

THE EULER-LAGRANGE PDE AND FINSLER METRIZABILITY

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ABSTRACT. In this paper we investigate the following question: under what conditions can a second-order homogeneous ordinary differential equation (spray) be the geodesic equation of a Finsler space. We show that the Euler-Lagrange partial differential system on the energy function can be reduced to a first order system on this same function. In this way we are able to give effective necessary and sufficient conditions for the local existence of a such Finsler metric in terms of the holonomy algebra generated by horizontal vector-fields. We also consider the Landsberg metrization problem and prove similar results. This reduction is a significant step in solving the problem whether or not there exists a non-Berwald Landsberg space.

1. INTRODUCTION

A Finsler structure on an n -manifold M is a nonnegative function $F : TM \rightarrow \mathbb{R}$ that is smooth and positive away from the zero section of TM , positively homogeneous of degree 1, and strictly convex on each tangent space. The energy function $E : TM \rightarrow \mathbb{R}$ associated to a Finsler structure F is defined as $E := \frac{1}{2}F^2$. This is a direct generalization of a Riemannian structure. The fundamental tensor g_E associated to E is formally analogous to the metric tensor in Riemannian geometry. It is defined by

$$(1.1) \quad (g_E)_{ij} := \frac{\partial^2 E}{\partial y^i \partial y^j},$$

in an induced standard coordinate system (x, y) on TM .

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As in Riemannian geometry, a canonical connection Γ can be defined for a Finsler space [5].

However, since the energy function is not necessarily quadratic and only homogeneous, the connection is in general non-linear. We mention two special types of Finsler spaces: *Berwald spaces*, where the connection Γ is linear, and *Landsberg spaces*, where the connection Γ is metric, i.e. the parallel transport preserves the norm defined by g_E .

Suppose that M is an n -manifold endowed with a Finsler structure. The geodesics are the extremals of the variational problem in which the Lagrangian is the energy function. Since g_E is non-degenerate, the parametrization of the extremals is fixed. The geodesic equation associated to a Finsler structure is described by the Euler-Lagrange equations

$$(1.2) \quad \frac{d}{dt} \frac{\partial E}{\partial y^i} - \frac{\partial E}{\partial x^i} = 0, \quad i = 1, \dots, n.$$

Recently several papers were devoted to the problem of characterizing second-order differential equations coming from a Finsler, a special Finsler, or a generalized Finsler structure (see for example [1], [2], [9], [11], [12], [13], [14]). In this paper we offer a contribution to the solution of this problem. Now we formulate the problem from our point of view.

Definition 1. A second-order differential equation on M , locally given by

$$(1.3) \quad \ddot{x}^i = f^i(x, \dot{x}), \quad i = 1, \dots, n,$$

where the functions f^i are positive homogeneous of degree 2 in the \dot{x} variable, is called *Finsler metrizable*, if there exists a Finsler structure whose geodesics are described by (1.3). Moreover, (1.3) is *Landsberg metrizable*, if it is Finsler metrizable, and in addition we also have

$$(1.4) \quad \frac{\partial g_{jk}}{\partial x^i} - \Gamma_i^l \frac{\partial g_{jk}}{\partial y^l} - \Gamma_{ik}^l g_{lj} - \Gamma_{ij}^l g_{lk} = 0,$$

where $\Gamma_j^i := -\frac{1}{2} \frac{\partial f^i}{\partial y^j}$ are the components of the connection Γ associated to (1.3), $\Gamma_{jk}^i := \frac{\partial \Gamma_j^i}{\partial y^k}$, and $g_{ij} = (g_E)_{ij}$.

It follows that a second-order system (1.3) is Finsler metrizable if and only if there exists a function $E : TM \rightarrow \mathbb{R}$ (energy function), so that

- (1) E is homogeneous of degree 2,
- (2) E is a solution of the Euler-Lagrange system (1.2) considered as a second-order partial differential equation with respect to E ,
- (3) the quadratic form g_E defined by (1.1) is positive definite.

Euler's theorem for homogeneous functions implies that the homogeneity condition on E can be described by the equation

$$(1.5) \quad y^i \frac{\partial E}{\partial y^i} - 2E = 0.$$

The Euler-Lagrange partial differential equation associated to (1.3) is

$$(1.6) \quad y^j \frac{\partial^2 E}{\partial x^j \partial y^i} + f^j \frac{\partial^2 E}{\partial y^j \partial y^i} - \frac{\partial E}{\partial x^i} = 0, \quad i = 1, \dots, n,$$

in induced local coordinates (x, y) on TM . We arrive at the reformulation of the metrizable property in terms of a partial differential system:

Proposition 1. *A second-order differential equation (1.3) is*

- (1) *Finsler metrizable, if and only if, there exists a solution $E : TM \rightarrow \mathbb{R}$ to the second-order PDE system formed by the equations (1.5) and (1.6) so that the quadratic form g_E defined in (1.1) is positive definite;*
- (2) *Landsberg metrizable, if and only if there exists a solution $E : TM \rightarrow \mathbb{R}$ to the third-order¹ PDE system (1.4), (1.5) and (1.6) so that the quadratic form g_E is positive definite.*

The main results of this paper can be found in Sections 3 and 4.

In Section 3 we consider the problem of Finsler metrizable. Using the integrability conditions of the corresponding PDE, we show that the system is equivalent to a first order PDE on the same unknown function (Theorem 1). We formulate a necessary and sufficient condition for the local metrizable in terms of a distribution \mathcal{H} associated to the spray (Theorem 2 and 3). \mathcal{H} is called *holonomy distribution* or *holonomy algebra* [8], and it is generated by the horizontal vector fields and their successive Lie-brackets.

In Section 4 we consider the problem of Landsberg metrizable. We show that the corresponding third-order system can be reduced to a first order PDE on the same energy function (Theorem 4). As in the previous case, we are able to formulate a necessary and sufficient condition for the metrizable in terms of a distribution \mathcal{L} (Theorem 5). The distribution \mathcal{L} is generated by the holonomy algebra and the image of the Berwald curvature.

In Sections 5 and 6 we illustrate some consequences of the results on Finsler and Landsberg metrizable. We also discuss the famous problem of whether there exists a non-Berwald Landsberg space. As we show through several examples, Theorem 5 offers a promising alternative approach to solve this problem.

¹The equation (1.4) is a 3rd order PDE, taking into account of (1.1).

