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INVARIANT SHEN CONNECTIONS AND GEODESIC ORBIT SPACES

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Dedicated to the 65th birthday of Professor Joseph Grifone

Abstract

The geodesic graph of Riemannian spaces all geodesics of which are orbits of 1-parameter isometry groups was constructed by J. Szenthe in 1976 and it became a basic tool for studying such spaces, called g.o. spaces. This infinitesimal structure corresponds to the reductive complement $\mathfrak{m}$ in the case of naturally reductive spaces. The systematic study of Riemannian g.o. spaces was started by O. Kowalski and L. Vanhecke in 1991, when they introduced the most important definitions, classified the low-dimensional examples and described the basic constructions of this theory. The aim of this paper is to investigate a connection theoretical analogue of the concept of the geodesic graph.

1. Introduction

Let $M = G/H$ be a homogeneous space equipped with an invariant connection ∇. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebra of the Lie group G and H, respectively. The space $(M = G/H,\ \nabla)$ is called *affine reductive* if there exists an Ad_H invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ such that any geodesic $\gamma(t)$ emanating from the origin $o = H \in M$ is the orbit of a 1-parameter subgroup $\{\exp tX,\ t \in \mathbb{R}\}$ of G, where $X \in \mathfrak{m}$, and the parallel translation $\tau^{\gamma}_{0,t} : T_{\gamma(0)}M \to T_{\gamma(t)}M$ along the geodesic $\gamma(t)$ is the same as the left translation by the 1-parameter subgroup $\{\exp tX,\ t \in \mathbb{R}\}$. (cf. [6]). A homogeneous manifold $M = G/H$ with an invariant connection ∇ is called *affine geodesic orbit space (g.o. space)* if it has the more general property: each geodesic of M is an orbit of a one-parameter subgroup $\exp tZ$ $(t \in \mathbb{R})$, $Z \in \mathfrak{g}$.

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If $M = G/H$ is a homogeneous space equipped with an invariant Riemannian metric g then there exists always an Ad_H invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. In general one can find more than one such decomposition. The Riemannian homogeneous space $(M = G/H, g)$ is called *naturally reductive homogeneous space* if there exists an Ad_H invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ such that any geodesic $\gamma(t)$ emanating from the origin $o = H \in M$ is the orbit of a 1-parameter subgroup $\{\exp tX,\ t \in \mathbb{R}\}$ of G, where $X \in \mathfrak{m}$. The Riemannian homogeneous space $(M = G/H,\ g)$ is called a *Riemannian g.o. space* if it is an affine g.o. space with respect to its Levi–Civita connection. Obviously, a naturally reductive space $(G/H, g)$ is a Riemannian g.o. space. Finally, the Riemannian manifold (M, g) is said to be *naturally reductive or g.o. space* respectively, if it is naturally reductive or Riemannian g.o. space for some connected subgroup G of the full isometry group of (M, g).

The first example of a Riemannian g.o. space which is in no way naturally reductive was given by A. Kaplan in 1983 [5]. Before this work it was generally believed that the Riemannian geodesic orbit property is just equivalent to the natural reductivity (cf. [1], Theorem 5.4). J. Szenthe in 1976 [14] proposed a deep construction for the study of affine g.o. spaces with compact isotropy group H, his construction results the reductive complement $\mathfrak{m}$ in the special case of affine reductive spaces. The systematic study of Riemannian g.o. spaces was started by O. Kowalski and L. Vanhecke in [9], where they introduced the most important definitions, classified the low-dimensional examples and described the basic constructions of this theory. They called *geodesic graph* the infinitesimal structure generalizing the notion of a reductive complement of a subalgebra $\mathfrak{h}$ in the Lie algebra $\mathfrak{g}$ the construction of which was proposed by J. Szenthe for the investigation of affine g.o. spaces. In the last years interesting papers were devoted to the study of geodesic graphs of Riemannian g.o. spaces (cf. e.g. [8], [7]). This notion is generalized in Appendix to [8] which more general structure can be interpreted as an infinitizemial version of some invariant connection.

The aim of our paper is to show that a connection theoretical version of the concept of the generalized geodesic graph occurs as a natural canonical connection of Finsler spaces. This type of connection has been introduced in an early paper of L. Berwald ([2]) and it is strongly related to the Finsler connection theory of S. S. Chern (cf. [13]) and of H. Rund ([10], [11]), but it is different from the connections of Finsler type named as Berwald, Rund or Chern connection. In this paper we give an invariant treatment of this generalized linear connection, a version of which is used systematically by Z. Shen for the investigation of Finsler manifolds (cf. [12], [13]). We reinterpret some results on Riemannian g.o. spaces as informations on invariant Shen connections. In a following paper we will give a treatment of the curvature theory of invariant Shen connections and of homogeneous Finsler manifolds.

