomorphism between the vertical subspace T_z^v of T_zTM and T_xM that we denote by ξ_z. Indeed the definition of J at z is simply the composition of π_{*z} followed by $(\xi_z)^{-1}$.

Definition 3.4. Let M a manifold with connection Γ. Suppose that $z \in \mathfrak{X}(M)$ and $w \in T_xM$. Then recalling that v is vertical projector, the *covariant derivative* of z with respect to w is defined by:

$$D_{w(x)}z = \xi_{z(x)}\big(v \circ z_*(w)\big). \tag{13}$$

A vector field $z \in TM$ along a curve γ is called *parallel* if $D_{\frac{d}{dt}} z = 0$. A *geodesic* is a curve $\gamma : [a, b] \to M$ such that $D_{\frac{d}{dt}} \gamma' = 0$.

If $w, z \in \mathfrak{X}(M)$, we have locally

$$D_w z = w^\alpha\left(\frac{\partial z^\beta}{\partial x^\alpha} + \Gamma_\alpha^\beta\big(x, z(x)\big)\right)\frac{\partial}{\partial x^\beta}. \tag{14}$$

J. Douglas introduced a tensor now called the Jacobi endomorphism that plays an essential role in the theory of the inverse problem of the calculus of variations [4]. Its coordinate-free presentation was given by J. Klein in [11]:

Definition 3.5. The *Jacobi endomorphism* is the (1–1) tensor Φ on TM defined by

$$\Phi := v \cdot [h, S], \tag{15}$$

where h and v are the horizontal and vertical projectors and $[h, S]$ is simply the negative of the Lie derivative of h with respect to S.

It is easy to see that Φ is semi-basic and

$$\Phi = [h, S] + F + J, \tag{16}$$

where F is the almost complex structure associated to Γ. The endomorphism Φ is related to the curvature by the formula

$$R = \frac{1}{3}[J, \Phi]. \tag{17}$$

Definition 3.6. A *Lagrangian* is a map $E : TM \to \mathbb{R}$ that is smooth on at least an open subset of TM. The Lagrangian E is said to be *regular* if the 2-form

$$\Omega_E := dd_J E \tag{18}$$

has maximal rank.

The Lagrangian E is regular if and only if $\det(\frac{\partial^2 E}{\partial y^\alpha \partial y^\beta}) \neq 0$. If $E : TM \to \mathbb{R}$ is a regular Lagrangian, then the vector field S on TM defined by the equation

$$i_S \Omega_E = d(E - \mathcal{L}_C E) \tag{19}$$

is a spray and the paths of S are the solutions to

$$\frac{d}{dt}\frac{\partial E}{\partial \dot{x}^\alpha} - \frac{\partial E}{\partial x^\alpha} = 0, \quad \alpha = 1 \ldots n,$$

the Euler–Lagrange equations [6].

Definition 3.7. A spray S is called *variational* if there exists a smooth, regular Lagrangian E which satisfies (19), the Euler–Lagrange equation.

To every Lagrangian E and spray S a scalar 1-form ω_E can be associated by

$$\omega_E := i_S \Omega_E + d\mathcal{L}_C E - dE, \tag{20}$$

which is called *Euler–Lagrange* form. It is easy to see that ω_E is semi-basic, and the local expression in the standard coordinate system on TM is

$$\omega_E = \sum_{i=1}^n \left(S\left(\frac{\partial E}{\partial y^i}\right) - \frac{\partial E}{\partial x^i}\right) dx^i.$$

Therefore along a curve $\gamma = x(t)$ associated with S we have

$$\omega_E|_\gamma = \sum_{i=1}^n \left(\frac{d}{dt}\frac{\partial E}{\partial \dot{x}^i} - \frac{\partial E}{\partial x^i}\right) dx^i,$$

where d/dt denotes the differentiation along γ. So, in order to solve the inverse problem of the calculus of variations for a given spray, we have to look for a regular Lagrangian E for which $\omega_E \equiv 0$. For this purpose we must study the local integrability of the second order partial differential operator $P_1 : \mathcal{C}^\infty(TM) \longrightarrow \operatorname{Sec} T_v^*$ called the *Euler–Lagrange operator* defined by

$$P_1 := i_S\, dd_J + d\mathcal{L}_C - d. \tag{21}$$

4. The geometry of Lie groups

In this section we will describe the connection and geometric structures associated to the canonical covariant derivation ∇ of a Lie group G. Since our goal is to study the inverse problem, it will be very convenient to suppose that G is a linear Lie group, that is, a subgroup of $Gl(n,\mathbb{R})$ for some n.

Let (x) be coordinates on G, (x, y) be the standard associated coordinate system on TG. We will also use the "left-invariant" coordinates (x, α) on $TG \simeq G \times \mathfrak{g}$, where $\alpha = (L_{x^{-1}})_* y$ is the Maurer–Cartan form. The corresponding coordinates on TTG are (x, α, X, A), that is,

$$(x, \alpha, X, A) = X \frac{\partial}{\partial x}\bigg|_{(x,\alpha)} + A \frac{\partial}{\partial \alpha}\bigg|_{(x,\alpha)}.$$

Since the coordinates α and A are left-invariant, we find that left translation by a group element g induces on TTG

$$L_g(x, \alpha, X, A) = (gx, \alpha, gX, A) = gX \frac{\partial}{\partial x}\bigg|_{(gx,\alpha)} + A \frac{\partial}{\partial \alpha}\bigg|_{(gx,\alpha)}.$$

The canonical projection $\pi : TG \to G$ is $(x, \alpha) \to x$, therefore $\pi_* : TTG \to TG$ is given by $(x, \alpha, X, A) \to (x, x^{-1}X)$ and the vertical subspace on $(x, \alpha) \in TG$ is

$$V_{(x,\alpha)}TG := \operatorname{Ker} \pi_* = \big\{(x, \alpha, 0, b) \mid b \in \mathfrak{g}\big\}.$$

4.1. Horizontal and vertical lifts of left-invariant fields

A vector-field $X \in \mathfrak{X}(G)$ can be represented as a map $X : G \to G \times \mathfrak{g}$, $x \to (x, a(x))$, where $a(x) \in \mathfrak{g}$. A vector-field X is left-invariant, if and only if $a(x) = a$ is a constant map.

Let $X, Y : G \to G \times \mathfrak{g}$ be two left-invariant vector-field on G, $X : x \to (x, a)$, $Y : x \to (x, b)$. The integral curve of X passing through a point $x \in G$ is $L_x(\exp(tX)) = x(1 + ta + \frac{t^2}{2}a^2 + \cdots)$, and therefore

$$\begin{aligned} Y_* X(x) &= \frac{d}{dt}\bigg|_{t=0} Y_{L_x(\exp(tX))} = \frac{d}{dt}\bigg|_{t=0} L_x\big(\exp(tX)\big)_* Y \\ &= \left(x, b, \frac{d}{dt}\bigg|_{t=0} \left(x\left(1 + ta + \frac{t^2}{2}a^2 + \cdots\right), \frac{d}{dt}\bigg|_{t=0} b\right)\right) = (x, b, xa, 0). \end{aligned}$$

By the definition of ∇, we have

$$(\nabla_X Y)_x = \left(x, \frac{1}{2}[a, b]\right) \tag{22}$$

so from (13) taking $D = \nabla$ we obtain that

$$v(x, b, xa, 0) = \left(x, b, 0, \frac{1}{2}[a, b]\right) = \frac{1}{2}[a, b]\frac{\partial}{\partial\alpha}\bigg|_{(x,b)}, \tag{23}$$

and

$$h(x, b, xa, 0) = \left(x, b, xa, -\frac{1}{2}[a, b]\right) = xa\frac{\partial}{\partial x}\bigg|_{(x,b)} - \frac{1}{2}[a, b]\frac{\partial}{\partial\alpha}\bigg|_{(x,b)}. \tag{24}$$

Therefore the horizontal and vertical lifts of a left-invariant vector-field $X = (x, a)$ are

$$X^h_{(x,\alpha)} = h(x, \alpha, xa, 0) = \left(x, \alpha, xa, -\frac{1}{2}[a, \alpha]\right) = xa\frac{\partial}{\partial x}\bigg|_{(x,\alpha)} - \frac{1}{2}[a, \alpha]\frac{\partial}{\partial\alpha}\bigg|_{(x,\alpha)},$$
$$X^v_{(x,\alpha)} = JX^h_{(x,\alpha)} = (x, \alpha, 0, a) = a\frac{\partial}{\partial\alpha}\bigg|_{(x,\alpha)}.$$

We remark that at each point (x, α) of TG, the horizontal subspace $H_{(x,\alpha)}$ associated to ∇ is the average of the horizontal space determined by ∇^- and ∇^+.