2. PRELIMINARIES

2.1. Notations, conventions. Throughout this paper M will denote an n -dimensional smooth manifold. $C^\infty(M)$ denotes the ring of real-valued smooth functions, $\mathfrak{X}(M)$ is the $C^\infty(M)$ -module of vector fields on M , $\pi : TM \rightarrow M$ is the tangent bundle of M , $\mathcal{T}M = TM \setminus 0$ is the slit tangent space. We will essentially work on the manifold TM and on its tangent space TTM . When there is no danger of confusion, TTM and T^*TM will simply be denoted by T and T^* , respectively. $T^v = \text{Ker } \pi_*$ will be the vertical sub-bundle of T .

The exterior differential, the Lie differential (with respect to $X \in \mathfrak{X}(M)$) and the interior product (induced by X) are denoted by d , \mathcal{L}_X and i_X , respectively.

We denote by $\Lambda^k(M)$ and $S^k(M)$ the $C^\infty(M)$ -modules of the skew-symmetric and symmetric k -forms. The Frölicher-Nijenhuis theory provides a complete description of the derivation of $\Lambda(M)$ with the help of vector-valued differential forms, for details we refer to [4]. The i_* and the d_* type derivation associated to a vector valued l -form L will be denoted by i_L and d_L . They can be defined in the following way:

- (1) if $\text{deg } L = 0$, i.e. $L \in \mathfrak{X}(M)$, then $i_L \omega := \omega(L)$, and $d_L \omega := \mathcal{L}_L \omega$;
- (2) if $\text{deg } L = l > 1$, then

$$\begin{aligned} i_L \omega(X_1, \dots, X_l) &:= \omega(L(X_1, \dots, X_l)), & \text{for } \omega \in \Lambda^1(M); \\ d_L f(X_1, \dots, X_l) &:= df(L(X_1, \dots, X_l)), & \text{for } f \in C^\infty(M). \end{aligned}$$

2.2. Geometry associated to a spray. Let J be the canonical vertical endomorphism of T ($= TTM$) and $C \in \mathfrak{X}(TM)$ the canonical vertical vector field. In an induced local coordinate system (x^i, y^i) on TM we have

$$J = dx^i \otimes \frac{\partial}{\partial y^i}, \quad C = y^i \frac{\partial}{\partial y^i}.$$

Remark 1. Using the canonical vector-field, equation (1.5) can be written in the form $P_c E = 0$, where $P_c : C^\infty(TM) \rightarrow C^\infty(TM)$ is a first-order differential operator defined on a function $E : TM \rightarrow \mathbb{R}$ by

$$(2.1) \quad P_c E := \mathcal{L}_C E - 2E.$$

A *spray* is a vector field $S \in \mathfrak{X}(TM)$ on TM satisfying the relations $JS = C$ and $[C, S] = S$. The coordinate representation of a spray S takes the form

$$(2.2) \quad S = y^i \frac{\partial}{\partial x^i} + f^i(x, y) \frac{\partial}{\partial y^i},$$

where $f^i(x, y)$ is positive-homogeneous of degree 2 in $y = (y^j)$. The integral curves of a spray are curves $\gamma : I \rightarrow M$ so that $S \circ \dot{\gamma} = \ddot{\gamma}$. They are the solutions of the equations $\ddot{x}^i = f^i(x, \dot{x})$.

To every spray S a connection $\Gamma := [J, S]$ can be associated [5]. We have $\Gamma^2 = \text{id}_T$, and the eigenspace of Γ corresponding to the eigenvalue -1 is the vertical space T^v . We denote the eigenspace belonging to the eigenvalue $+1$ of Γ by T^h and we call it the *horizontal space*. Then

$$T = T^h \oplus T^v.$$

The horizontal and the vertical projector belonging to Γ are $h := \frac{1}{2}(\text{I} + \Gamma)$, and $v := \text{id}_T - h$. The almost complex structure associated to Γ is the vector valued 1-form F on TM such that $FJ = h$ and $Fh = -J$. The *curvature* of the connection Γ is the vector-valued 2-form

$$(2.3) \quad R := -\frac{1}{2}[h, h].$$

A linear connection on TM , called the *Berwald connection*, can also be associated to S . It is defined by:

$$\nabla \Gamma = 0, \quad \nabla_{hX} JY = [h, JY]X, \quad \nabla_{JX} JY = [J, JY]X;$$

$X, Y \in \mathfrak{X}(TM)$. In an induced coordinate system (x, y) we have

$$(2.4) \quad \begin{cases} \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 0, \\ \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^j} = \nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial x^i} = \Gamma_{ij}^k \frac{\partial}{\partial y^k}, \\ \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k} + \left(\frac{\partial \Gamma_{ij}^l}{\partial x^i} + \Gamma_j^k \Gamma_{ik}^l - \Gamma_k^l \Gamma_{ij}^k \right) \frac{\partial}{\partial y^l}. \end{cases}$$

where $\Gamma_i^k := -\frac{1}{2} \frac{\partial f^k}{\partial y^i}$ and $\Gamma_{ij}^k := \frac{\partial \Gamma_i^k}{\partial y^j}$. Considering the (h, v, v) components of the classical curvature of the Berwald connection we obtain a tensor-field

$$(2.5) \quad \mathcal{R}(X, Y, Z) = \nabla_{hX} \nabla_{JY} JZ - \nabla_{JY} \nabla_{hX} JZ - \nabla_{[hX, JY]} JZ$$

called the *Berwald curvature* in Shen's monograph [13].

Remark 2. Using the coordinate expressions (2.4), it is easy to see that locally we have

$$\mathcal{R} = -\frac{1}{2} \frac{\partial^3 f^l}{\partial y^i \partial y^j \partial y^k} dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial y^l}.$$

Therefore the connection Γ is linear, and the corresponding Finsler space is of Berwald type, if and only if, $\mathcal{R} = 0$.

Remark 3. Using the Berwald connection, we can introduce a third-order differential operator $P_g : C^\infty(TM) \rightarrow Sec(T^* \otimes S^2T^*)$, given by

$$(2.6) \quad (P_g E)(X, Y, Z) := \nabla_{h_X} g_E(JY, JZ),$$

for $X, Y, Z \in \mathfrak{X}(TM)$. Then (1.4) takes the form $P_g E = 0$.

2.3. Lagrangian and spray. A Lagrangian $E : TM \rightarrow \mathbb{R}$ is called *regular*, if the 2-form

$$\Omega_E := dd_J E$$

is symplectic. This holds if and only if $\det\left(\frac{\partial^2 E}{\partial y^\alpha \partial y^\beta}\right) \neq 0$. Let $S \in \mathfrak{X}(TM)$ be a spray. We introduce a second-order differential operator $P_e : C^\infty(TM) \rightarrow Sec T^*$, given by

$$(2.7) \quad P_e E := i_S \Omega_E + d\mathcal{L}_C E - dE.$$

It is not difficult to see that $P_e E$ is a semi-basic 1-form for all $E \in C^\infty(TM)$, and its coordinate representation takes the form $P_e E = \omega_i dx^i$ where the coefficients ω_i are the functions appearing in the left-hand side of the Euler-Lagrange equation (1.6). Therefore S corresponds to the geodesic equation of E if and only if the equation $P_e E = 0$ is valid. So we have the

Remark 4. If S is a spray, then $P_e E = 0$ is the coordinate-free expression of the Euler-Lagrange partial differential equation (1.6) associated to S .