2. Shen connections

2.1. Shen connections on the frame bundle

Let M be a differentiable manifold, let $\pi : TM \to M$ and $\pi_L : LM \to M$ be the tangent bundle and the frame bundle of M, respectively. We denote by $\mathcal{T}M$ the open submanifold of TM consisting of nonzero vectors. We consider the direct products $TM \times_{_M} LM$ and $\mathcal{T}M \times_{_M} LM$ of the bundles TM and LM, respectively $\mathcal{T}M$ and LM, over the base manifold M. Let $p_1 : TM \times_{_M} LM \to TM$ and $p_2 : TM \times_{_M} LM \to LM$ be the projections of $TM \times_{_M} LM$ onto the first and the second components. We have the commutative diagram:

$$\begin{array}{ccc} TM \times_{_M} LM & \xrightarrow{\;p_2\;} & LM \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_L} \\ TM & \xrightarrow{\;\pi\;} & M \end{array}$$

If $x \to (x^1, \dots, x^n)$ is a local coordinate map on M then we denote by $(x, y) \to (x^1, \dots, x^n; y^1, \dots, y^n)$ and $(x, z) \to (x^1, \dots, x^n; z^1_1, \dots, z^n_n)$ the associated coordinate maps on TM and LM, respectively, where $y^1, \dots, y^n$ are the coordinates of the tangent vector $y \in T_xM$ and $z^1_1, \dots, z^n_n$ are the coordinates of the vectors of the frame $z \in L_xM$ with respect to the coordinate basis $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$.

Definition 2.1. A map

$$\sigma : (x, y, z) \mapsto \sigma(x, y, z) : TM \times_{_M} LM \to TLM,$$

which is $\mathcal{C}^\infty$-differentiable on $\mathcal{T}M \times_{_M} LM$ and continuous on $TM \times_{_M} LM$, is called *Shen connection* on the manifold M, if

$$\sigma(x, y, z) \in T_{(x,z)}LM, \tag{C1}$$

$$\pi_{L*}\sigma = p_1, \tag{C2}$$

$$\sigma(x, \lambda\, y, z) = \lambda\, \sigma(x, y, z), \qquad \forall\ \lambda \in \mathbb{R}, \tag{C3}$$

$$(R_h)_*\sigma(x, y, z) = \sigma\big(x, y, R_h\, z\big), \tag{C4}$$

where $R : LM \times GL_n(\mathbb{R}) \to LM$ denotes the right action of the linear group $GL_n(\mathbb{R})$ on the fibers. The Shen connection σ is called K-*invariant* with respect to a differentiable transformation group $\mu : K \to \mathrm{Diff}(M)$ of M if

$$(\mu^{_L}_k)_*\sigma(x, y, z) = \sigma\big(\mu_k x,\ (\mu_k)_* y,\ \mu^{_L}_k z\big)$$

for any $k \in K$, where $\mu^{_L} : K \to \mathrm{Diff}(LM)$ denotes the associated representation of K in $\mathrm{Diff}(LM)$. If the manifold M is a homogeneous space G/H and σ is invariant with respect to the group G then we call σ an *invariant Shen connection* on G/H.

For the sake of simplicity, in the rest of the paper we denote the map $\mu^{_L}_k$ simply by $(\mu_k)_*$.

We remark that every linear connection can be interpreted as a Shen connection. Indeed, considering the horizontal lift of TM into TLM associated to a linear connection we obtain a map σ satisfying the above conditions. In this particular case the map σ is linear with respect to the variable y and it is $\mathcal{C}^\infty$-differentiable on the whole manifold $TM \times_M LM$.

Using local coordinates (x^i) on M and the associated coordinates (x^i, y^i) and (x^i, z^i_j) on TM and on LM respectively, we find that the local coordinate expression of σ has the shape

$$\sigma(x^i, y^i, z^i_j) = y^i \frac{\partial}{\partial x^i} + \Gamma^j_k(x,y) z^k_l \frac{\partial}{\partial z^j_l} \tag{1}$$

with some functions $\Gamma^j_k(x,y),\ j,k = 1,\dots,n$, which are homogeneous of degree 1 with respect to the variable y, i.e. $\Gamma^j_k(x, \lambda y) = \lambda\, \Gamma^j_k(x,y)$ for any $\lambda \in \mathbb{R}$.

Remark 2.2. The notion of Shen connection can be defined analogously for any principal subbundle $\pi_L : Q \to M$ of the frame bundle $\pi_L : LM \to M$. If we consider such case we say that the Shen connection is *defined in the subbundle* $\pi_L : Q \to M$.

Clearly, if a Shen connection is defined in a subbundle of the linear frame bundle then it can be extended to the linear frame bundle in a natural way. We can also define the notion of the reduction of Shen connections.

Definition 2.3. Let $\pi_L : Q \to M$ be a principal subbundle of the frame bundle $\pi_L : LM \to M$ and let $\sigma : TM \times_M LM \to TLM$ be a Shen connection. We say that σ *can be reduced* to the subbundle $\pi_L : Q \to M$ if the restriction of the map σ to the submanifold $TM \times_M Q$ determines a Shen connection in the subbundle $\pi_L : Q \to M$. The restricted map $TM \times_M Q \to TQ$ is called the *Shen connection reduced* to $\pi_L : Q \to M$.

2.2. Parallelism

A connection gives us the possibility to introduce the notion of *parallel frame field* and *parallel vector field* along a curve of M. Moreover we can introduce parallel translation of tangent vectors along curves as well as the notion of geodesics on M.

Definition 2.4. Let $\gamma : [a,b] \to M$ be a differentiable curve on M.

(a) A frame field $z_t,\ t \in [a,b]$, along γ is called *parallel*, if

$$\dot z_t = \sigma(\gamma_t, \dot\gamma_t, z_t)$$

for every $t \in [a,b]$.

