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Tangent Lie algebras to the holonomy group of a Finsler manifold

Zoltán Muzsnay, Péter T. Nagy

Abstract. Our goal in this paper is to make an attempt to find the largest Lie algebra of vector fields on the indicatrix such that all its elements are tangent to the holonomy group of a Finsler manifold. First, we introduce the notion of the *curvature algebra*, generated by curvature vector fields, then we define the *infinitesimal holonomy algebra* by the smallest Lie algebra of vector fields on an indicatrix, containing the curvature vector fields and their horizontal covariant derivatives with respect to the Berwald connection. At the end we introduce conjugates of infinitesimal holonomy algebras by parallel translations with respect to the Berwald connection. We prove that this holonomy algebra is tangent to the holonomy group.

1 Introduction

The notion of the holonomy group of Riemannian manifolds can be generalized very naturally for Finsler manifolds: it is the group generated by canonical homogeneous (nonlinear) parallel translations along closed loops. Until now the holonomy groups of non-Riemannian Finsler manifolds were described only in special cases: the Berwald manifolds have the same holonomy group as some Riemannian manifolds (cf. Z.I. Szabó, [8]) and the holonomy groups of Landsberg manifolds are compact Lie groups (cf. L. Kozma, [3]). A thorough study of the holonomy algebras of homogeneous (nonlinear) connections was initiated by W. Barthel [1], he gave a successive extension by Berwald's covariant derivation of the Lie algebras generated by the curvature vector fields. A general setting for the study of infinite dimensional holonomy groups and holonomy algebras of nonlinear connections was initiated by P. Michor in [5], but the tangential properties of the holonomy algebras to the holonomy group were not clarified.

Recently, the authors introduced in [6] the notion of tangent Lie algebras to the holonomy group and proved that the curvature algebra (the Lie algebra generated

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by curvature vector fields) is a tangent algebra to the holonomy group. With this technique we have constructed a Finsler manifold (with singular metric) with infinite dimensional curvature algebra, which implies that the holonomy group can not be a finite dimensional Lie group in this case. We suspect that for most of non-Riemannian Finsler manifolds, the holonomy group is not a finite dimensional Lie group.

In a recent paper [2] M. Crampin, D.J. Saunders carried on a deep analysis of the holonomy structures of bundles with fibre metrics, and in particular the holonomy structures of Landsbergian type Finsler manifolds. In these cases, the holonomy groups are finite dimensional Lie groups. They introduced the notion of holonomy algebra and proved a version of Ambrose-Singer Theorem for such spaces. Reflecting to our results, they noticed that in the general Finslerian framework the holonomy algebra should contain the parallel translated curvature algebras. They showed that in this case the topological closure of this holonomy algebra contains the covariant derivatives of curvature vector fields, but the tangent properties of the successive covariant derivatives of curvature vector fields are not obvious from this approach in the cases, when the holonomy group is not a finite dimensional Lie group. The difficulty comes from the fact, that a topologically non-closed infinite dimensional Lie algebra of vector fields may expand, if we add the covariant derivatives of its elements.

Our goal in this paper is to make an attempt to find the right notion of the holonomy algebra of Finsler spaces. The holonomy algebra should be the largest Lie algebra such that all its elements are tangent to the holonomy group. In our attempt we are building successively Lie algebras having the tangent properties. First, we introduce the notion of the *curvature algebra* (the Lie algebra generated by curvature vector fields) which is a tangent Lie algebra to the holonomy group (cf. [6]). Then we define the *infinitesimal holonomy algebra* by the smallest Lie algebra of vector fields on an indicatrix, containing the curvature vector fields and their horizontal covariant derivatives with respect to the Berwald connection and prove the tangential property of this Lie algebra to the holonomy group. At the end we introduce the notion of the *holonomy algebra* of a Finsler manifold by all conjugates of infinitesimal holonomy algebras by parallel translations with respect to the Berwald connection. We prove that this holonomy algebra is tangent to the holonomy group. The question of whether the holonomy algebra introduced in this way is the largest Lie algebra, which is tangent to the holonomy group, is still open.

2 Preliminaries

Let M be an n-dimensional C^∞ manifold and let $\mathfrak{X}^\infty(M)$ denote the vector space of smooth vector fields on M. For a local coordinate system $(x^1, \ldots, x^n)$ on M we denote by $(x^1, \ldots, x^n; y^1, \ldots, y^n)$ the induced local coordinate system on the tangent bundle TM.

Finsler manifold, canonical connection, parallelism

A *Finsler manifold* is a pair $(M, \mathcal{F})$, where the Finsler function $\mathcal{F}\colon TM \to \mathbb{R}$ is continuous, smooth on $\hat{T}M := TM \setminus \{0\}$, its restriction $\mathcal{F}_x = \mathcal{F}|_{T_xM}$ is a positively homogeneous function of degree 1 and the symmetric bilinear form (the Finsler metric)

$$g_{x,y}\colon (u,v) \mapsto g_{ij}(x,y)u^i v^j = \frac{1}{2}\frac{\partial^2 \mathcal{F}_x^2(y+su+tv)}{\partial s\,\partial t}\Big|_{t=s=0}$$

is positive definite at every $y \in \hat{T}_x M$.