4.2. *The spray and the geodesics*

Second order differential equations are described by vector-fields $\tilde{S}$ on TG that have the characteristic property $J\tilde{S} = C$. In the local coordinate system (x, α), $\tilde{S}$ is a spray if and only if it has the local form $\tilde{S}_{(x,\alpha)} = (x, \alpha, x\alpha, \tilde{f})$, that is,

$$\tilde{S}_{(x,\alpha)} = x\alpha\frac{\partial}{\partial x}\bigg|_{(x,\alpha)} + \tilde{f}\frac{\partial}{\partial\alpha}\bigg|_{(x,\alpha)}$$

for some functions $\tilde{f} = (f^i_j)$.

The spray S associated to the connection Γ can be computed as $S = h\tilde{S}$, where $\tilde{S}$ is an arbitrary spray on G. It, like any spray, has the characteristic property that the projection of its integral curves are the geodesics with respect to the associated covariant derivative ∇. In order to compute S we can choose for $\tilde{S}$ simply the spray corresponding to the functions $\tilde{f} = 0$, and we find that

$$S = h\tilde{S} = h(x, \alpha, x\alpha, 0) = \left(x, \alpha, x\alpha, \frac{1}{2}[\alpha, \alpha]\right) = (x, \alpha, x\alpha, 0),$$

that is, S and $\tilde{S}$ coincide. The equation of the geodesics is $S_{\dot{\gamma}} = \ddot{\gamma}$, where $\gamma = x_t$ is a path on G. Its first and second derivatives are $\dot{\gamma} = (x_t, \alpha_t)$ and $\ddot{\gamma} = (x, \alpha, x\alpha, \dot{\alpha}) = (x, \alpha, x\alpha, -x^{-1}\dot{x}x^{-1}\dot{x} + x^{-1}\ddot{x})$, since $\alpha = x^{-1}\dot{x}$. Therefore the spray corresponding to ∇ is encoded in the system

$$\ddot{x} = \dot{x}x^{-1}\dot{x}. \tag{25}$$

Of course to obtain the spray proper, one has to select a certain number of independent coordinates in the matrix x.

In order to solve the inverse problem we have to find necessary and sufficient conditions for a spray to be the Euler–Lagrange equations of a regular Lagrangian function. In practice we must obtain conditions that will ensure integrability of an over-determined system of partial differential equations. The first compatibility conditions are determined by the curvature tensor R, the Jacobi endomorphism Φ and a double hierarchy of tensors obtained from them by recursion [1,8,13]. In this section we will examine them in the case of a Lie group.

4.3. The curvature

The curvature R of a connection Γ is described by the Nijenhuis torsion of the associated horizontal projection: $R := -\frac{1}{2}[h,h]$. Let us compute $R(X^h, Y^h)$, where the X^h and Y^h are horizontal lifts of the left-invariant vector fields $X = (x, a)$ and $Y = (x, b)$. Using the formula

$$M\frac{\partial[\alpha, N]}{\partial\alpha}\frac{\partial}{\partial\alpha} = [M, N]\frac{\partial}{\partial\alpha},$$

where N denotes a constant matrix, we obtain

$$\begin{aligned}
[X^h, Y^h] &= \left[xa\frac{\partial}{\partial x} - \frac{1}{2}[a,\alpha]\frac{\partial}{\partial\alpha},\ xb\frac{\partial}{\partial x} - \frac{1}{2}[b,\alpha]\frac{\partial}{\partial\alpha}\right] \\
&= x[a,b]\frac{\partial}{\partial x} + \frac{1}{4}\big(\big[[\alpha,a],b\big] - \big[[\alpha,b],a\big]\big)\frac{\partial}{\partial\alpha} \\
&= x[a,b]\frac{\partial}{\partial x} + \frac{1}{4}\big[\alpha,[a,b]\big]\frac{\partial}{\partial\alpha} = \left(x, \alpha, x[a,b], \frac{1}{4}\big[\alpha,[a,b]\big]\right).
\end{aligned}$$

Now, using (23) we find that

$$\begin{aligned}
R_{(x,\alpha)}(X^h, Y^h) &= -\frac{1}{2}[h,h](X^h, Y^h) = -v[X^h, Y^h] \\
&= -\left(x, \alpha, 0, \frac{1}{2}\big[[a,b],\alpha\big] + \frac{1}{4}\big[\alpha,[a,b]\big]\right) \\
&= \left(x, \alpha, 0, \frac{1}{4}\big[\alpha,[a,b]\big]\right) = R^{\nabla}(X, Y, \alpha)^v,
\end{aligned}$$

that is, roughly speaking the curvature $R_{(x,\alpha)}$ as defined above is the vertical lift of the usual curvature R^{∇} associated to ∇.

4.4. The Jacobi endomorphism

Let Φ be the Jacobi endomorphism, $\Phi := v \circ [h, S]$ and $X := (x, a)$ a left-invariant field. Then

$$\Phi(X^h) = v\left[xa\frac{\partial}{\partial x} - \frac{1}{2}[a,\alpha]\frac{\partial}{\partial\alpha}, x\alpha\frac{\partial}{\partial x}\right] = v\left(\frac{1}{2}x[a,\alpha]\frac{\partial}{\partial x}\right) = \frac{1}{4}\big[[a,\alpha],\alpha\big]\frac{\partial}{\partial\alpha}$$

and so

$$\Phi(X^h) = R^{\nabla}(\alpha, X, \alpha)^v.$$

4.5. The higher order curvature and Jacobi tensors

The higher order tensors obtained from R and Φ are obtained by induction by means of the following formulas:

$$R^{(k+1)}(X,Y) := v[S, R^{(k)}](hX, hY), \qquad \Phi^{(k+1)}(X) := v[S, \Phi^{(k)}](hX).$$

(The bracket in the above formula is the Frölicher–Nijenhuis bracket. In the case when the first argument is a vector-field it means simply the Lie derivative with respect to that vector-field.)

Let $X = (x, a)$ and $Y = (x, b)$ left-invariant vector-fields. We have

$$R^1(X^h, Y^h) = v\big(\big[R(X^h, Y^h), S\big] - R\big([X^h, S], Y^h\big) - R\big(X^h, [Y^h, S]\big)\big).$$

Hence

$$[X^v, S] = \left[a\frac{\partial}{\partial\alpha}, x\alpha\frac{\partial}{\partial x}\right] = xa\frac{\partial}{\partial x},$$
$$[X^h, S] = \left[xa\frac{\partial}{\partial x} - \frac{1}{2}[a,\alpha]\frac{\partial}{\partial\alpha}, x\alpha\frac{\partial}{\partial x}\right] = \frac{1}{2}x[a,\alpha]\frac{\partial}{\partial x},$$

so using the fact that R is semi-basic we find

$$\begin{aligned}
R^1(X^h, Y^h) &= v\left[\frac{1}{4}\big[\alpha, [a,b]\big]\frac{\partial}{\partial\alpha}, x\alpha\frac{\partial}{\partial x}\right] - R\left(\frac{1}{2}x[a,\alpha]\frac{\partial}{\partial x}, xb\frac{\partial}{\partial x}\right) - R\left(xa\frac{\partial}{\partial x}, \frac{1}{2}x[b,\alpha]\frac{\partial}{\partial x}\right)\\
&= v\left(\frac{1}{4}x\big[\alpha, [a,b]\big]\frac{\partial}{\partial x}\right) - \frac{1}{8}\big[\alpha, [[a,\alpha], b]\big]\frac{\partial}{\partial\alpha} - \frac{1}{8}\big[\alpha, [a, [b,\alpha]]\big]\frac{\partial}{\partial\alpha}\\
&= \frac{1}{8}\big[[\alpha, [a, [b,\alpha]]], \alpha\big]\frac{\partial}{\partial\alpha} - \frac{1}{8}\big[\alpha, [[a,\alpha], b]\big]\frac{\partial}{\partial\alpha} - \frac{1}{8}\big[\alpha, [a, [b,\alpha]]\big]\frac{\partial}{\partial\alpha}\\
&= \frac{1}{8}\big[\alpha, \big[[a,b],\alpha\big] + \big[[b,\alpha],a\big] + \big[[\alpha,a],b\big]\big] = 0.
\end{aligned}$$