(b) Let be given a parallel frame field z_t along γ. A vector field $U(t)$ along γ is called *parallel* if the function $t \mapsto z_t^{-1}(U(t)) : [a,b] \to \mathbb{R}^n$ is constant for the parallel frame field z_t along γ. (Here z_t is considered as the canonical linear map $\mathbb{R}^n \to T_{\gamma(t)}M$.)

The following assertion is an immediate consequence of this definition:

REMARK 2.5. If a vector field $U(t)$ along γ is parallel with respect to a parallel frame field z_t then it is parallel with respect to any parallel frame field. Hence the notion of parallel vector fields along γ is well defined, it is independent of the parallel frame field used in the definition.

Using the local form (1) of the connection σ we obtain that a frame field $z(t) = z^i_j(t)$ is parallel along $\gamma = x(t)$ if and only if for any $i, j = 1, \dots, n$ it satisfies the equation

$$\dot z^i_j(t) - \Gamma^i_k(x, \dot x) z^k_j = 0.$$

Now, we can introduce the notion of covariant derivative.

DEFINITION 2.6. The *covariant derivative* of a vector field $U(x) = U^i(x)\frac{\partial}{\partial x^i}$ along the vector field $V(x) = V^i(x)\frac{\partial}{\partial x^i}$ is given by

$$\nabla_V U = \left(\frac{\partial U^i}{\partial x^j} V^j - \Gamma^i_j(x^1, \dots, x^1; V^1, \dots, V^n)\, U^j\right) \frac{\partial}{\partial x^i} \tag{2}$$

A vector field $U(t) = U^i(t)\frac{\partial}{\partial x^i}$ is parallel along γ if and only if we have

$$\begin{aligned} \nabla_{\dot x} U &= \left(\dot U^i - \Gamma^i_j(x^1, \dots, x^1; \dot x^1, \dots, \dot x^n)\, U^j\right) \frac{\partial}{\partial x^i} \\ &= \left(\dot U^i(t) - \Gamma^i_j(x, \dot x)\, U^j(t)\right) \frac{\partial}{\partial x^i} = 0. \end{aligned}$$

REMARK 2.7. Canonical covariant derivatives ∇ of Finsler spaces having the local form (2) occured first in [2], p. 45, equation 14, (cf. also [10], p. 83, equation 2.4). In the last years Z. Shen uses the canonical covariant derivative of type (2) for the global investigation of Finsler manifolds (cf. [13]).

Using parallelism the notion of the parallel translation $\tau^\gamma_{a,t}$, $t \in [a,b]$, along a differentiable curve $\gamma : [a,b] \to M$ can be introduced as follows:

DEFINITION 2.8. Let $z(t)$ be an arbitrary parallel frame along γ. The *parallel translation* $\tau^\gamma_{a,t} : T_{\gamma(a)}M \to T_{\gamma(t)}M$ is defined by the equation

$$\tau^\gamma_{a,t} = z(t) \cdot z(a)^{-1} : T_{\gamma(a)}M \to T_{\gamma(t)}M, \quad t \in [a,b].$$

Clearly, the parallel translation is a linear map.

Definition 2.9. A curve γ is called a *geodesic* if $\dot\gamma$ is parallel along itself.

The differential equation of a geodesic $x = (x^1(t), \dots, x^n(t))$ in a local coordinate system has the shape

$$\nabla_{\dot x}\dot x = \big(\ddot x^i(t) - \Gamma^i_j(x(t), \dot x(t))\,\dot x^j(t)\big)\,\frac{\partial}{\partial\, x^i} = 0. \tag{3}$$

At the end of this section we extend the notion of the holonomy group to Shen connections.

Definition 2.10. The *linear holonomy group at the point $o \in M$* of the Shen connection σ is the linear group acting on T_oM which is generated by the parallel translation of tangent vectors from T_oM along piecewise differentiable curves starting and ending at o.

Clearly, if the manifold is connected then the linear holonomy groups at different points are isomorphic.

2.3. Spray associated with a Shen connection

Definition 2.11. A vector field $S : TM \to TTM$ on TM is called a *spray* if $\pi_*(S) = \mathrm{id}_{TM}$.

A vector field S is a spray if and only if its local expression has the form

$$S(x^i, y^i) = y^i\frac{\partial}{\partial\, x^i} + f^i(x,y)\frac{\partial}{\partial\, y^i} \tag{4}$$

with some function $f^i(x,y)$, $i = 1, \dots, n$. The notion of spray can be used for coordinate free formulation of second order ordinary differential equations. Indeed, a differentiable curve $\gamma : [a,b] \to M$ is called a *path of the spray S* if its speed curve $\dot\gamma : [a,b] \to TM$ is an integral curve of S. Clearly, a curve $\gamma(t) = x^i(t)$ is a path of S if and only if it satisfies the second order differential equation

$$\ddot x^i = f^i(x, \dot x), \qquad i = 1, \dots, n. \tag{5}$$

If there is given a Shen connection σ then one can associate a spray canonically by the following way: let us denote the $\mathbb{R}^n$-valued map

$$\vartheta_x(v,z) := z^{-1}(v), \qquad v \in T_xM,\ z \in L_xM$$

by $\vartheta : TM \times_{M} LM \longrightarrow \mathbb{R}^n$ and the map

$$\beta(\eta, z) := \eta^i z_i, \qquad \eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n,\ z = (z_1, \dots, z_n) \in L_xM$$

by $\beta : \mathbb{R}^n \times LM \longrightarrow TM$. We denote by $\beta^{(2)}_* : \mathbb{R}^n \times TLM \longrightarrow TTM$ the tangent map of β with respect to its second variable. Then the diagram

$$TM \times_{M} LM \xrightarrow{\vartheta\times\sigma} \mathbb{R}^n \times TLM \xrightarrow{\beta^{(2)}_*} TTM, \tag{6}$$

determines the map $\hat{S} = \beta_*^{(2)} \circ (\vartheta \times \sigma) : TM \times_M LM \longrightarrow TTM$.