Geodesics of Finsler manifolds are determined by a system of second order ordinary differential equation $\ddot{x}^i + 2G^i(x,\dot{x}) = 0$, $i = 1,\dots,n$, where $G^i(x,\dot{x})$ are locally given by

$$G^i(x,y) := \frac{1}{4}g^{il}(x,y)\Big(2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\Big)y^j y^k. \tag{1}$$

The associated *homogeneous (nonlinear) parallel translation* $\tau_c\colon T_{c(0)}M \to T_{c(1)}M$ along a curve $c\colon [0,1] \to \mathcal{R}$ is defined by vector fields $X(t) = X^i(t)\frac{\partial}{\partial x^i}$ along $c(t)$ which are solutions of the differential equation

$$D_{\dot{c}}X(t) := \Big(\frac{dX^i(t)}{dt} + G^i_j(c(t),X(t))\dot{c}^j(t)\Big)\frac{\partial}{\partial x^i} = 0, \quad \text{where} \quad G^i_j = \frac{\partial G^i}{\partial y^j}. \tag{2}$$

Horizontal distribution, Berwald connection, curvature

Let (TM,π,M) and (TTM,ρ,TM) denote the first and the second tangent bundle of the manifold M, respectively. The horizontal distribution $\mathcal{H}TM \subset TTM$ associated with the Finsler manifold $(M,\mathcal{F})$ can be defined as the image of the horizontal lift which is an isomorphism $X \to X^h$ between T_xM and $\mathcal{H}_yTM$ at $y \in T_xM$ defined by

$$\Big(X^i\frac{\partial}{\partial x^i}\Big)^h := X^i\left(\frac{\partial}{\partial x^i} - G^k_i(x,y)\frac{\partial}{\partial y^k}\right). \tag{3}$$

If $\mathcal{V}TM := \operatorname{Ker}\pi_* \subset TTM$ denotes the vertical distribution on TM, then for any $y \in TM$ we have $T_yTM = \mathcal{H}_yTM \oplus \mathcal{V}_yTM$. The projectors corresponding to this decomposition will be denoted by $h\colon TTM \to \mathcal{H}TM$ and $v\colon TTM \to \mathcal{V}TM$. We note that the vertical distribution is integrable.

Let $(\hat{\mathcal{V}}TM,\rho,\hat{T}M)$ be the vertical bundle over $\hat{T}M := TM \setminus \{0\}$. We denote by $\mathfrak{X}^\infty(M)$, respectively by $\hat{\mathfrak{X}}^\infty(TM)$ the vector space of smooth vector fields on M and of smooth sections of the bundle $(\hat{\mathcal{V}}TM,\tau,\hat{T}M)$, respectively. The *horizontal Berwald covariant derivative* of a section $\xi \in \hat{\mathfrak{X}}^\infty(TM)$ by a vector field $X \in \mathfrak{X}^\infty(M)$ is $\nabla_X\xi := [X^h,\xi]$.

In an induced local coordinate system (x^i,y^i) on TM for vector fields $\xi(x,y) = \xi^i(x,y)\frac{\partial}{\partial y^i}$ and $X(x) = X^i(x)\frac{\partial}{\partial x^i}$ we have (3) and hence

$$\nabla_X\xi = \left(\frac{\partial\xi^i(x,y)}{\partial x^j} - G^k_j(x,y)\frac{\partial\xi^i(x,y)}{\partial y^k} + \frac{\partial G^i_j(x,y)}{\partial y^k}(x,y)\xi^k(x,y)\right)X^j\frac{\partial}{\partial y^i}. \tag{4}$$

Let $(\pi^*TM,\bar{\pi},\hat{T}M)$ be the pull-back bundle of $(\hat{T}M,\pi,M)$ by the map $\pi\colon TM \to M$. Clearly, the mapping

$$\Big(x,y,\xi^i\frac{\partial}{\partial y^i}\Big) \mapsto \Big(x,y,\xi^i\frac{\partial}{\partial x^i}\Big)\colon \hat{\mathcal{V}}TM \to \pi^*TM \tag{5}$$

is a canonical bundle isomorphism. In the following we will use the isomorphism (5) for the identification of these bundles.