Of course, the higher order tensors $R^{(k)}$ vanish also. Moreover

$$\begin{aligned}
\Phi^1(X^h) &= v\circ[\Phi, S](hX^h) = v[\Phi X^h, S] - v\Phi[X^h, S]\\
&= v\left[\frac{1}{4}\big[[a,\alpha],\alpha\big]\frac{\partial}{\partial\alpha}, x\alpha\frac{\partial}{\partial x}\right] - \Phi\left(\frac{1}{2}x[a,\alpha]\frac{\partial}{\partial x}\right)\\
&= v\left(\frac{1}{4}x\big[[a,\alpha],\alpha\big]\frac{\partial}{\partial x}\right) - \frac{1}{8}\big[[[a,\alpha],\alpha],\alpha\big]\frac{\partial}{\partial\alpha}\\
&= \frac{1}{8}\big[[[a,\alpha],\alpha],\alpha\big]\frac{\partial}{\partial\alpha} - \frac{1}{8}\big[[[a,\alpha],\alpha],\alpha\big]\frac{\partial}{\partial\alpha} = 0
\end{aligned}$$

and also, the higher order tensors $\Phi^{(k)}$ vanish.

5. The inverse problem of the calculus of variations on Lie groups

As we explained at the end of Section 2, in order to solve the inverse problem of the Lagrange dynamics in the analytic category we have to study the integrability of the Euler–Lagrange differential operator P_1.

The method consists in principal of deriving successively a number of compatibility conditions and when no more can be obtained at a certain level, checking if the symbol of the system is involutive. If it is then the Cartan–Kähler ensures the existence of a solution. If not then the system must be prolonged and the entire process started over. According to the Cartan–Kuranishi theorem, the process must terminate at a finite stage at which point the Cartan–Kähler theorem is applicable or else the system is incompatible and there are no solutions. As a comprehensive reference we cite [2] and also [9] as it pertains to the inverse problem. In Sections 5 and 6 we shall adopt the following convention so as to ease the notation: the tangent and cotangent bundle of G will be denoted simply by T and T^*. Similarly instead of TTG we shall write TT.

The computations made in [8] or [9] show that the first compatibility condition for P_1 is given by the equation

$$i_\Gamma \Omega_E = 0, \tag{26}$$

where $\Omega_E := dd_J E$. The geometric meaning of this condition is the following: if ∇ is variational and E is a regular Lagrangian associated to it, then the horizontal distribution associated to ∇ must be Lagrangian with respect to the symplectic 2-form Ω_E. If the dimension of G is greater than one, the above condition is not satisfied identically. In order to incorporate the condition (26) we introduce the operator

$$P_\Gamma : C^\infty(T) \to \Lambda^2 T^*, \qquad P_\Gamma(E) = i_\Gamma\, dd_J E, \tag{27}$$

and define the system $P_2 := (P_1, P_\Gamma)$.

We refer to [8] and [9] once again, where the compatibility condition of P_2 is calculated; namely, a second order formal solution E can be lifted into a third order formal solution if and only if the equations

$$i_\Phi \Omega_E = 0, \tag{28}$$

$$i_R \Omega_E = 0, \tag{29}$$

hold, where Φ is the Jacobi endomorphism and R is the curvature introduced in Section 2. The operator $P_2 = (P_1, P_\Gamma)$ is regular and involutive (see [9]). Therefore in the case where the curvature is zero or (that is $\Phi = \lambda J$ for some scalar function λ that in fact must be identically zero in the case of a linear connection), the system is formally integrable. It is clear from the formula for the curvature given in the introduction that the canonical connection is flat if and only if G is nilpotent of order two. Hence:

Proposition 5.1. *The canonical connection of a two-step nilpotent Lie group is variational. In particular The canonical connection of a commutative Lie group is variational.*

If ∇ is non-flat, then Eqs. (28) and (29) are not satisfied identically. Therefore we must study the integrability of the system

$$\begin{cases} i_S \Omega_E + d\mathcal{L}_C E - dE = 0, \\ i_\Gamma \Omega_E = 0, \\ i_\Phi \Omega_E = 0, \\ i_R \Omega_E = 0, \end{cases} \tag{30}$$

where $\Omega_E := dd_J E$. The differential operator corresponding to the system (30) is

$$P_3 : C^\infty(TG) \to F_3,$$

defined by

$$P_3 := (P_1, P_\Gamma, P_\Phi, P_R),$$

where $F_3 := T^* \oplus \Lambda^2 T^* \oplus \Lambda^2 T^* \oplus \Lambda^3 T^*$ and

$$P_\Phi : C^\infty(TG) \to \Lambda^2 T^*, \qquad P_\Phi(E) = i_\Phi\, dd_J E, \tag{31}$$

$$P_R : C^\infty(TG) \to \Lambda^3 T^*, \qquad P_R(E) = i_R\, dd_J E, \tag{32}$$

are also second order linear differential operators.

The linear partial differential operator P_3 is of second order. Its symbol is the map $\sigma_2(P_3) : S^2 T^* \to F_3$ given by the product of the symbols of the operators out of which it is formed:

$$\sigma_2(P_3) = \sigma_2(P) \times \sigma_2(P_\Gamma) \times \sigma_2(P_\Phi) \times \sigma_2(P_R),$$

where

$$\begin{aligned}
&\big(\sigma_2(P_1)B\big)(X) = B(S, JX),\\
&\big(\sigma_2(P_\Gamma)B\big)(X, Y) = 2\big(B(hX, JY) - B(hY, JX)\big),\\
&\big(\sigma_2(P_\Phi)B\big)(X, Y) = B(\Phi X, JY) - B(\Phi Y, JX),\\
&\big(\sigma_2(P_R)B\big)(X, Y, Z) = B\big(R(X, Y), JZ\big) + B\big(R(Y, Z), JX\big) + B\big(R(Z, X), JY\big)
\end{aligned}$$

for every $B \in S^2 T^*$, and $X, Y, Z \in T$.

Remark 5.2. Although the curvature tensor R^∇ is constant on the Lie group G, the rank of the Jacobi endomorphism Φ and the curvature tensor R on TG can change depending on the point in TG. Therefore the differential operator P_3 is regular only in a neighborhood of a generic point.