Proposition 2.12. *The value $\hat{S}(v,z)$ of the map $\hat{S} : TM \times_M LM \to TTM$ depends only on the vector $v \in T_xM$, but it is independent of the frame $z \in L_xM$. The vector field $S : TM \to TTM$ defined on TM by*

$$S(v) := \hat{S}(v,z)$$

is a spray on M.

Proof. Indeed, using the local expression of σ given by (1) we obtain immediately for $v = (x^i, y^j)$, $z = (x^i, z^i_j)$, $v = (x^i, \eta^j z^i_j)$ that

$$(\vartheta \times \sigma)(v,z) = \left(\eta, y^i \frac{\partial}{\partial x^i}\bigg|_{(x,z)} + \Gamma^j_k(x,y) z^k_i \frac{\partial}{\partial z^j_i}\bigg|_{(x,z)} \right)$$

and therefore

$$\begin{aligned} \hat{S}(v,z) &= \beta_*^{(2)}\,(\vartheta \times \sigma(v,z)) \\ &= y^i \frac{\partial}{\partial x^i}\bigg|_{(x,\eta^i z_i)} + \Gamma^j_k(x,y)\eta^i z^k_i \frac{\partial}{\partial y^j}\bigg|_{(x,\eta^i z_i)} \\ &= y^i \frac{\partial}{\partial x^i}\bigg|_{(x,y)} + \Gamma^j_k(x,y) y^k \frac{\partial}{\partial y^j}\bigg|_{(x,y)} . \end{aligned}$$

It is clear from this formula that $\hat{S}$ is independent of the choice of the reference frame z. Moreover, we can also see from this computation that the local expression of S is

$$S(x,y) = y^i \frac{\partial}{\partial x^i}\bigg|_{(x,y)} + \Gamma^j_k(x,y) y^k \frac{\partial}{\partial y^j}\bigg|_{(x,y)} . \tag{7}$$

Comparing this with (4) we obtain that S is a spray with the functions $f^i(x,y) = \Gamma^j_k(x,y)\, y^k$. □

Definition 2.13. The spray *associated with the Shen connection σ* is the vector field $\hat{S} = \beta_*^{(2)} \circ (\vartheta \times \sigma) : TM \to TTM$ investigated in the previous proposition.

Comparing the differential equation (3) of geodesics of the Shen connection σ, the differential equation (5) of paths of a spray and the local expression (7) of the spray associated with the Shen connection σ we obtain the following

Corollary 2.14. *A differentiable curve is geodesic with respect to the Shen connection σ if and only if it is a path of the spray associated with σ.*

3. Invariant Shen connections

3.1. Reductive Shen connections

Let M be a differentiable manifold on which the Lie transformation group G acts transitively. Let us fix an origin $o \in M$ and denote by H the stabilizer of $o \in M$ in the group G and by $p : G \to G/H$ the projection map. As usual we call H the isotropy group of the homogeneous space G/H. Then M is isomorphic to the factor space G/H with origin H and its tangent space at $o \in M$ is isomorphic to $\mathfrak{g}/\mathfrak{h}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie-algebras of the Lie groups G and H respectively. The action of G on M is determined by the map

$$\lambda : (g, m) \mapsto \lambda_g m = g \cdot m : G \times M \to M.$$

We denote by

$$\varphi : gH \to g \cdot o : G/H \to M,$$

the isomorphism between G/H and M, its tangent map

$$\varphi_* : (v + \mathfrak{h}) \mapsto \frac{d}{dt}\Big|_{t=0} \exp(tv) \cdot o : \mathfrak{g}/\mathfrak{h} \to T_oM$$

gives the isomorphism between $\mathfrak{g}/\mathfrak{h}$ and T_oM.

In the following we identify the manifold M with G/H and the tangent space T_oM with $\mathfrak{g}/\mathfrak{h}$ by the described isomorphisms φ and φ_*. Now, we investigate Shen connections defined on the homogeneous space $M = G/H$ which are invariant under the transitive transformation group G.

Proposition 3.1. *Let $M = G/H$ be a connected homogeneous space equipped with an invariant Shen connection $\sigma : TM \times_M LM \to TLM$. Let $z_0 \in L_oM$ be a fixed reference frame at the origin o. Then the mapping*

$$\Lambda_{z_0} : g \mapsto (\lambda_g)_* z_0 : G \to LM$$

is an imbedding of the Lie group G into the frame bundle LM.

Proof. First, we notice that the maps λ_g, $g \in G$ are transformations of M preserving the Shen connection σ. Hence the transformations λ_g, $g \in G$ map geodesics of M into geodesics. It follows that if the transformation λ_g fixes the origin o and its tangent map $(\lambda_g)_*$ fixes the reference frame z_0 then $(\lambda_g)_* : T_oM \to T_oM$ is the identity map id_{T_oM} on T_oM. Hence λ_g preserves the geodesics emanated from the origin $o \in M$ from which follows that λ_g is locally the identity map. Let F be the set fixed points $f \in M$ of λ_g such that $(\lambda_g)_*$ induces the identity map on the tangent space T_fM. Clearly, this set F is closed in M. But it follows from the previous arguments that for any $f \in F$ the map λ_g induces the identity map in a suitable neighbourhood of f. It follows that $F = M$ since F is open and closed in the connected manifold M, consequently $\lambda_g : M \to M$ is the identity map. Hence Λ_{z_0} is an injective imbedding. $\square$

Now we investigate a class of Shen connections which have similar properties as the canonical connections of reductive homogeneous spaces.