2.4. Formal integrability. In order to solve the metrizable problems formulated above, we have to deal with partial differential systems. We shall use Spencer's technique of formal integrability in the form explained in [7]; for a detailed account see [3]. We recall here only some basic notions in order to fix the terminology.

Let B be a vector bundle over M . If s is a section of B , then $j_{k,p}s = (j_k s)_p$ will denote the k th order jet of s at the point $p \in M$. The bundle of k th order jets of the sections of B is denoted by $J_k B$. In particular $J_k(\mathbb{R}_M)$ will denote the k th order jet of the sections of the trivial line bundle, i.e. the real valued functions. If B_1 and B_2 are two vector bundles over the same manifold M and

$$P : Sec(B_1) \rightarrow Sec(B_2)$$

is a linear differential operator of order k , then the morphism $p_{k+l}(P) : J_{k+l}(B_1) \rightarrow J_l(B_2)$ defined by

$$p_{k+l}(P)(j_{k+l,p}(s)) := j_{l,p}(Ps), \quad l = 0, 1, 2, \dots$$

is called the l th order prolongation of P . $R_{k+l,p}(P) := \text{Ker } p_{k+l}(P)_p$ will denote the bundle of the formal solutions of order $k+l$ at p . A differential operator P is called *formally integrable* at $p \in M$, if $R_{k+l}(P)$ is a vector bundle for all $l \geq 0$, and $\bar{\pi}_{k+l,p} : R_{k+l,p}(P) \rightarrow R_{k+l-1,p}(P)$ is onto for every $l \geq 1$. In analytical terms, formal integrability implies for arbitrary initial data the existence of solutions (see. [3], p. 397).

$\sigma_k(P) : S^k T^*M \otimes B_1 \rightarrow B_2$ is the symbol of P , defined as the highest order terms of the operator, and $\sigma_{k+l}(P) : S^{k+l} T^*M \otimes B_1 \rightarrow S^l T^*M \otimes B_2$ is the symbol of the l -th order prolongation of P . We write

$$g_{k,p}(P) = \text{Ker } \sigma_{k,p}(P),$$

$$g_{k,p}(P)_{e_1 \dots e_j} = \{A \in g_{k,p}(P) \mid i_{e_1} A = \dots = i_{e_j} A = 0\}, \quad j = 1, \dots, n,$$

where $\{e_1, \dots, e_n\}$ is a basis of $T_p M$. A basis $\{e_i\}_{i=1}^n$ of $T_p M$ is called *quasi-regular* if

$$\dim g_{k+1,p}(P) = \dim g_{k,p}(P) + \sum_{j=1}^n \dim g_{k,p}(P)_{e_1 \dots e_j}.$$

A symbol is called *involutive*² at p , if there exists a quasi-regular basis at p . The notion of involutivity allows us to check the formal integrability in quite a simple way:

Theorem (Cartan-Kähler). *Let P be a linear partial differential operator. Suppose that $g_{k+1}(P)$ is regular, i.e. $R_{k+1}(P)$ is a vector bundle on $R_k(P)$. If the map $\bar{\pi}_k : R_{k+1}(P) \rightarrow R_k(P)$ is onto and the symbol is involutive, then P is formally integrable.*

3. FINSLER METRICS WITH PRESCRIBED GEODESICS

In this paragraph we are going to investigate the following problem: *under which conditions can a second order differential equation (1.3) be the geodesic equation of a Finsler metric.* As we explained in Section 1 (Proposition 1) we have to look for a solution of the PDE comprised of (1.5) and (1.6). Therefore we have to deal with the second-order system

$$(3.1) \quad P_F := (P_c, P_e)$$

²There is a slight problem of language here. In the works of Cartan, and more generally in the theory of exterior differential systems, “involutivity” means more than the existence of a quasi-regular basis and it refers to “integrability” (cf. [3], p.107, 140). Here we are following the terminology of Goldschmidt (cf. [3], p. 409).

where P_c and P_e are defined in (2.1) and (2.7). We will prove the following theorems:

Theorem 1. (Reduction of P_F .) *A Lagrangian $E : TM \rightarrow \mathbb{R}$ is a solution of the second order operator P_F , if and only if, it is a solution of the first order system*

$$(3.2) \quad \begin{cases} \mathcal{L}_C E - 2E = 0, \\ d_{\mathfrak{h}} E = 0, \end{cases}$$

where $\mathcal{H} \subset T(= TTM)$ is the holonomy algebra generated by the horizontal vector fields and their successive Lie-brackets, and $\mathfrak{h} : T \rightarrow \mathcal{H}$ is an arbitrary projection on \mathcal{H} .

Remark 5. *For $X \in \mathfrak{X}(TM)$ we have $d_{\mathfrak{h}} E(X) = \mathfrak{h}X(E) = \mathcal{L}_{\mathfrak{h}X} E$, so the second equation of (3.2) means simply that the Lie-derivative of E with respect to vector-fields in the holonomy distribution $\mathcal{H} = \text{Im } \mathfrak{h}$ is zero. This property is independent of the projection \mathfrak{h} of \mathcal{H} chosen.*

PROOF OF THEOREM 1. Let us suppose that $E : TM \rightarrow \mathbb{R}$ is a solution of (3.2). Since $T^h \subset \mathcal{H}$, we have $\mathfrak{h} \circ h = h$. Therefore

$$d_h E = d_{\mathfrak{h} \circ h} E = i_h d_{\mathfrak{h}} E - d_{\mathfrak{h}} i_h E + i_{[h, \mathfrak{h}]} E = i_h d_{\mathfrak{h}} E = 0$$

since the action of an i_* -type derivation is trivial on functions. Moreover as S is homogeneous, $hS = S$ and

$$\mathcal{L}_S E = \mathcal{L}_{hS} E = d_h E(S) = 0.$$

Writing the Euler-Lagrange operator in the form

$$P_e E = i_S d d_J E + d \mathcal{L}_C E - dE = d_J \mathcal{L}_S E - i_{[J, S]} dE = d_J \mathcal{L}_S E - 2d_h E$$

we obtain that $P_e E = 0$ and E is a solution of (3.1).