For the symbol of the first prolongation $\sigma_3(P_3)$ of P_3 we find that $\sigma_3(P_3) : S^3 T^* \to T^* \otimes F_3$, where $\sigma_3(P_3) = \sigma_3(P) \times \sigma_3(P_\Gamma) \times \sigma_3(P_\Phi) \times \sigma_3(P_R)$ and the component maps are

$$\begin{aligned}
&\big(\sigma_3(P_1)B\big)(X, Y) = B(X, S, JY),\\
&\big(\sigma_3(P_\Gamma)B\big)(X, Y, Z) = 2\big(B(X, hY, JZ) - B(X, hZ, JY)\big),\\
&\big(\sigma_3(P_\Phi)B\big)(X, Y, Z) = B(X, \Phi Y, JZ) - B(X, \Phi Z, JY),\\
&\big(\sigma_3(P_R)B\big)(X, Y, Z, W) = \sum_{YZW}^{\text{cycl}} B\big(X, R(Y, Z), JW\big).
\end{aligned}$$

Let us consider the map $\tau : T^* \otimes F_3 \to K$ with

$$\tau := \tau_\Gamma \oplus \tau_\Phi \oplus \tau_R \oplus \tau_{[J,J]} \oplus \tau_{\Phi'} \oplus \tau_{[h,\Phi]} \oplus \tau_{[\Phi,\Phi]} \oplus \tau_{[J,\Phi]} \oplus \tau_{R'} \oplus \tau_{[J,R]} \oplus \tau_{[h,R]} \oplus \tau_{[\Phi,R]}$$

where the corresponding maps are defined as follows: for every

$$\xi := (B, C_\Gamma, C_\Phi, D_R) \in T^* \otimes F_3 \equiv (T^* \otimes T^*) \oplus (T^* \otimes \Lambda^2 T^*) \oplus (T^* \otimes \Lambda^2 T^*) \oplus (T^* \otimes \Lambda^3 T^*)$$

we set

$$(\tau_\Gamma \xi)(X, Y) := B(JX, Y) - B(JY, X),$$

$$
\begin{aligned}
&(\tau_\Phi \xi)(X,Y) := B(hX,Y) - B(hY,X) - \frac{1}{2} C_\Gamma(S,X,Y),\\
&(\tau_R \xi)(X,Y,Z) := C_\Gamma(hX,Y,Z) + C_\Gamma(hY,Z,X) + C_\Gamma(hZ,X,Y),\\
&(\tau_{[J,J]} \xi)(X,Y,Z) := C_\Gamma(JX,Y,Z) + C_\Gamma(JY,Z,X) + C_\Gamma(JZ,X,Y),\\
&(\tau_{\Phi'} \xi)(X,Y) := B(\Phi X,Y) - B(\Phi Y,X) - C_\Phi(S,X,Y),\\
&(\tau_{[h,\Phi]} \xi)(X,Y,Z) := \sum_{XYZ}^{\text{cyc}} C_\Phi(hX,Y,Z) - \frac{1}{2} \sum_{XYZ}^{\text{cyc}} C_\Gamma(\Phi X,Y,Z),\\
&(\tau_{[\Phi,\Phi]} \xi)(X,Y,Z) := \sum_{XYZ}^{\text{cyc}} C_\Phi(\Phi X,Y,Z),\\
&(\tau_{[J,\Phi]} \xi)(X,Y,Z) := \sum_{XYZ}^{\text{cyc}} C_\Phi(JX,Y,Z),\\
&(\tau_{R'} \xi)(X,Y,Z) := \sum_{XYZ}^{\text{cyc}} B(R(X,Y),W) - C_R(S,X,Y,W),\\
&(\tau_{[J,R]} \xi)(X,Y,Z,W) := \sum_{XYZ}^{\text{cyc}} C_R(JX,Y,Z,W),\\
&(\tau_{[h,R]} \xi)(X,Y,Z,W) := \sum_{XYZ}^{\text{cyc}} C_R(hX,Y,Z,W),\\
&(\tau_{[\Phi,R]} \xi)(X,Y,Z,W) := \sum_{XYZ}^{\text{cyc}} C_\Phi(R(X,Y),Z,W) - \sum_{XYZ}^{\text{cyc}} C_R(\Phi X,Y,Z,W).
\end{aligned}
$$

A simple computation shows that $\operatorname{Im}\sigma_3(P_3) \subset \operatorname{Ker}\tau$. Taking an arbitrary linear connection D on TG we can compute the map $\varphi := \tau \circ D \circ p_0(P_3)$ defined on the space of second order solutions of P_3: it gives compatibility conditions on P_3 (cf. [2,9]).

Remark 5.3. The map τ is "universal", in the sense that it has the smallest kernel that can be constructed in the general situation without imposing any restriction on G.

It is not difficult to show that if E is a second order solution of P_3 at a point $v \in TG$, that is, $P_3E(v) = 0$, then

$$
\begin{aligned}
(\varphi_E)_v = \big(&i_\Gamma \Omega_E, i_\Phi \Omega_E, i_R \Omega_E, i_{[J,J]} \Omega_E, i_{\Phi'} \Omega_E, i_{[h,\Phi]} \Omega_E,\\
&i_{[\Phi,\Phi]} \Omega_E, i_{[J,\Phi]} \Omega_E, i_{R'} \Omega_E, i_{[J,\Phi]} \Omega_E, i_{[h,R]} \Omega_E, i_{[\Phi,R]} \Omega_E\big),
\end{aligned}
$$

and the new conditions that permit a second order solution E to be lifted to a third order solution are:

$$i_{\Phi^1} \omega_E = 0, \tag{33}$$
$$i_{R^1} \Omega_E = 0, \tag{34}$$
$$i_{[h,R]} \Omega_E = 0, \tag{35}$$
$$i_{[R,R]} \Omega_E = 0, \tag{36}$$

$$i_{[J,R]}\Omega_E = 0, \tag{37}$$
$$i_{[h,\Phi]}\Omega_E = 0, \tag{38}$$
$$i_{[\Phi,R]}\Omega_E = 0, \tag{39}$$
$$i_{[\Phi,\Phi]}\Omega_E = 0. \tag{40}$$

Eqs. (33) and (34) hold since all derivatives of R and Φ vanish. From the Bianchi identity ($[h,[h,h]] = 0$) we have also $[R,R]=0$, and therefore $i_{[h,R]}\Omega = 0$ and $i_{[R,R]}\Omega = 0$. For the other compatibility conditions we have:

$$[J,R] = -\left[J, \frac{1}{2}[h,h]\right] = \big[h,[J,h]\big] \overset{t=0}{=} 0,$$

so (37) holds. The vectorial 2-form $[h,\Phi]$ is semi-basic, hence $d_{[\Phi,h]}d_J E = i_{[\Phi,h]}\Omega_E$. Also the torsion $t = \frac{1}{2}[J,\Gamma]$ is zero, so

$$\begin{aligned}[h,\Phi] &= \big[h,[h,S]-h[h,S]\big] = \big[h,[h,S]\big] + [h, F+J] \\ &= \big[h,[h,S]\big] + [h,F] + [h,J] \overset{t=0}{=} -[R,S] + FR - R \barwedge F = R^1.\end{aligned} \tag{41}$$

Since $R^1 = 0$, Eq. (38) is satisfied. Using once again that $R^1 = 0$ we find that

$$[\Phi,R] = \left[\Phi, \frac{1}{2}[h,h]\right] = -\frac{1}{2}\big[h,[h,\Phi]\big] - \frac{1}{2}\big[h,[\Phi,h]\big] = \big[h,[\Phi,h]\big] = [h,R^1] = 0.$$

and (39) is satisfied.

In order to compute (40) we remark that if N is a constant matrix, then

$$M\frac{\partial[[N,x],x]}{\partial x}\frac{\partial}{\partial x} = \big(\big[[N,M],x\big] + \big[[N,x],M\big]\big)\frac{\partial}{\partial x}.$$

Indeed,

$$\begin{aligned}M\frac{\partial[[N,x],x]}{\partial x}\frac{\partial}{\partial x} &= M^i_j \frac{\partial}{\partial x^i_j}\big(N^\alpha_k x^\beta_\alpha x^l_\beta - 2x^\alpha_k N^\beta_\alpha x^l_\beta + x^\alpha_k x^\beta_\alpha N^l_\beta\big)\frac{\partial}{\partial x^l_k} \\ &= M^i_j\big(N^\alpha_k \delta^j_\alpha \delta^i_\beta x^l_\beta + N^\alpha_k x^\beta_\alpha \delta^j_\beta \delta^i_l - 2\delta^j_k \delta^i_\alpha N^\beta_\alpha x^l_\beta - 2x^\alpha_k N^\beta_\alpha \delta^j_\beta \delta^i_l \\ &\quad + \delta^j_k \delta^i_\alpha x^\beta_\alpha N^l_\beta + x^\alpha_k \delta^j_\alpha \delta^i_\beta N^l_\beta\big)\frac{\partial}{\partial x^l_k} \\ &= \big(\big[[N,M],x\big] + \big[[N,x],M\big]\big)\frac{\partial}{\partial x}.\end{aligned}$$