Definition 3.2. An invariant Shen connection $\sigma : TM \times_{M} LM \to TLM$ given on the manifold $M = G/H$ is called *reductive* if for any geodesic $\gamma(t)$ emanating from the origin $o \in M$ there exists a suitable $X \in \mathfrak{g}$ such that

(a) $\gamma(t)$ is the orbit of a 1-parameter subgroup $\{\exp tX,\ t \in \mathbb{R}\}$ of G, i.e.

$$\gamma(t) = \lambda_{\exp tX}\, o = (\exp tX) \cdot o,$$

(b) the parallel translation $\tau^{\gamma}_{o,t} : T_{\gamma(o)}M \to T_{\gamma(t)}M$ along $\gamma(t)$ is the same as the translation by the 1-parameter subgroup $\{\exp tX,\ t \in \mathbb{R}\}$, i.e.

$$\tau^{\gamma}_{o,t} = (\lambda_{\exp tX})_{*} : T_oM \to T_{(\exp tX)\cdot o}\,.$$

According to 3.1 the Lie group G is diffeomorphic to the total space $\Lambda_{z_0}G$ of the subbundle $\pi_L : \Lambda_{z_0}G \to M$, where identity element $e \in G$ corresponds to the reference frame $z_0 \in L_oM$ and the Lie algebra $\mathfrak{g}$ corresponds to the tangent space $T_{z_0}LM$. The diffeomorphism Λ_{z_0} induces a bundle isomorphism between the principal bundles $p : G \to G/H$ and $\pi_L : \Lambda_{z_0}G \to M$.

Proposition 3.3. *Let $\sigma : TM \times_{M} LM \to TLM$ be a reductive invariant Shen connection. Then it can be reduced to the subbundle $\pi_L : \Lambda_{z_0}G \to M$ of $\pi_L : LM \to M$.*

Proof. We have to prove that the image $\sigma(TM \times_{M} \Lambda_{z_0}G)$ of the connection map σ is contained in the manifold $T\Lambda_{z_0}G$. We know from the definition of parallel translation of frames that the subset $\sigma(TM \times \{z_0\})$ of the image $\sigma(TM \times_{M} \Lambda_{z_0}G)$ consists of the tangent vectors of parallel translated frames along curves emanated from the origin o. Hence it follows from the definition of the g.o. property that $\sigma(TM \times \{z_0\}) \subset (\Lambda_{z_0})_{*}\mathfrak{g}$. Both of the connection map σ and the manifold $T\Lambda_{z_0}G$ are invariant with respect to the action of the transformation group G from which we obtain that $\sigma(TM \times_{M} \Lambda_{z_0}G)$ is contained in $T\Lambda_{z_0}G$. □

Clearly for a reductive invariant Shen connection σ the parallel translation of frames leaves invariant the submanifold $T\Lambda_{z_0}G \subset TLM$ of frames. It follows that the linear holonomy group of σ is isomorphic to a subgroup of the fibre group of the principal bundle $\pi_L : \Lambda_{z_0}G \to M$. Hence we obtain:

Corollary 3.4. *If σ is a reductive invariant Shen connection on the homogeneous space $M = G/H$ then its linear holonomy group is isomorphic to a subgroup of the isotropy group H.*

3.2. (Non-linear) horizontal lift

Let $M = G/H$ be a homogeneous space, where $H \subset G$ are Lie groups and $\mathfrak{h} \subset \mathfrak{g}$ are the corresponding Lie algebras. The tangent space T_oM at the origin o can be identified by the factor space $\mathfrak{g}/\mathfrak{h}$. The linear isotropy representation of the subgroup H on the tangent space T_oM is given by

$$h \mapsto (\lambda_h)_* : H \to GL(T_oM), \quad h \in H.$$

The corresponding action of H on the factor space $\mathfrak{g}/\mathfrak{h}$ is the induced action of the adjoint representation $h \mapsto \mathrm{Ad}_h : H \to GL(\mathfrak{g})$ of H on the factor space $\mathfrak{g}/\mathfrak{h}$, i.e. an arbitrary $h \in H$ acts on a coset $X + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$ by the map $X + \mathfrak{h} \mapsto \mathrm{Ad}_h(X + \mathfrak{h}) = \mathrm{Ad}_h(X)+\mathfrak{h}$. We denote this induced action of $h \in H$ on $\mathfrak{g}/\mathfrak{h}$ by $\mathrm{Ad}_h^{(\mathfrak{g}/\mathfrak{h})} : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$. If there is given a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ then a natural map $\mathfrak{g}/\mathfrak{h} \to \mathfrak{m} \subset \mathfrak{g}$ can be defined. In the case of reductive invariant Shen connections we define the notion of (non-linear) reductive lift which is a not necessarily linear version of the previous map. Such kind of maps are introduced by J. Szenthe [14] for the investigation of affine g.o. spaces:

Definition 3.5. A map $\xi : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ is called *(non-linear) horizontal lift* if the following conditions are satisfied:

(a) ξ is $\mathrm{Ad}(H)$-invariant, which means

$$\xi(\mathrm{Ad}_h^{(\mathfrak{g}/\mathfrak{h})}(X + \mathfrak{h})) = \mathrm{Ad}_h(\xi(X + \mathfrak{h})), \quad \text{for all } h \in H,\ X \in \mathfrak{g}.$$

(b) ξ is homogeneous, i.e. $\xi(\lambda X + \mathfrak{h}) = \lambda\xi(X + \mathfrak{h})$, for every $X \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$.
(c) $(\xi(X + \mathfrak{h}) - X) \in \mathfrak{h}$ for any $X \in \mathfrak{g}$.