Let us suppose now that $E : TM \rightarrow \mathbb{R}$ is a solution of (3.1). We have

$$(3.3) \quad i_S \Omega_E = d(E - \mathcal{L}_C E) = -dE.$$

Since $[J, J] = 0$, we have $d_J^2 = d_J \circ d_J = d_{[J, J]} = 0$, and $i_J \Omega_E = 0$, so

$$(3.4) \quad i_C \Omega_E = i_{JS} \Omega_E = i_S i_J \Omega_E - i_J i_S \Omega_E = i_J dE.$$

On the other hand, for every $X \in \mathfrak{X}(TM)$ we have

$$i_S \Omega_E(X) = \Omega_E(S, X) = -\Omega_E(C, FX) = -i_F i_C \Omega_E(X),$$

i.e.

$$(3.5) \quad i_S \Omega_E = i_F i_C \Omega_E.$$

Putting (3.4) into (3.5) we obtain

$$(3.6) \quad i_S \Omega_E = -i_F i_C \Omega_E = -i_F i_J dE = -d_v E = -dE + d_h E.$$

Comparing (3.6) with (3.3) we obtain that $d_h E = 0$. It follows that $hX(E) = 0$, i.e. E is constant with respect to horizontal vector fields. Therefore it must be constant on the distribution generated by the horizontal sub-bundle taking the recursive Lie-bracket operations, i.e. on \mathcal{H} . This means that we have $d_{\mathfrak{h}} E = 0$ and E is a solution of (3.2). \square

Remark 6. E is a solution of (3.2) if and only if it is a solution of

$$(3.2') \quad \begin{cases} \mathcal{L}_C E - 2E = 0, \\ d_h E = 0, \end{cases}$$

where h is simply the horizontal projection associated to Γ , so (3.2) and (3.2') are equivalent. However, as we will see in Proposition 2, under regularity assumption the system (3.2) is integrable while (3.2') is not, unless the curvature is zero. Indeed, we have

$$d_R E = -\frac{1}{2} d_{[h,h]} E = -\frac{1}{2} d_h d_h E,$$

therefore $d_R E = 0$ is a compatibility condition for (3.2').

Remark 7. Let us introduce the first order differential operator $P_{\mathfrak{h}} : C^\infty(TM) \longrightarrow Sec(T^*)$ by the rule

$$(3.7) \quad P_{\mathfrak{h}} E(X) := \mathfrak{h}X(E),$$

$E \in C^\infty(TM)$, $X \in \mathfrak{X}(TM)$, and the differential operator

$$(3.8) \quad P_F^2 := (P_c, P_{\mathfrak{h}})$$

corresponding to the system (3.2). Theorem 1 shows that a Lagrangian is a solution of P_F if and only if it is a solution of P_F^2 .

Theorem 2. *Let S be a spray over the manifold M . If $C \in \mathcal{H}$, then there is no Finsler metric whose geodesics are given by S .*

PROOF. Let S be a spray and $E : TM \rightarrow \mathbb{R}$ a Lagrangian. From Proposition 1 we know that if E is an energy function associated to S , then it is a solution of $P_F = (P_c, P_e)$, and by Theorem 1 we obtain that E satisfies the equations $\mathcal{L}_C E - 2E = 0$ and $d_{\mathfrak{h}} E = 0$. If $C \in \mathcal{H}$, we have also

$$0 = P_{\mathfrak{h}} E(C) = (\mathfrak{h}C)E = CE = \mathcal{L}_C E,$$

therefore $E = 0$. Since E has to be a regular Lagrangian, this is impossible and the proposition is proved. \square

Let us consider the case when $C \notin \mathcal{H}$. We have the following

Theorem 3. *Let S be an analytical spray over the analytical manifold M . If $C \notin \mathcal{H}$ and \mathcal{H} has constant rank in a neighbourhood of $v \in \mathcal{TM}$, then there exists an analytical Finsler metric in a neighbourhood of v such that the geodesics are given by S if and only if the kernel of the first prolongation of (3.2) at v contains positive definite initial data.*

Remark 8. Let (x^i) be a local coordinate system on M , (x^i, y^i) the associated coordinate system on \mathcal{TM} in the neighborhood of v . If $p := j_k(E)_v \in J_2(\mathbb{R}_{\mathcal{TM}})$ is a k th order jet of a real valued function E on \mathcal{TM} we set

$$(3.9) \quad s_{i_1 \dots i_a i_{a+1} \dots i_l}(p) := \frac{\partial^l E}{\partial x^{i_1} \dots \partial x^{i_a} \partial y^{i_{a+1}} \dots \partial y^{i_l}}(v), \quad 1 \leq l \leq k.$$

Then $(s, s_j, s_{\underline{j}}, s_{jk}, s_{\underline{jk}}, s_{\underline{jk}})$ gives a coordinate system on $J_{2,v}(\mathbb{R}_{\mathcal{TM}})$. Using the notation of (3.8) introduced in Remark 7, positive definite initial data for the first prolongation of (3.2) at v is simply an element $s_{2,v} \in J_{2,v}(\mathbb{R}_{\mathcal{TM}})$ represented as $s_{2,v} = (s, s_i, s_{\underline{i}}, s_{ij}, s_{\underline{ij}}, s_{\underline{ij}}) \in \mathbb{R}^{1+(n+n)+\frac{n(n+1)}{2}+n^2+\frac{n(n+1)}{2}}$ such that $(s_{\underline{ij}})_{1 \leq i, j \leq n}$ determines a positive definite quadratic form, and $s_{2,v}$ is a second order solution of P_F^2 at v . This last condition gives *linear algebraic* equations on the coordinates of $s_{2,v}$.

PROOF OF THEOREM 3. The proof is based on Theorem 1 and on Proposition 2 proved below. Indeed, if P_F^2 is formally integrable (see Proposition 2), then for every initial condition we have an infinite order formal solution of P_F^2 . In the analytic case, this formal solution gives an analytical solution in an open neighborhood of \mathcal{TM} . Theorem 1 shows that this solution is also a solution of the operator P_F . In this way we obtain an analytical solution of P_F , i.e. a homogeneous function which satisfies the Euler-Lagrange equation associated to S . Therefore S is locally Finsler metrizable. \square

Proposition 2. *Let S be a spray over M so that $C \notin \mathcal{H}$ and the rank of \mathcal{H} is locally constant. Then the differential operator $P_F^2 = (P_c, P_{\mathfrak{h}})$ is formally integrable.*

PROOF. First of all remark that P_F^2 is a regular differential operator because, by the hypothesis, rank of $\mathcal{H} = \text{Im } \mathfrak{h}$ is locally constant. Moreover, using Lemma 1, Lemma 2 and the Cartan-Kähler theorem on formal integrability (see page 85), we obtain the proposition. \square