Now let $X = (x,a)$ and $Y = (x,b)$ be two left-invariant vector fields on G and X^h and Y^h their horizontal lifts. Using the fact that Φ is semi-basic we have $\Phi \circ \Phi = 0$ and also since

$$\begin{aligned}\Phi[\Phi X^h, Y^h]_{(x,\alpha)} &= \Phi\left[\left(x,\alpha,0,\frac{1}{4}\big[[a,\alpha],\alpha\big]\right), \left(x,\alpha,xb,\frac{1}{2}[b,\alpha]\right)\right] \\ &= \Phi\left[\frac{1}{4}\big[[a,\alpha],\alpha\big]\frac{\partial}{\partial\alpha}, xb\frac{\partial}{\partial x} + \frac{1}{2}[b,\alpha]\frac{\partial}{\partial\alpha}\right] = 0,\end{aligned}$$

we have $[\Phi,\Phi](X^h,Y^h)=[\Phi X^h,\Phi Y^h]$. Therefore

$$\begin{aligned}
[\Phi,\Phi](X^h,Y^h)=[\Phi X^h,\Phi Y^h]&=\left[\frac{1}{4}\big[[a,\alpha],\alpha\big]\frac{\partial}{\partial\alpha},\frac{1}{4}\big[[b,\alpha],\alpha\big]\frac{\partial}{\partial\alpha}\right]\\
&=\frac{1}{16}\Big(\Big[\Big(\big[[[b,\alpha],\alpha],a\big]-\big[[[a,\alpha],\alpha],b\big]-\big[[a,\alpha],[b,\alpha]\big]\Big),\alpha\Big]\Big)\frac{\partial}{\partial\alpha}\\
&=\frac{1}{16}\Big(\big[[\Phi(b),a],\alpha\big]-\big[[\Phi(a),b],\alpha\big]-\big[[[a,\alpha],[b,\alpha]],\alpha\big]\Big)\frac{\partial}{\partial\alpha}\\
&=\frac{1}{16}\Big(R\big(\Phi(X^h),Y^h,\alpha\big)-R\big(\Phi(Y^h),X^h,\alpha\big)+R\big([X^h,\alpha],[Y^h,\alpha],\alpha\big)\Big)\frac{\partial}{\partial\alpha}.
\end{aligned}$$

In the generic case the vectorial 2-form $[\Phi,\Phi]$ is not linearly related to R and Φ and therefore Eq. (40) represents potentially a new compatibility condition for the system. As we will see in the next section, a nontrivial example where this condition is identically satisfied is given by ASO_n, the group of Euclidean transformations of $\mathbb{R}^n$.

6. Worked example: The Euclidean group

In this section we shall apply the theory developed in the previous sections to the Euclidean group ASO_n of $\mathbb{R}^n$, that is, the group generated by all rotations and translations. It consists of maps of the form:

$$\begin{aligned}
&\mathbb{R}^n\to\mathbb{R}^n\\
&x\mapsto Ax+t
\end{aligned}$$

where $A\in SO(n)$ is a rotation matrix. We represent ASO_n as a matrix Lie group, in fact as a subgroup of $GL(n+1,\mathbb{R})$ by

$$\mathrm{ASO}_n=\left\{\begin{pmatrix}A & t\\ 0 & 1\end{pmatrix}\,\middle|\, t\in\mathbb{R}^n,\ A\in SO(n)\right\},$$

and the action on $\mathbb{R}^n$ is given as a matrix action when we write an element of $\mathbb{R}^n$ as $x={}^t(x^1,\dots,x^n,1)$. Therefore the Lie algebra of ASO_n can be represented as

$$\mathfrak{aso}_n=\left\{\begin{pmatrix}M & t\\ 0 & 0\end{pmatrix}\,\middle|\, t\in\mathbb{R}^n,\ M\in so(n)\right\}.$$

The system $\mathfrak{B}:=\{E_{i,j},E_k\mid j<i,\ i,j,k=1\dots n\}$ give us a basis of $\mathfrak{aso}_n$, where E_{ij} denotes a skew-symmetric matrix with 1 in the ith row and jth column, -1 in the jth row and ith column, and 0 elsewhere, and E_i denotes the matrix with 1 in the ith row and $(n+1)$th column and 0 elsewhere. The bracket relations of the Lie algebra are given by:

$$\begin{aligned}
&[E_i,E_j]=0,\\
&[E_{ij},E_k]=-\delta_{ik}E_j+\delta_{jk}E_i,\\
&[E_{ij},E_{kl}]=\delta_{jk}E_{il}+\delta_{il}E_{jk}-\delta_{jl}E_{ik}-\delta_{ik}E_{jl}.
\end{aligned}$$

Let us compute the components of the curvature tensor R_v at a tangent vector $v=(x,\alpha)\in T_x\mathrm{ASO}_n$, where $\alpha=\alpha^{mn}E_{mn}+\alpha^mE_m$. In order to make the computation as simple as possible, we use the notation

$E_{ji} = -E_{ij}$ for $i > j$. Thus:

$$\begin{aligned} R_v(E_{ij}, E_{kl}) = R^\nabla(E_{ij}, E_{kl}, v) = {}& \frac{1}{4}\delta_k^j\big(\alpha^{lm}E_{mi} - \alpha^{im}E_{ml} + \alpha^i E_l - \alpha^l E_i\big) \\ &+ \frac{1}{4}\delta_l^i\big(\alpha^{km}E_{mj} - \alpha^{jm}E_{mk} + \alpha^j E_k - \alpha^k E_j\big) \\ &- \frac{1}{4}\delta_k^i\big(\alpha^{lm}E_{mj} - \alpha^{jm}E_{ml} + \alpha^j E_l - \alpha^l E_j\big) \\ &- \frac{1}{4}\delta_l^j\big(\alpha^{km}E_{mi} - \alpha^{im}E_{mk} + \alpha^i E_k - \alpha^k E_i\big), \end{aligned}$$

$$\begin{aligned} &R_v(E_{ij}, E_j) = R^\nabla(E_{ij}, E_j, v) = \frac{1}{4}\delta_j^k(\alpha^{mi} - \alpha^{im})E_m - \frac{1}{4}\delta_i^k(\alpha^{mj} - \alpha^{jm})E_m, \\ &R_v(E_i, E_j) = R^\nabla(E_i, E_j, v) = 0. \end{aligned}$$

Remark 6.1. It is not difficult to check that the Nijenhuis bracket of the Jacobi endomorphism is proportional to the curvature; in fact $[\Phi, \Phi] = \frac{1}{4}R$. Since from the integrability conditions (33)–(40) only the last one can be non-trivial, we conclude that all the compatibility conditions found in the previous section are satisfied.

In the sequel we focus on the three dimensional group ASO_2 and we show:

Theorem 6.2. *The canonical connection of* ASO_2 *is variational.*

To prove the theorem we have to examine the integrability of the system (30) which contains the Euler–Lagrange equations and its first compatibility conditions. We denote by e_1, e_2 and e_{12} the left-invariant vector-fields on ASO_2 corresponding to the element E_1, E_2 and E_{12} of $\mathfrak{aso}_2$.

Remark 6.3. Let v be an arbitrary vector of $T\mathrm{ASO}_2$ represented as (x, α) in $\mathrm{ASO}_2 \times \mathfrak{aso}_2$. From the formulae of the curvature above we obtain that

$$R_v = \omega \wedge J,$$

where $\omega_v = \alpha_{12}e^{12}$, e^{12} being the dual one-form corresponding to e_{12}.