A (non-linear) horizontal lift $\xi : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ is called $\mathcal{C}^\infty$*-differentiable* (or *analytic*) if it is continuous on $\mathfrak{g}/\mathfrak{h}$ and $\mathcal{C}^\infty$-differentiable (or analytic) on $\mathfrak{g}/\mathfrak{h} \setminus \{0\}$.

Clearly, ξ is differentiable linear map if and only if it is differentiable at the origin $0 \in \mathfrak{g}/\mathfrak{h}$, too. In this case the image $\xi(\mathfrak{g}/\mathfrak{h})$ is a reductive complement of the Lie subalgebra $\mathfrak{h}$ in $\mathfrak{g}$.

One can associate (non-linear) horizontal lifts with reductive invariant Shen connections.

Proposition 3.6. *Let $M = G/H$ be a homogeneous space, where $H \subset G$ are Lie groups and $\mathfrak{h} \subset \mathfrak{g}$ are the corresponding Lie algebras. Let σ be a reductive invariant Shen connection. Then the restriction*

$$\sigma\big|_{T_oM \times \{z_0\}} : T_oM \times \{z_0\} \to T_{z_0}\Lambda_{z_0}G$$

of the map σ to the tangent space T_oM at the reference frame z_0 determines a differentiable (non-linear) horizontal lift by the use of the identifications $M = G/H$, $T_oM = \mathfrak{g}/\mathfrak{h}$ and $T_{z_0}\Lambda_{z_0}G = \mathfrak{g}$.

Proof. Since the Shen connection map σ is reduced to the frame subbundle $\pi_L : \Lambda_{z_0}G \to M$ of $\pi_L : LM \to M$, it is left invariant. The property (C4) in the definition 2.1 of Shen connections implies that σ is right invariant on the fiber $\pi_L^{-1}(o)$ and hence the map $\sigma\big|_{T_oM\times\{z_0\}}$ satisfies the property (a) of a (non-linear) horizontal lift. The conditions (b) and (c) for a (non-linear) horizontal lift follow from the properties (C3) and (C2) in the definition 2.1 of a Shen connection. □

Definition 3.7. Let σ be a reductive invariant Shen connection given on the homogeneous space $M = G/H$. The differentable (non-linear) horizontal lift $\xi : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ determined by the restriction $\sigma\big|_{T_oM\times\{z_0\}}$ of the map σ to the tangent space T_oM at the reference frame z_0 is called the *(non-linear) horizontal lift associated with the Shen connection σ.*

Proposition 3.8. *Let us identify the manifold M with the homogeneous space G/H, the tangent space T_oM with the factor space $\mathfrak{g}/\mathfrak{h}$, the subbundle $\pi_L : \Lambda_{z_0}G \to M$ of frames with the principal bundle $p : G \to G/H$. If $\xi : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ is a $\mathcal{C}^\infty$-differentiable (non-linear) horizontal lift then the map $\sigma : TM \times_M \Lambda_{z_0}G \to T\Lambda_{z_0}G$ defined at the point $(x,u,z) \in TM \times_M \Lambda_{z_0}G$ by the value $\sigma(x,u,z) = (x, (\lambda_g)_*\xi((\lambda_g)_*^{-1}(u)))$ is a reductive Shen connection in the bundle $\pi_L : \Lambda_{z_0}G \to M$, where λ_g $(g \in G)$ is the unique map such that $\lambda_g : o \mapsto x$ and $(\lambda_g)_* : z_0 \mapsto z$.*

Proof. Clearly, the map $\sigma : TM\times_M \Lambda_{z_0}G \to T\Lambda_{z_0}G$ is invariant with respect to the action of G on $\Lambda_{z_0}G$. It follows from the defining properties of ξ that the map σ satisfies the properties (C1) - (C3) of a Shen connection. The property (C4) is fulfilled, too, since the (non-linear) horizontal lift is Ad_H-invariant. Hence the map $\sigma : TM \times_M \Lambda_{z_0}G \to T\Lambda_{z_0}G$ determines an invariant Shen connection in the principal bundle.