Lemma 1. *Every first order solution of P_F^2 can be lifted to a second order solution.*

PROOF. It is easy to see from their local description that the symbol of P_c and $P_{\mathfrak{h}}$ can be interpreted as a map

$$\begin{aligned}\sigma_1(P_c) : T^* &\rightarrow \mathbb{R}, & \sigma_1(P_c)B_1 &= B_1(C) \\ \sigma_1(P_{\mathfrak{h}}) : T^* &\rightarrow T^* & (\sigma_1(P_{\mathfrak{h}})B_1)(X) &= B_1(\mathfrak{h}X)\end{aligned}$$

for all $B_1 \in T^*$, $X \in T$. The symbol of the first prolongations are defined by

$$\begin{aligned}\sigma_2(P_c) : S^2T^* &\rightarrow T^*, & (\sigma_2(P_c)B_2)(X) &= B_2(X, C) \\ \sigma_2(P_{\mathfrak{h}}) : S^2T^* &\rightarrow T^* \otimes T^* & (\sigma_2(P_{\mathfrak{h}})B_2)(X, Y) &= B_2(X, \mathfrak{h}Y)\end{aligned}$$

for all $B_2 \in S^2T^*$, $X, Y \in T$. Comparing the first prolongation of the symbols, we can easily find that for every $B_2 \in S^2T^*$ and $X \in T$ we have

$$(\sigma_2(P_c)B_2)(\mathfrak{h}X) - (\sigma_2(P_{\mathfrak{h}})B_2)(C, X) = B_2(\mathfrak{h}X, C) - B_2(C, \mathfrak{h}X) = 0,$$

and there is no more relation between the two symbols. That is, if we consider the map $\tau : T^* \oplus (T^* \otimes T^*) \rightarrow T^*$ defined for $B_1 \in T^*$, $B_2 \in T^* \otimes T^*$ and for $X \in T$ as

$$(\tau(B_1, B_2))(X) := B_1(\mathfrak{h}X) - B_2(C, X),$$

then we find the commutative diagram:

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & g_2(P_F^2) & \xrightarrow{i} & S^2T^* & \xrightarrow{\sigma_2(P_F^2)} & T^* \oplus (T^* \otimes T^*) & \xrightarrow{\tau} & T^* & \longrightarrow & 0 \\ & \downarrow & & \downarrow & & \downarrow & & & & & \\ 0 & \longrightarrow & R_2 & \xrightarrow{i} & J_2(\mathbb{R}_{TM}) & \xrightarrow{p_2(P_F^2)} & J_1(\mathbb{R}_{TM} \oplus T^*) & & & & \\ & \downarrow \bar{\pi} & & \downarrow \pi & & \downarrow \pi & & & & & \\ 0 & \longrightarrow & R_1 & \xrightarrow{i} & J_1(\mathbb{R}_{TM}) & \xrightarrow{p_1(P_F^2)} & \mathbb{R} \oplus T^* & & & & \\ & & & & \downarrow & & \downarrow & & & & \\ & & & & 0 & & 0 & & & & \end{array}$$

where the successive arrows represent exact sequences. (R_1 and R_2 denote the spaces of the first and second order formal solutions of P_F^2 .)

Every first order solution of P_F^2 can be lifted into a second-order formal solution if and only if the map $\bar{\pi} : \mathcal{R}_2 \rightarrow \mathcal{R}_1$ is onto. We know by a lemma of homological

algebra that there exists a map $\varphi : R_1 \longrightarrow T^*$ ($= \text{Im } \tau$) such that

$$(3.10) \quad \text{Im } \bar{\pi} = \text{Ker } \varphi.$$

This map can be constructed for a first order formal solution $j_{1,v}(E) \in R_1$ of P_F^2 at $v \in TM$ as follows:

$$(3.11) \quad \varphi_v(E) := \tau(\nabla P_F^2 E)_v.$$

Let us compute how this map acts. If $E : TM \rightarrow \mathbb{R}$ is a function such that $j_{1,v}(E) \in R_{1,v}$, then $(P_F^2 E)_v = 0$, that is $(P_{\mathfrak{h}} E)_v = 0$ and $(P_c E)_v = 0$. Evaluating $\varphi_v(E)$ on an arbitrary vector $X \in T$ we find that

$$\begin{aligned} \varphi_v(E)(X) &= \tau(\nabla P_F^2 E)_v(X) = (\nabla P_c E)_v(\mathfrak{h}X) - (\nabla P_{\mathfrak{h}} E)_v(C, X) \\ &= (\mathcal{L}_{\mathfrak{h}X}(\mathcal{L}_C E - 2E) - \mathcal{L}_C(\mathcal{L}_{\mathfrak{h}X} E))_v = (\mathcal{L}_{\mathfrak{h}X}(\mathcal{L}_C E) - \mathcal{L}_C(\mathcal{L}_{\mathfrak{h}X} E) - 2\mathcal{L}_{\mathfrak{h}X} E)_v \\ &= (\mathcal{L}_{[\mathfrak{h}X, C]} E)_v - 2(P_{\mathfrak{h}} E)_v(X) = (\mathcal{L}_{[\mathfrak{h}X, C]} E)_v. \end{aligned}$$

Now we can remark, that if $X \in \mathfrak{X}(TM)$, then $[\mathfrak{h}X, C] \in \mathcal{H}$. Indeed \mathcal{H} can be generated by the successive brackets of the horizontal basis $\{h_1, \dots, h_n\}$, where $h_i := h(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} - \Gamma_i^\alpha \frac{\partial}{\partial y^\alpha}$. Since S is homogeneous, we have $[h_i, C] = 0$. By the Jacobi identity, this is also true for the successive brackets of the h_i 's. If we consider an arbitrary $Y \in \mathcal{H}$, then it can be written as a linear combination of the elements $Y = g^\alpha Y_\alpha$, where Y_α can be obtained by successive brackets of the h_i 's. Thus we have

$$[Y, C] = [g^\alpha Y_\alpha, C] = -(Cg^\alpha)Y_\alpha + g^\alpha [Y_\alpha, C] = -(Cg^\alpha)Y_\alpha,$$

which shows that $[Y, C] \in \mathcal{H}$.