For this reason the operator P_R can be removed from P_3 because the equation represented by it is identically satisfied. Indeed, for every Lagrangian E we have

$$i_R\Omega_E = i_{\omega\wedge J}\Omega_E = \omega \wedge i_J\, dd_J E = \omega \wedge d_J d_J E = \omega \wedge d_{[J,J]}E = 0$$

because $[J, J] = 0$. So it is sufficient to consider the operator

$$P_{\mathrm{ASO}_2} : C^\infty(T\mathrm{ASO}_2) \to F_{\mathrm{ASO}_2}, \tag{42}$$

where $P_{\mathrm{ASO}_2} := (P_1, P_\Gamma, P_\Phi)$ and $F_{\mathrm{ASO}_2} := T^* \oplus \Lambda^2T^* \oplus \Lambda^2T^*$.

Therefore, to prove Theorem 6.2 it is sufficient to show that there exists a second order regular solution for P_{ASO_2} (Lemma 6.4), and that P_{ASO_2} is formally integrable, that is, every second order solution can be lifted to an infinite order solution (Lemmas 6.5 and 6.6).

Lemma 6.4. *For every $v \neq 0$ in $T\mathrm{ASO}_2$ there exists a second order regular formal solution to P_{ASO_2}.*

Proof. Let (x^i) be a local coordinate system on ASO_2, and (x^i, y^i) the associated coordinate system on $T\mathrm{ASO}_2$ in the neighborhood of v. If E is a function on $T\mathrm{ASO}_2$, then its second order jet at v will be denoted by

$$(j_2E)_x = \big(x^i, v^i, p, p_j, p_{\underline{j}}, p_{jk}, p_{j\underline{k}}, p_{\underline{jk}}\big),$$

where $p = E(v)$, and $p_i = \frac{\partial E}{\partial x^i}$, $p_{\underline{i}} = \frac{\partial E}{\partial y^i}$, $p_{ij} = \frac{\partial^2 E}{\partial x^i \partial x^j}$, $p_{i\underline{j}} = \frac{\partial^2 E}{\partial x^i \partial y^j}$, $p_{\underline{ij}} = \frac{\partial^2 E}{\partial y^i \partial y^j}$ are the corresponding derivatives computed at v. Now $(j_2E)_x$ is a second order regular solution of P_{ASO_2} at v if and only if

$$\det(p_{\underline{ij}}) \neq 0, \tag{43}$$

and $(P_1E)_v = 0$, $(P_\Gamma E)_v = 0$ and $(P_{\mathrm{ASO}_2}E)_v = 0$ are satisfied, that is, if we have (43) and the *linear system*:

$$v^\alpha p_{\alpha\underline{i}} + f^\alpha p_{\underline{\alpha i}} - p_i = 0, \tag{44}$$

$$p_{j\underline{i}} - p_{i\underline{j}} + \Gamma_i^\alpha p_{\underline{\alpha j}} - \Gamma_j^\alpha p_{\underline{\alpha i}} = 0, \tag{45}$$

$$\Phi_i^\alpha p_{\underline{\alpha j}} - \Phi_j^\alpha p_{\underline{\alpha i}} = 0, \tag{46}$$

where f^α are the components of the spray and Γ_β^α are the coefficients of the connection $\Gamma = [J, S]$. Computing the Jacobi endomorphism Φ with the help of the formula $\Phi = i_S R$ we find that in the basis $\mathcal{B} = \{e_1^h, e_2^h, S, e_1^v, e_2^v, C\}$ at $v = (x, \alpha)$ the matrix of Φ is $\begin{pmatrix} 0 & 0 \\ \Phi_i^j & 0 \end{pmatrix}$, where 0 denotes the 3×3 zero matrix and

$$(\Phi_i^j) = \begin{pmatrix} \alpha_{12} & 0 & 0 \\ 0 & \alpha_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now let g be a scalar product of T_x^v so that the basis $\{e_1^v, e_2^v, C\}$ is orthogonal. If $(p_{\underline{ij}})$ is the matrix of g with respect to the basis $\{\frac{\partial}{\partial y^i}\}_{i=1,\ldots,n}$, we find that (43) and (46) are satisfied. Solving the system (44) and (45) with respect to the pivot terms p_i and $p_{j\underline{i}}$ we arrive at a regular second order formal solution of P_{ASO_2} at v. $\square$

Lemma 6.5. *Every second order solution of P_{ASO_2} can be lifted into a 3rd order solution.*

Proof. Let $\tau_{\mathrm{ASO}_2} := (\tau_\Gamma \oplus \tau_A \oplus \tau_{[J,J]} \oplus \tau_{A'} \oplus \tau_{[h,A]} \oplus \tau_{[A,A]} \oplus \tau_{[J,A]} \oplus \rho)$ defined on $T^* \otimes F_{\mathrm{ASO}_2}$, where the maps τ_Γ, τ_A, $\tau_{[J,J]}$, $\tau_{A'}$, $\tau_{[h,A]}$, $\tau_{[A,A]}$ and $\tau_{[J,A]}$ and corresponding formulas are defined on page 270 and let

$$\big(\rho(B, C_\Gamma, C_\Phi)\big)(X, Y) := C_\Phi(hX, Y, S) - \frac{1}{2}C_\Gamma(X, Y, S) - B(AY, X)$$

for every $(B, C_\Gamma, C_\Phi) \in (T^* \otimes T^*) \oplus (T^* \otimes \Lambda^2 T^*) \oplus (T^* \otimes \Lambda^2 T^*)$ and $X, Y \in \mathcal{H} := \langle e_1^h, e_2^h \rangle$: the space generated by the horizontal lift of the infinitesimal translations e_1 and e_2. It is easy to show that $\mathrm{Im}\,\sigma_3(P_{\mathrm{ASO}_2}) = \mathrm{Ker}\,\tau_{\mathrm{ASO}_2}$. Therefore, taking an arbitrary linear connection D on TM we can compute the map $\varphi := \tau_{\mathrm{ASO}_2} \circ D \circ p_0(P_{\mathrm{ASO}_2})$ defined on the space of second order solutions of P_3: it gives all the compatibility conditions of P_3 (cf. [9]).

Using the computation of the previous section and the Remark 6.1 we obtain

$$\varphi_E = \big(0, 0, 0, 0, 0, 0, 0, \rho(DP_{\mathrm{ASO}_2}E)\big)$$

for every second order solution $j_2(E)$ of P_{ASO_2}. Let us compute $\rho(DP_{\mathrm{ASO}_2}E)$: if $X, Y \in \mathcal{H}$, then

$$\begin{aligned}\rho(DP_{\mathrm{ASO}_2}E)(X,Y) &= X\big(\Omega_E(\Phi Y, S)\big) - \frac{1}{2}\Phi Y\big(i_\Gamma \Omega_E(X,S)\big) - \Phi Y\big(\omega_E(X)\big)\\ &= X\Omega_E(\Phi Y, S) - \Phi Y \Omega_E(X,S) - d\omega_E(\Phi Y, X).\end{aligned}$$

Since

$$d\omega_E = d(i_S dd_J E + d\mathcal{L}_C E - dE) = di_S dd_J E = \mathcal{L}_S \Omega_E,$$

we obtain

$$\begin{aligned}\rho(DP_{\mathrm{ASO}_2}E)(X,Y) &= \Omega_E\big([S,\Phi Y], X\big) + \Omega_E\big(\Phi Y, [S,X]\big) + \sum_{X,\Phi Y,S} X\Omega_E(\Phi Y, S)\\ &= d\Omega_E(X, \Phi Y, S) + \Omega_E\big([X,\Phi Y], S\big) = \Omega_E\big([X,\Phi Y], S\big)\end{aligned}$$

because $d\Omega_E = d^2 d_J E = 0$. Replacing X and Y by the generators e_i^h and e_j^h in the above formula and using the fact that $[e_i^h, e_j^v] = 0$ we find that at v

$$\begin{aligned}\rho(DP_{\mathrm{ASO}_2}E)\big(e_i^h, e_j^h\big) &= \Omega_E\big(\big[e_i^h, \Phi e_j^h\big], S\big) = \Omega_E\big(\big[e_i^h, \alpha_{12} e_j^v\big], S\big)\\ &= \big(e_i^h \alpha_{12}\big)\Omega_E\big(e_j^v, S\big) = \frac{(e_i^h\alpha_{12})}{\alpha_{12}} i_A \Omega_E\big(e_j^v, S\big) = 0,\end{aligned}$$

because, E being a second order solution at v, $i_A\Omega_E$ vanishes on $T_v\mathrm{ASO}_2$. Hence for every second order solution E the map φ_E is zero and so every second order solution can be lifted into a third order solution. This proves the lemma. □