Now, we prove that the Shen connection σ is reductive. Let be given a tangent vector $u \in T_oM$ and let $z(t) = (\lambda_{\exp t\xi(u)})_*z_0$ be the orbit of z_0 with respect to the 1-parameter subgroup $\exp t\xi(u)$, $t \in \mathbb{R}$. This orbit corresponds to the 1-parameter subgroup $\exp t\xi(u)$, $t \in \mathbb{R}$ of G by the identification of the subbundle $\pi_L : \Lambda_{z_0}G \to M$ of frames with the principal bundle $p : G \to G/H$. We show that $z(t)$ is a parallel frame field along the curve $\pi_L(z(t))$, or equivalently, that its tangent vector is given by the value of the map σ at the tangent vector of the projection curve $\pi_L(z(t))$ and at the frame $z(t)$. Indeed, the tangent vector at $t \in \mathbb{R}$ of the corresponding curve $\exp t\xi(u)$ in G is $(\lambda_{\exp t\xi(u)})_*\xi(u)$ which corresponds to the value $\sigma\big(\exp t\xi(u), (\lambda_{\exp t\xi(u)})_*u, (\lambda_{\exp t\xi(u)})_*\xi(u)\big)$ by the identification, which means that $z(t)$ is a parallel frame field. Similarly we obtain that the tangent vector of $p(\exp t\xi(u))$, corresponding to $\pi_L(z(t))$, is the vector $(\lambda_{\exp t\xi(u)})_*u$ which is a parallel vector field along $p(\exp t\xi(u))$. Hence the projected curve $\pi_L(z(t))$ is a geodesic tangent to $u \in T_oM$ and the parallel translation along this geodesic coincides with the action $(\lambda_{\exp t\xi(u)})_*$ of the 1-parameter group $\exp t\xi(u)$. This result means that the Shen connection σ is reductive. □

The previous propositions give the following

THEOREM 3.9. *There exists a bijective correspondance between differentiable (non-linear) horizontal lifts of the factor space* $\mathfrak{g}/\mathfrak{h}$ *and reductive Shen connections on the homogeneous space* G/H.

3.3. Riemannian g.o. spaces

Let $(M = G/H, g)$ be a homogeneous Riemannian manifold with origin $o = H \in M = G/H$ and let $\pi_O : OM \to M$ denote the orthonormal subbundle of $OM \subset LM$ of the linear frame bundle $\pi_L : LM \to M$ of M. Let $z_0 \in O_oM$ be a fixed orthonormal reference frame at the origin o. Then the mapping

$$\Lambda_{z_0} : g \mapsto (\lambda_g)_* z_0 : G \to OM$$

is an imbedding of the Lie group G into the frame bundle OM. Since the mapping Λ_{z_0} induces an isomorphism of the isotropy subgroup H onto a subgroup of the orthogonal group $O(n, \mathbb{R})$ the Cartan–Killing form of $O(n, \mathbb{R})$ determines an invariant Riemannian metric on the subgroup H. The following statement is a reformulation of a

THEOREM OF J. SZENTHE. *Let* $(M = G/H,\ g)$ *be a Riemannian g.o. space. For a given* Ad_H*-invariant decomposition* $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ *there exists a canonical (non-linear) horizontal lift* $\xi : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ *such that for any* $X + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h} \setminus \{0\}$ *the orbit of* $\{\exp t\xi(X + \mathfrak{h}),\ t \in \mathbb{R}\}$ *through the origin* $o = H \in G/H$ *is a geodesic.*

(Cf. [14], [9].) We notice that the construction of J. Szenthe depends on the chosen Ad_H-invariant decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$. The construction depends on the choice of the invariant Riemannian metric on the isotropy group H, but for the simplicity we fixed this metric previously.

The canonical horizontal lift of J. Szenthe can be described by the following construction:

For each given coset $X + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h},\ X \neq 0$ put

$$\mathfrak{q}_{(X+\mathfrak{h})} = \{A \in \mathfrak{h} : [A, X + \mathfrak{h}] \subset \mathfrak{h}\}.$$

Clearly, $\mathfrak{q}_{(X+\mathfrak{h})}$ is a subalgebra of $\mathfrak{h}$. Next, let $N_{(X+\mathfrak{h})}$ be the normalizer of $\mathfrak{q}_{(X+\mathfrak{h})}$ in $\mathfrak{h}$, i.e.

$$N_{(X+\mathfrak{h})} = \left\{B \in \mathfrak{h} : [B, A] \in \mathfrak{q}_{(X+\mathfrak{h})} \text{ for all } A \in \mathfrak{q}_{(X+\mathfrak{h})}\right\}.$$

PROPOSITION 3.10. *Let* $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ *be an* Ad_H*-invariant decomposition of the Lie algebra* $\mathfrak{g}$. *We identify the tangent space* T_oM *with the subspace* $\mathfrak{m} \subset \mathfrak{g}$. *Let* $X \in \mathfrak{m} \setminus \{0\}$ *be the tangent vector of a geodesic through* $o \in M$ *which is the orbit of the 1-parameter isometry group* $\exp t(X + A)$ *where* $A \in \mathfrak{h}$. *Then* $A \in N_{(X+\mathfrak{h})}$.

Proof. Using the Ad_H-invariant decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ one has that if $X \in \mathfrak{m}$ then $\mathfrak{q}_{(X+\mathfrak{h})} = \{A \in \mathfrak{h} : [A, X] = 0\}$. The detailed proof of the statement $A \in N_{(X+\mathfrak{h})}$ is given in [14].