Continuing the above computation of $\varphi_v(E)$ we find that

$$\varphi_v(E)(X) = (\mathcal{L}_{[\mathfrak{h}X, C]} E)_v = (\mathcal{L}_{\mathfrak{h}[\mathfrak{h}X, C]} E)_v = (P_{\mathfrak{h}} E)_v([\mathfrak{h}X, C]) = 0,$$

since $(P_{\mathfrak{h}} E)_v$ vanishes on \mathcal{H}_v . It follows that φ_v is identically zero, by (3.10) we conclude that $\text{Ker } \varphi_v = R_1$, and $\bar{\pi}$ is onto. Hence every first order solution of P_F^2 can be lifted into a second order solution. \square

Lemma 2. *The symbol of P_F^2 is involutive.*

PROOF. Let k be the co-dimension of \mathcal{H} . Since $\dim \mathcal{H} \geq n$ and $C \notin \mathcal{H}$, we have $n \leq \dim \mathcal{H} \leq 2n - 1$ and $1 \leq k \leq n$. Let us consider the basis

$$(3.12) \quad \{e_1, \dots, e_{2n}\} := \{v_1, \dots, v_n, h_1, \dots, h_n\}$$

of T at $v \in TM$ where v_1, \dots, v_n are vertical, h_1, \dots, h_n are horizontal, the last $2n - k$ vectors generate \mathcal{H} and $v_k := C$. Then we have

$$\begin{aligned} g_1(P_F^2) &:= \text{Ker}(\sigma_1(P_F^2)) = \{B_1 \in T^* \mid B_1(e_i) = 0, i = k, \dots, 2n\}, \\ g_2(P_F^2) &:= \text{Ker}(\sigma_2(P_F^2)) = \{B_2 \in S^2T^* \mid B_2(e_i, e_j) = 0, i = 1, \dots, 2n, j = k, \dots, 2n\} \end{aligned}$$

and for $1 \leq m < k$,

$$\begin{aligned} g_{1, e_1, \dots, e_m}(P_F^2) &:= \text{Ker}(\sigma_1(P_F^2)) \cap \{B_1 \in T^* \mid B_1(e_i) = 0, i = 1, \dots, m\} \\ &= \{B_1 \in T^* \mid B_1(e_i) = 0, i \in \{1, \dots, m\} \cup \{k, \dots, 2n\}\}. \end{aligned}$$

The dimension of these spaces are

$$\begin{aligned} \dim(g_1(P_F^2)) &= k - 1, \\ \dim(g_2(P_F^2)) &= \frac{k(k-1)}{2}, \\ \dim(g_{1, e_1, \dots, e_m}(P_F^2)) &= \begin{cases} k-1-m, & \text{for } m = 1, \dots, k-1, \\ 0, & \text{for } m = k, \dots, 2n, \end{cases} \end{aligned}$$

therefore

$$\begin{aligned} \dim(g_1(P_F^2)) + \sum_{m=1}^{2n} \dim(g_{1, e_1, \dots, e_m}(P_F^2)) &= (k-1) + \sum_{m=1}^{k-1} (k-1-m) = \frac{(k-1)k}{2} \\ &= \dim(g_2(P_F^2)). \end{aligned}$$

This shows that (3.12) is a quasi-regular basis for P_F^2 . The existence of such a basis proves Lemma 2. \square

4. LANDSBERG METRIZABILITY

In this paragraph we will investigate the following problem: *under what conditions can a given second order differential equation (1.3) be the geodesic equation of a Finsler metric of Landsberg type?* As we explained in Proposition 1, to answer this question we have to look for a solution of the PDE system consisting of (1.5), (1.6) and (1.4). Let us consider the third order system

$$(4.1) \quad P_L = (P_c, P_e, P_g)$$

where P_c , P_g and P_e are defined by (2.1), (2.6) and (2.7). We will prove the following theorems:

Theorem 4. (Reduction of P_L) *The third-order partial differential system $P_L E = 0$ is equivalent to the first order system*

$$(4.2) \quad \begin{cases} \mathcal{L}_C E - 2E = 0, \\ d_\ell E = 0, \end{cases}$$

where \mathfrak{L} is the distribution generated by the horizontal vector fields, the image of the Berwald curvature and their successive Lie-brackets and $\ell : TTM \rightarrow \mathfrak{L}$ is an arbitrary projection of TTM onto \mathfrak{L} .

Remark 9. The second equation of (4.2) means simply that the Lie-derivative of E with respect to vector-fields in the distribution $\mathfrak{L} = \text{Im } \ell$ is zero. This property is independent of the projection ℓ of \mathfrak{L} chosen.

Remark 10. E is a solution of (4.2) if and only if it is a solution of

$$(4.2') \quad \begin{cases} \mathcal{L}_C E - 2E = 0, \\ d_h E = 0, \\ d_{\mathcal{R}} E = 0, \end{cases}$$

where h is simply the horizontal projection associated to Γ . However, under the assumption of regularity, the system (4.2) is integrable but (4.2') in general is not, because it is not containing its compatibility conditions.

Theorem 5. *Let S and M be analytical, and suppose that rank of \mathfrak{L} constant in a neighborhood of $v \in TM$. Then there exists a Finsler metric of Landsberg type in a neighborhood of v whose geodesics are given by S , if and only if, $C \notin \mathfrak{L}$, and the kernel of the first prolongation of (4.2) at v contains a positive definite initial condition.*

In order to prove the above theorems, we need the following

Lemma 3. *Let us consider the differential operator $d_{\mathcal{R}} : C^\infty(TM) \rightarrow \text{Sec}(S^3 T^*)$, where \mathcal{R} is the Berwald curvature. For all $X, Y, Z \in T$ we have*

$$(4.3) \quad P_g E(X, Y, Z) = \nabla^2 P_{\mathfrak{b}} E(JY, JZ, hX) + d_{\mathcal{R}} E(X, Y, Z),$$

where ∇ is the Berwald connection and $P_{\mathfrak{b}}$ is introduced in (3.7).

PROOF. The three terms in (4.3) are all semi-basic in X , Y and Z . Putting $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$ and $Z = \frac{\partial}{\partial x^k}$, we have

$$\begin{aligned} (P_g E) \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) - (\nabla^2 P_{\mathfrak{h}} E) \left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}, h \left(\frac{\partial}{\partial x^k} \right) \right) &= \frac{\partial^3 E}{\partial x^i \partial y^j \partial y^k} \\ &\quad - \Gamma_i^l \frac{\partial^3 E}{\partial y^l \partial y^j \partial y^k} - \Gamma_{ij}^l \frac{\partial^2 E}{\partial y^l \partial y^k} - \Gamma_{ik}^l \frac{\partial^2 E}{\partial y^l \partial y^j} - \frac{\partial}{\partial y^j} \frac{\partial}{\partial y^k} \left(\frac{\partial E}{\partial x^i} - \Gamma_i^l \frac{\partial E}{\partial y^l} \right) \\ &= \frac{\partial^2 \Gamma_i^l}{\partial y^j \partial y^k} \frac{\partial E}{\partial y^l} = -\frac{1}{2} \frac{\partial^3 f^l}{\partial y^i \partial y^j \partial y^k} \frac{\partial E}{\partial y^l} = d_{\mathcal{R}} E \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right). \end{aligned}$$

□

Remark 11. Since $\mathfrak{h} \circ h = h$, we have $P_{\mathfrak{h}}(hX) = d_h E(hX)$, and we have the relation

$$(4.3') \quad P_g E(X, Y, Z) = \nabla^2 d_h E(JY, JZ, hX) + d_{\mathcal{R}} E(X, Y, Z).$$

expressed in terms of the horizontal projection h .