Lemma 6.6. *The symbol of P_{ASO_2} is involutive.*

Proof. To prove the lemma we have to find a quasi-regular basis [2,9], that is, a basis $\{v_i\}_{i=1}^6$ of T^* such that

$$\dim g_2(P_{\mathrm{ASO}_2}) + \sum_{k=1}^{6} \dim g_2(P_{\mathrm{ASO}_2})_{v_1\dots v_k} = \dim g_3(P_{\mathrm{ASO}_2}), \tag{47}$$

where $g_2(P_{\mathrm{ASO}_2}) = \operatorname{Ker}\sigma_2(P_{\mathrm{ASO}_2})$ is the kernel of the symbol of P_{ASO_2}, $g_3(P_{\mathrm{ASO}_2}) := \operatorname{Ker}\sigma_3(P_{\mathrm{ASO}_2})$ is the kernel of the symbol of the first prolongation, and $g_2(P_{\mathrm{ASO}_2})_{v_1\dots v_k} := \{B \in g_2(P_{\mathrm{ASO}_2}) \mid B(v_i, .) = 0,\ i = 1\dots k\}$.

An easy computation shows that the basis $\{v_i\}_{i=1}^6$ satisfies the condition (47), where

$$\begin{aligned}&v_1 := e_1^h + C, \qquad v_2 := e_2^h + e_1^v, \qquad v_3 := S + e_2^v,\\ &v_4 := e_1^v, \qquad v_5 := e_2^v, \qquad v_6 := C. \qquad \square\end{aligned}$$

7. The Helmholtz conditions and more examples

In the previous two sections we have applied Spencer theory to study the existence of Lagrangians for the canonical symmetric connection of a Lie group. Spencer theory is probably the best theoretical tool for establishing existence of such Lagrangians and we have given details for the Euclidean group of the plane as a means of providing a nice worked example of the theory. It has been known since the work of Douglas [4] however, in the context of the general inverse problem for second order systems, that an alternative to studying the Euler–Lagrange operator directly is to work with the Helmholtz conditions. Indeed apart from [8,9] almost all recent investigations into the inverse problem have proceeded through the Helmholtz conditions.

The Helmholtz conditions are formulated in terms of "the multiplier" or Hessian g_{ij} which is related to the two-form of Definition 3.6 by

$$\Omega_E = g_{\alpha\beta}\, dx^\alpha \wedge \theta^\beta \tag{48}$$

where θ^β are the horizontal one-forms of the connection, that is, $dy^\beta + \Gamma^\beta_{\alpha\gamma} y^\alpha\, dx^\gamma$. They consist of in addition to (28)

$$\frac{dg_{\alpha\beta}}{dt} - y^\delta \Gamma^\gamma_{\alpha\delta} g_{\gamma\beta} - y^\delta \Gamma^\gamma_{\beta\delta} g_{\gamma\alpha} = 0 \tag{49}$$

and

$$\frac{\partial g_{\alpha\beta}}{\partial y^\gamma} - \frac{\partial g_{\alpha\gamma}}{\partial y^\beta} = 0. \tag{50}$$

Of course the Helmholtz conditions can easily be formulated for general second order systems but we shall have no need to do so here. Notice that (28) is now to be regarded as an algebraic condition whereas (49) and (50) are systems of ordinary and partial differential equations, respectively. In the Helmholtz approach if a non-degenerate solution can be found, the Lagrangian can found by means of two quadratures and the addition of suitable terms linear in y^α; there is no obstruction to the construction of these terms and the only ambiguity in the Lagrangian is the addition of a total time derivative. It turns out that (29) arises as an integrability condition so it may as well be appended to the other conditions from the outset. Now (28) and (29) are linear algebraic conditions that in principle can be solved. Likewise there is no obstruction to solving (49), though it may be difficult to solve them in practice. The hardest part by far is to solve (50) of which there are in general $2\binom{n+1}{3}$ conditions for an n-dimensional Lie group. On the other hand if one is interested in finding just a single Lagrangian it may not be necessary to integrate or analyze (50) in complete generality.

Throughout this article we have assumed that we are given a linear group G at the outset. However, it is interesting to observe that (28) and (29) can be solved purely at a Lie algebra level where in differential geometry one often prefers to start. In fact in some cases one is able to conclude that the multiplier is singular and hence the canonical connection of any associated Lie group is not variational. If one does start with an n-dimensional Lie algebra $\mathfrak{g}$ and the multiplier is not forced to be singular one is then faced with the problem of finding a local vector field representation of $\mathfrak{g}$ in terms of vector fields on R^n, if one wants to obtain the equations for the geodesics of ∇ locally. Such a representation is guaranteed by Lie's third theorem [10].

There is one final general remark that we shall make assuming that we have been able to find a local representation for the geodesics of ∇. One is at liberty to change coordinates so as to simplify the geodesics before solving the Helmhlotz conditions. Again, if only a single Lagrangian is required the transformation may lead easily to the construction of such a Lagrangian.

We proceed in this final section by considering some special kinds of Lie algebra that occur in the inverse problem. The results complement the theory of the previous sections.

Example 7.1 (*Lie algebras nilpotent of order two*). Let $\mathfrak{g}$ be a finite dimensional Lie algebra associated to a Lie group G. We know that the canonical connection on G is flat if and only if g is nilpotent of order two and Proposition 5.1 showed that a Lagrangian exists in this case. We shall now construct one concretely. The necessary and sufficient condition for g to be nilpotent of order two is that $\mathfrak{g}' \subset Z(\mathfrak{g})$, where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is the derived algebra and $Z(\mathfrak{g})$ denotes the center of $\mathfrak{g}$. We can introduce a basis $\{e_i, e_A\}$ for $\mathfrak{g}$ where $1 \leqslant i \leqslant p$ and $p+1 \leqslant A \leqslant n$ such that $\{e_i\}$ is a basis for $Z(\mathfrak{g})$. The Lie bracket relations of $\mathfrak{g}$ are given by

$$[e_i, e_j] = 0, \qquad [e_i, e_A] = 0, \qquad [e_A, e_B] = \gamma^i_{AB} e_i,$$

where $\gamma^i_{AB} = -\gamma^i_{BA}$. Clearly the Jacobi identity is satisfied because all second order Lie brackets vanish.

We can obtain a faithful local representation for $\mathfrak{g}$ in terms of vector fields on $\mathbb{R}^n$ as follows: put

$$e_i = \frac{\partial}{\partial x^i}, \qquad e_A = \frac{\partial}{\partial w^A} - \frac{1}{2}\gamma^i_{AB}\frac{\partial}{\partial x^i}.$$

If we compute the connection components of the canonical connection we find that they are all zero. Hence the geodesic equations are given by

$$\ddot{x}^i = 0, \qquad \ddot{w}^A = 0.$$

Clearly then $h := \sum (dx^i)^2 + \sum (dw^A)^2$ is a Riemannian metric compatible with ∇. However, h is not bi-invariant because, according to [12], a bi-invariant metric exists on a Lie group G if and only if G is a product of compact and abelian Lie groups.