Let $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ and $\mathfrak{g} = \mathfrak{m}' + \mathfrak{h}$ be two different Ad_H-invariant decompositions and let $X \in \mathfrak{m} \setminus \{0\}$ and $X' = X + C \in \mathfrak{m}' \setminus \{0\}$ represent the same tangent vector from T_oM, i.e. $C \in \mathfrak{h}$. Then the subalgebras $\{A \in \mathfrak{h} : [A, X] = 0\}$ and $\{A' \in \mathfrak{h} : [A', X'] = 0\}$ coincide since $[A', X'] = [A', X + C] = 0$ implies $[A', X] = 0$ and conversely. Hence the result is independent of the chosen Ad_H-invariant decomposition of $\mathfrak{g}$. (Cf. [8], p. 225.) □

Let $N_{(X+\mathfrak{h})} = \mathfrak{q}_{(X+\mathfrak{h})} + \mathfrak{c}_{(X+\mathfrak{h})}$ be the orthogonal decomposition with respect to the Ad_H-invariant scalar product $(\ ,\ )$ on $\mathfrak{h}$ determined by the given invariant Riemannian metric of the isotropy group H. Then the (non-linear) horizontal lift $\xi : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ which is determined by the construction of J. Szenthe can be identified with a map $\xi : \mathfrak{m} \to \mathfrak{g}$ uniquely characterized by the condition $\xi(X) - X \in \mathfrak{c}_{(X+\mathfrak{h})}$ for any $X \in \mathfrak{m} \setminus \{0\}$. (Cf. [14]), or equivalently, the value of the map $X \to (\xi(X) - X) : \mathfrak{m} \to \mathfrak{h}$ is orthogonal to the subalgebra $\mathfrak{q}_{(X+\mathfrak{h})}$ at each vector $X \in \mathfrak{m}$.

Definition 3.11. The map $X \to (\xi(X) - X) : \mathfrak{m} \to \mathfrak{h}$ the value of which is orthogonal to the subalgebra $\mathfrak{q}_{(X+\mathfrak{h})}$ at each vector $X \in \mathfrak{m} \setminus \{0\}$ is called the *geodesic graph*.

Explicit expression for geodesic graphs are described in [4], [8], [3], [7]. All known examples of Riemannian g.o. spaces which are not naturally reductive show that the (non-linear) horizontal lifts $\xi : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ determined by the Szenthe's construction have essential singularities. Hence we introduce the following notion:

Definition 3.12. A (non-linear) horizontal lift $\xi : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ is called *differentiable with singularities* if the map $\xi : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ is differentiable on an open dense Ad_H-invariant subset of $\mathfrak{g}/\mathfrak{h}$. The map $\sigma : TM \times_M \Lambda_{z_0}G \to T\Lambda_{z_0}G$ determined by the construction of Proposition 3.8 is called a *differentiable reductive Shen connection with singularities*.

Definition 3.13. A reductive Shen connection σ (with singularities) defined on the orthonormal frame bundle of a homogeneous Riemannian manifold $(M = G/H, g)$ is called *naturally reductive Shen connection (with singularities)* if the geodesic spray of the Riemannian manifold (M, g) coincides with the spray associated with the Shen connection σ.

From the previous definition follows that the spray associated with a naturally reductive Shen connection with singularities is differentiable. One can formulate the following

Theorem 3.14. *Let $(G/H, g)$ be a Riemannian g.o. space. For a given* Ad_H-*invariant decomposition* $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ *the Szenthe's construction determines a unique* *naturally reductive Shen connection (with singularities) of the Riemannian g.o. space* $(G/H, g)$.

Proof. According to Szenthe's theorem there exists a canonical (non-linear) horizontal lift $\xi : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ such that for any $X + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h} \setminus \{0\}$ the orbit of $\{\exp t\xi(X + \mathfrak{h}),\ t \in \mathbb{R}\}$ through the origin $o = H \in G/H$ is a geodesic. This (non-linear) horizontal lift is differentiable, possibly with singularities. Indeed, according to Theorem 2.1 in [8] the corresponding geodesic graph $X \to (\xi(X) - X) : \mathfrak{m} \to \mathfrak{h}$ can be uniquely expressed by a rational map on an open dense subset in $\mathfrak{m}$. Hence the map $\sigma : TM \times_M \Lambda_{z_0}G \to T\Lambda_{z_0}G$ determined by the construction of Proposition 3.8 determines a differentiable reductive Shen connection, eventually with singularities. □

O. Kowalski and Ž. Nikčević in Appendix to [8] generalized the notion of geodesic graph as follows:

Definition 3.15. Let $(G/H, g)$ be a Riemannian g.o. space and $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ an Ad_H-invariant decomposition of the corresponding Lie algebra $\mathfrak{g}$. A *general geodesic graph* for G/H is an Ad_H-equivariant map $\eta : \mathfrak{m} \to \mathfrak{h}$ which is analytic on a dense open subset of $\mathfrak{m}$ and such that for each vector $X \in \mathfrak{m} \setminus \{0\}$ the orbit of $\{\exp t(X + \eta(X)),\ t \in \mathbb{R}\}$ through the origin $o = H \in G/H$ is a geodesic.

In the case of naturally reductive spaces $U(3)/U(2)$ and $U(2,1)/U(2)$ they constructed examples of general (non-linear) geodesic graphs which are analytic on $\mathfrak{m}\setminus\{0\}$ and hence the associated naturally reductive Shen connection is differentiable (cf. equation (A) in Proposition 1). This motivates the following

Problem. *Find classes of Riemannian g.o. spaces* $(G/H, g)$ *which are not naturally reductive but their geodesic graphs are* $\mathcal{C}^\infty$*-differentiable or analytic on* $\mathfrak{m} \setminus \{0\}$ *and hence there exists an associated naturally reductive Shen connection in the frame bundle corresponding to* $p : G \to G/H$.

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