PROOF OF THEOREM 4. 1) If $E : TM \rightarrow \mathbb{R}$ is a solution of $P_L = (P_c, P_e, P_g)$, then by Theorem 1 we obtain that $P_{\mathfrak{h}} E = 0$. In particular, E is constant on the horizontal distribution. Moreover, we can find from (4.3) that $d_{\mathcal{R}} E = 0$, i.e. E is constant on the image of the Berwald curvature. Consequently E has to be constant on the distribution \mathfrak{L} generated by the horizontal vector-field and the image of Berwald curvature.

2) Conversely, let $E : TM \rightarrow \mathbb{R}$ be a solution of (4.2). By the construction $\mathcal{H} \subset \mathfrak{L}$, we obtain that E is a solution of $P_{\mathfrak{h}}$. By Theorem 1, E is also a solution of P_c and P_e . Moreover, $\text{Im } \mathcal{R} \subset \mathfrak{L}$ implies $d_{\mathcal{R}} E = 0$ and by Lemma 3 we have $P_g E = 0$. Therefore E is a solution of the system $P_L = (P_c, P_e, P_g)$.

By 1) and 2) we conclude that the system $P_L = (P_c, P_e, P_g)$ is equivalent to the 1st order system (4.2) which proves Theorem 4. □

PROOF OF THEOREM 5. The reasoning is completely analogous to the proof of Theorem 3. Indeed, $E : TM \rightarrow \mathbb{R}$ is a Landsberg-type Finsler metric associated to S , if and only if, g_E is positive definite and E is a solution of the system $P_L = (P_c, P_e, P_g)$.

If $C \in \mathfrak{L}$ and $E : TM \rightarrow \mathbb{R}$ is a solution of (4.2), then $dE = 0$. So E is not a regular Lagrangian, and S cannot be variational.

Suppose that $C \notin \mathfrak{L}$ and that \mathfrak{L} has constant rank in a neighbourhood of $v \in TM$. By Theorem 4 we know that E is a solution of P_L if and only if it is a

solution of (4.2). Therefore, it is sufficient to consider this first order PDE and show that it has a solution.

By the hypotheses, \mathfrak{L} is of constant rank in a neighbourhood of $v \in TM$, the system (4.2) is regular.

A computation, completely analogous to that of made in the proof of Proposition 2, shows that (4.2) is formally integrable. Consequently, for every initial condition, there exists an analytical solution to (4.2) in a neighbourhood of $v \in TM$. Using Theorem 4, this function will be a solution of the system $P_L = (P_c, P_e, P_g)$, and therefore it will be a Landsberg type Finsler metric in a neighborhood of v with geodesics determined by S . \square

5. REMARKS AND EXAMPLES OF FINSLER AND LANDSBERG METRIZABILITY

Theorems 1, 2, 3, 4 and 5 give us a powerful method to test the metrizability of a second order ordinary differential system. We mention here only some direct consequences.

Proposition 3. *A quadratic second order differential equation is Landsberg metrizable if and only if it is Finsler metrizable.*

Indeed, in the quadratic case, the functions $f^i(x, \dot{x})$ are quadratic in the \dot{x} variable and the Berwald curvature \mathcal{R} vanishes identically. Therefore the distribution \mathfrak{L} coincides with \mathcal{H} . \square

Theorem 6. *If $\text{rank } \mathfrak{L} = 2n$ (resp. $\text{rank } \mathcal{H} = 2n$), then the spray is not Landsberg (resp. Finsler) metrizable.*

Indeed, in this cases $\mathfrak{L} = T$ (resp. $\mathcal{H} = T$). If $E : TM \rightarrow \mathbb{R}$ is a solution of (4.2) (resp. (3.2)), then $dE = 0$, and E cannot be a regular Lagrangian. \square

Examples

- (1) For a generic spray, the image of the curvature R and the image of \mathcal{R} generate the whole vertical space. In this case $\mathfrak{L} = T$, and therefore there is no a regular solution to (4.2).
- (2) In some cases, even if the image of the curvature R and the image of \mathcal{R} do not generate the whole vertical space, nevertheless $\mathfrak{L} = T$. For example let $f(t) := a\sqrt{t^2 + bt + c}$ with a, b, c nonzero reals, and consider the system

$$(5.1) \quad \ddot{x}_1 = \dot{x}_1^2 f\left(\frac{\dot{x}_2}{\dot{x}_1}\right), \quad \ddot{x}_2 = \dot{x}_1 \dot{x}_2 f\left(\frac{\dot{x}_2}{\dot{x}_1}\right).$$

In this case $\text{Im } \mathcal{R} = \text{Im } R$ is a 1-dimensional distribution of T . However, by computing the Lie-brackets of horizontal vector fields with the generator of $\text{Im } \mathcal{R}$ we find that

$$\left[h \frac{\partial}{\partial x^i}, \mathcal{R} \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1} \right) \right]_{i=1,2} \in \text{Im } \mathcal{R} \quad \Leftrightarrow \quad b = c = 0.$$

Therefore we have $\dim \mathfrak{L} = 4$, so $\mathfrak{L} = T$ and (5.1) is not Landsberg metrizable.

Theorem 7. *If \mathcal{H} (resp. \mathfrak{L}) contains the vertical lift of a non-zero vector field on M , then the spray is not Finsler (resp. Landsberg) metrizable.*

PROOF. Let $X \in \mathcal{H}$ (resp. $X \in \mathfrak{L}$) be a vertical lift, namely $X = Z^v$, $Z \in \mathfrak{X}(M)$. Then, locally, $X = (X^\alpha \circ \pi) \frac{\partial}{\partial y^\alpha}$, where the functions X^α are defined on a domain of M . If S is Finsler (resp. Landsberg) metrizable, then the corresponding energy function $E : TM \rightarrow \mathbb{R}$ is a regular Lagrangian, and it is a solution of P_F (resp. P_L). Using Theorem 1 (resp. Theorem 4) we get that E is a solution of (3.2) (resp. (4.2)), and in particular E is constant on every vector fields of \mathcal{H} (resp. \mathfrak{L}).