A particular case of two-step nilpotent algebras are the Heisenberg algebras $\mathfrak{g}$ of dimension $2n+1$ which are characterized by the extra property that the derived algebra $\mathfrak{g}'$ is one-dimensional. There is a basis $\{e_1, e_2, e_3, \ldots, e_{2n+1}\}$ such that the only non-zero brackets are given by

$$[e_p, e_{n+p}] = e_{2n+1}$$

for $1 \leqslant p \leqslant n$. In this case it is easy to find a group which has $\mathfrak{g}$ as its Lie algebra; namely, for algebras of dimension $2n+1$ the $(n+2)\times(n+2)$ matrices:

$$x = \begin{bmatrix} 1 & x_1 & x_2 & \ldots & x_n & z \\ 0 & 1 & 0 & \ldots & 0 & y_1 \\ 0 & 0 & 1 & \ldots & 0 & y_2 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 & y_n \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}.$$

The corresponding system of geodesic equations is given by

$$\ddot{z} = \sum_{j=1}^{n} \dot{x}_j \dot{y}_j, \qquad \ddot{x}^i = 0, \qquad \ddot{y}^i = 0 \quad (1 \leqslant i \leqslant n). \tag{51}$$

A metric Lagrangian is given by

$$E = \left(\dot{z} - \sum_{j=1}^{n} \frac{(x_j \dot{y}_j + y_j \dot{x}_j)}{2} \right)^2 + \sum_{j=1}^{n} \left(\dot{x}_j^2 + \dot{y}_j^2 \right).$$

Example 7.2 (*The affine group of the plane*). The affine group of motions of $\mathbb{R}^2$ is the six-dimensional subgroup of $GL(3, \mathbb{R})$ given by

$$A(2) = \left\{ \begin{pmatrix} A & t \\ 0 & 1 \end{pmatrix} \,\middle|\, t \in \mathbb{R}^2,\ A \in GL(2, \mathbb{R}) \right\},$$

and the Lie algebra of $A(2)$ is given by

$$a(2) = \left\{ \begin{pmatrix} a & x \\ 0 & 0 \end{pmatrix} \,\middle|\, x \in \mathbb{R}^2,\ a \in gl(2, \mathbb{R}) \right\}.$$

When conditions (28) and (29) are imposed one finds that the multiplier g_{ij} has the following form:

$$g_{ij} = \begin{bmatrix} y_2^2 g_{22} + 2y_2 y_3 g_{23} + y_3^2 & -y_1(y_2 g_{22} + y_3 g_{23}) & -y_1(y_2 g_{23} + y_3 g_{33}) \\ -y_1(y_2 g_{22} + y_3 g_{23}) & y_1^2 g_{22} & y_1^2 g_{23} \\ -y_1(y_2 g_{23} + y_3 g_{33}) & y_1^2 g_{23} & y_1^2 g_{33} \end{bmatrix}.$$

The same conclusion can be reached starting from the Lie algebra basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ with non-zero bracket relations:

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = -e_3, \quad [e_1, e_5] = e_5, \quad [e_2, e_3] = e_1 - e_4, \quad [e_2, e_4] = e_2,$$
$$[e_2, e_6] = e_5, \quad [e_3, e_4] = -e_3, \quad [e_3, e_5] = e_6, \quad [e_4, e_6] = e_6.$$

However, it is clear that in this example g_{ij} is singular and therefore there can be no Lagrangian for the geodesics, which are given by:

$$\dot{y}_1 = \frac{x_4 y_1 y_5 - x_3 y_1 y_6 - x_6 y_2 y_5 + x_5 y_2 y_6}{\Delta}, \qquad \dot{y}_2 = \frac{x_4 y_1 y_3 - x_3 y_1 y_4 - x_6 y_2 y_3 + x_5 y_2 y_4}{\Delta},$$
$$\dot{y}_3 = \frac{-x_6 y_3^2 + x_5 y_3 y_4 + x_4 y_3 y_5 - x_3 y_4 y_5}{\Delta}, \qquad \dot{y}_4 = \frac{-x_6 y_3 y_4 + x_4 y_3 y_6 + x_5 y_4^2 - x_3 y_4 y_6}{\Delta},$$
$$\dot{y}_5 = \frac{-x_6 y_3 y_5 + x_5 y_3 y_6 + x_4 y_5^2 - x_3 y_5 y_6}{\Delta}, \qquad \dot{y}_6 = \frac{-x_6 y_4 y_5 + x_5 y_4 y_6 + x_4 y_5 y_6 - x_3 y_6^2}{\Delta},$$

where Δ is the determinant $x_4 x_5 - x_3 x_6$. We do not know if a similar conclusion holds for the affine group $A(n)$ in general.

Example 7.3 (*The Euclidean group of the plane*). To conclude this section we shall revisit the inverse problem for ASO_2, the Euclidean group of the plane. This example was treated in [14] but we are now in a position to complete the analysis. The corresponding Lie algebra has the basis e_1, e_2, e_3 with non-zero brackets,

$$[e_1, e_3] = -e_2, \qquad [e_2, e_3] = e_1.$$

As in [14] the geodesics are given by

$$\dot{u} = tv, \qquad \dot{v} = -tu, \qquad \dot{t} = 0, \tag{52}$$

where u, v and t denote $\dot{x}$, $\dot{y}$ and $\dot{z}$, respectively. The connection form θ is given by

$$-2\theta = \begin{bmatrix} 0 & -dz & -dy \\ dz & 0 & dx \\ 0 & 0 & 0 \end{bmatrix}$$

and the curvature two-form is given by

$$4\Omega = \begin{bmatrix} 0 & 0 & dx\,dz \\ 0 & 0 & dy\,dz \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence we see that the curvature tensor has essentially only the following non-zero components

$$R^1_{313} = \frac{1}{4}, \qquad R^2_{323} = \frac{1}{4}. \tag{53}$$

Conditions (28) and (29) entail that g_{ij} satisfies the conditions

$$g_{1q}u^q = g_{2q}u^q = 0,$$

where u^q is the 3-vector (u, v, t) and the solution of the ODE conditions (49) imply that g_{ij} is given by

$$\begin{aligned} g_{ij} = {} & M\begin{bmatrix} t^2u & t^2v & -t(u^2+v^2) \\ t^2v & -t^2u & 0 \\ -t(u^2+v^2) & 0 & u(u^2+v^2) \end{bmatrix} + P\begin{bmatrix} t^2 & 0 & -tu \\ 0 & t^2 & -tv \\ -tu & -tv & u^2+v^2 \end{bmatrix} \\ & + N\begin{bmatrix} -t^2v & t^2u & 0 \\ t^2u & t^2v & -t(u^2+v^2) \\ 0 & -t(u^2+v^2) & v(u^2+v^2) \end{bmatrix} + H\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F \end{bmatrix}, \end{aligned}$$

where H, M, N and P are arbitrary first integrals.

The closure conditions (50) turn out to be

$$uM_u + vM_v + tM_t + 4M = 0, \tag{54}$$

$$uN_u + vN_v + tN_t + 4N = 0, \tag{55}$$

$$uP_u + vP_v + tP_t + 3P = 0, \tag{56}$$

$$F_u = 0, \tag{57}$$

$$F_v = 0, \tag{58}$$

$$P_v + 2N + tN_t + uM_v - vM_u = 0, \tag{59}$$

$$P_u + 2M + tM_t - uN_v + vN_u = 0. \tag{60}$$

The closure conditions can be solved by introducing the following first integrals: $\alpha = \frac{u}{t} - y$, $\beta = \frac{v}{t} + x$, $\gamma = \frac{\cos(z)u - \sin(z)v}{t}$, $\delta = \frac{\cos(z)v + \sin(z)u}{t}$. Thus

$$F = F(t), \tag{61}$$

$$M = \frac{m(\alpha,\beta)}{t^4}, \tag{62}$$

$$N = \frac{n(\alpha,\beta)}{t^4}, \tag{63}$$

$$P = \frac{(2n + \delta m_\gamma - \gamma m_\delta)\beta + (2m - \delta n_\gamma + \gamma n_\delta)\alpha}{t^3}, \tag{64}$$

where m, n and F are arbitrary smooth functions. Two simple Lagrangians in this case are given by

$$L_1 = \frac{(u^2 + v^2)}{t} + xv - yu + t^2, \tag{65}$$

$$L_2 = \frac{(v^2 - u^2)\cos z + 2uv\sin z}{2t} + t^2. \tag{66}$$

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