Since $X \in \mathcal{H}$ (resp. $X \in \mathfrak{L}$) we have $\mathcal{L}_X E = 0$. Taking the derivatives with respect to the vertical directions and using the special form of X we obtain that

$$0 = \frac{\partial \mathcal{L}_X E}{\partial y^i} = \frac{\partial}{\partial y^i} \left((X^j \circ \pi) \frac{\partial E}{\partial y^j} \right) = (X^j \circ \pi) \frac{\partial^2 E}{\partial y^j \partial y^i},$$

so E cannot be a regular Lagrangian. This contradicts the hypothesis. Therefore the spray is not Finsler (resp. Landsberg) metrizable. \square

Example. Let us consider the system

$$(5.2) \quad \begin{cases} \ddot{x}^1 := \lambda_1(x) f(x, \dot{x}), \\ \ddot{x}^2 := \lambda_2(x) f(x, \dot{x}), \end{cases}$$

where $f(x, y)$ is an arbitrary second order homogeneous but non-quadratic function in $y = (y^1, y^2)$ and λ_1, λ_2 arbitrary functions of $x = (x^1, x^2)$. In this case the image of the Berwald curvature is generated by the vertically lifted vector field $X = \lambda_1 \frac{\partial}{\partial y^1} + \lambda_2 \frac{\partial}{\partial y^2}$. Thus, by Theorem 7, the system is not Landsberg metrizable.

6. ON THE EXISTENCE OF NON-BERWALD TYPE LANDSBERG SPACES

A Landsberg metric is said to be of Berwald type if the connection Γ is linear, that is in its geodesic equations $\ddot{x}^i = f^i(x, \dot{x})$ the functions are quadratic in \dot{x} . These types of spaces can be characterized in terms of the Berwald curvature: a Landsberg space is of Berwald type if and only if the Berwald curvature introduced in (2.5) vanishes. One of the most exciting questions in Finsler geometry is the following:

Are there any non-Berwald Landsberg metrics on a manifold?

To answer this question a promising strategy is to investigate the solvability of the system $P_L = 0$, which is a third order differential system. Theorems 4 and 5 can be useful for this purpose, because they provide a reduction of P_L to a much simpler first order differential system. Far from exploring fully the possibilities offered by the above theorems, we shall be content to make the following observations.

Theorem 8. *There is no a nontrivial analytic function f such that the equations*

$$(6.1) \quad \begin{aligned} \ddot{x}_1 &= \dot{x}_1^2 f(\dot{x}_2/\dot{x}_1), \\ \ddot{x}_2 &= \dot{x}_1 \dot{x}_2 f(\dot{x}_2/\dot{x}_1) \end{aligned}$$

constitute the geodesic system of a non-Berwald type Landsberg metric.

PROOF. If $f \neq 0$, then $\mathcal{R} \neq 0$. Unless f satisfies the equation $3f''f' + ff''' = 0$, we have $\text{Im } \mathcal{R} \neq \text{Im } R$. In this case \mathcal{L} is the entire second tangent bundle TTM , and consequently there is no corresponding Landsberg metric.

If f satisfies the above equation, then it has the form $f(t) = a\sqrt{t^2 + bt + c}$ with $a, b, c \in \mathbb{R}$. Computing the Lie brackets $[h(\frac{\partial}{\partial x^i}), \mathcal{R}(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1})]$ we find that they are in the subspace generated by $\text{Im } \mathcal{R}$ if and only if $b = c = 0$. But in this case $\mathcal{R} = 0$ which contradicts our hypotheses. \square

Proposition 4. *The system*

$$(6.2) \quad \begin{aligned} \ddot{x}_1 &= a \dot{x}_1^{2-t} \dot{x}_2^t, & a \in \mathbb{R}, t \in \mathbb{N}, \\ \ddot{x}_2 &= b \dot{x}_1^{2-s} \dot{x}_2^s, & b \in \mathbb{R}, s \in \mathbb{N}, \end{aligned}$$

cannot be the geodesic system of a non-Berwald type Landsberg metric.

PROOF. Let us consider the spray S corresponding to (6.2):

$$S = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} + a y_1^{2-t} y_2^t \frac{\partial}{\partial y^1} + b y_1^{2-s} y_2^s \frac{\partial}{\partial y^2}.$$

If $s \in \{0, 1, 2\}$ or $t \in \{0, 1, 2\}$, then S is not the geodesic equation of a non-Berwald type Landsberg metric.

Indeed, in case of $s, t \in \{0, 1, 2\}$, then $\mathcal{R} = 0$ and therefore the Berwald connection is linear. If the system is Finsler-metrizable, then it is also Landsberg metrizable (Corollary 3), and the corresponding Finsler space is of Berwald type. If $s \in \{0, 1, 2\}$ or $t \in \{0, 1, 2\}$, then $\text{Im } \mathcal{R}$ generated by $\frac{\partial}{\partial y^1}$ or $\frac{\partial}{\partial y^2}$. As we explained in Theorem 7, in these cases there is no regular Lagrangian associated to the system.

If $s, t \notin \{0, 1, 2\}$, then the image of \mathcal{R} is generated by the vector-fields

$$\frac{\partial}{\partial y_1} + \frac{bs(s-1)(s-2)y_2^{s-t}}{at(t-1)(t-2)y_1^{s-t}} \frac{\partial}{\partial y_2}.$$

If in addition $s = t$, then using Theorem 7 we obtain that there is no regular Lagrangian associated to the system. Let us suppose now that $s \neq t$. The image of the curvature R is generated by the vector field

$$\frac{\partial}{\partial y_1} + \frac{b y_2^s y_1^{2t-s+2} bs(2-s) + y_2^{t+1} y_1^{t+1} a(2s^2 - 4s + 2t - st)}{y_2^{t+1} y_1^{t+1} at(t-2) + y_2^s y_1^{2t-s+2} b(st - 2t^2 + 2t)} \left(\frac{y_2}{y_1}\right)^{s-t} \frac{\partial}{\partial y_2}.$$

If $s \neq t + 1$, or $s \neq t - 1$, then $\text{Im } R \neq \text{Im } \mathcal{R}$. Since $T^h \oplus \text{Im } R \oplus \text{Im } \mathcal{R} \subset \mathcal{L}$ we obtain that $\mathcal{L} = T$. Using Theorem 6 we find that S is not Landsberg-metrizable.

If $s = t + 1$ or $s = t - 1$, then $\text{Im } R = \text{Im } \mathcal{R}$. Computing the Lie-brackets of the horizontal vector-fields with the image of the Berwald curvature we find that

$$[T^h, \text{Im } \mathcal{R}] \not\subset T^h \oplus \text{Im } \mathcal{R}.$$

We arrive at $\mathcal{L} = T$, and using Theorem 6 we conclude again that S is not Landsberg-metrizable. \square

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