The *Riemannian curvature tensor* field $R_{(x,y)}(X,Y) := v\big[X^h, Y^h\big]$, $X, Y \in T_xM$, $(x,y) \in \hat{T}M$ characterizes the integrability of the horizontal distribution. Namely, if the horizontal distribution $\mathcal{H}\hat{T}M$ is integrable, then the Riemannian curvature is identically zero. The expression of the Riemannian curvature tensor

$$R_{(x,y)} = R^i_{jk}(x,y)dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^i}$$

on the pull-back bundle $(\pi^*TM, \bar{\pi}, \hat{T}M)$ is

$$R^i_{jk}(x,y) = \frac{\partial G^i_j(x,y)}{\partial x^k} - \frac{\partial G^i_k(x,y)}{\partial x^j} + G^m_j(x,y)G^i_{km}(x,y) - G^m_k(x,y)G^i_{jm}(x,y).$$

Indicatrix bundle

The *indicatrix* $\mathcal{I}_pM$ of an n-dimensional Finsler manifold $(M, \mathcal{F})$ at a point $p \in M$ is the compact hypersurface $\mathcal{I}_pM := \{y \in T_pM; \mathcal{F}(y) = 1\}$ in T_pM, diffeomorphic to the standard $(n-1)$-sphere. The *indicatrix bundle* $(\mathcal{I}M, \pi, M)$ of $(M, \mathcal{F})$ is a smooth subbundle of the tangent bundle (TM, π, M). The group $\mathsf{Diff}^\infty(\mathcal{I}_pM)$ of all smooth diffeomorphisms of an indicatrix $\mathcal{I}_pM$ is a regular infinite dimensional Lie group modeled on the vector space $\mathfrak{X}^\infty(\mathcal{I}_pM)$ of smooth vector fields on $\mathcal{I}_pM$. The Lie algebra of the infinite dimensional Lie group $\mathsf{Diff}^\infty(\mathcal{I}_pM)$ is the vector space $\mathfrak{X}^\infty(\mathcal{I}_pM)$, equipped with the negative of the usual Lie bracket, (cf. A. Kriegl and P.W. Michor [4], Section 43).

Let $c(t)$, $0 \le t \le a$ be a smooth curve joining the points $p = c(0)$ and $q = c(a)$ in the Finsler manifold $(M, \mathcal{F})$. Since the parallel translation $\tau_c \colon T_pM \to T_qM$ along the curve $c \colon [0,a] \to M$ is a differentiable map between $\hat{T}_pM$ and $\hat{T}_qM$ preserving the value of the Finsler function, it induces a parallel translation $\tau_c \colon \mathcal{I}_pM \to \mathcal{I}_qM$ in the indicatrix bundle.

Holonomy group

The notion of the holonomy group of Riemannian manifolds can be generalized very naturally for Finsler manifolds:

Definition 1. The *holonomy group* $\mathsf{Hol}(p)$ of a Finsler space $(M, \mathcal{F})$ at $p \in M$ is the subgroup of the group of diffeomorphisms $\mathsf{Diff}^\infty(\mathcal{I}_pM)$ of the indicatrix $\mathcal{I}_pM$ determined by parallel translation of $\mathcal{I}_pM$ along piece-wise differentiable closed curves initiated at the point $p \in M$.

Clearly, the holonomy groups at different points of M are isomorphic. We note that the holonomy group $\mathsf{Hol}(p)$ is a topological subgroup of the regular infinite dimensional Lie group $\mathsf{Diff}^\infty(\mathcal{I}_pM)$, but its differentiable structure is not known in general.

3 Tangent Lie algebras to diffeomorphism groups

Here we discuss the tangential properties of Lie algebras of vector fields to an abstract subgroup of the diffeomorphism group of a manifold. The results of this section will be applied in the following to the investigation of tangent Lie algebras of the holonomy subgroup of the diffeomorphism group of an indicatrix $\mathcal{I}_xM$ and to the fibred holonomy subgroup of the diffeomorphism group of the indicatrix bundle $\mathcal{I}(M)$.

Let P be a C^∞ manifold, let H be a (not necessarily differentiable) subgroup of the diffeomorphism group $\mathsf{Diff}^\infty(P)$ and let $\mathfrak{X}^\infty(P)$ be the Lie algebra of smooth vector fields on P.

Definition 2. A vector field $X \in \mathfrak{X}^\infty(P)$ is called *tangent* to the subgroup H of $\mathsf{Diff}^\infty(P)$, if there exists a $\mathcal{C}^1$-differentiable 1-parameter family $\{\phi_t \in H\}_{t\in(-\varepsilon,\varepsilon)}$ of diffeomorphisms of M such that $\phi_0 = \mathsf{Id}$ and $\frac{\partial \phi_t}{\partial t}\big|_{t=0} = X$. A Lie subalgebra $\mathfrak{g}$ of $\mathfrak{X}^\infty(P)$ is called *tangent* to H, if all elements of $\mathfrak{g}$ are tangent vector fields to H.

Unfortunately, it is not true, that tangent vector fields to the group H generate a tangent Lie algebra to H. This is why we have to introduce a stronger tangency property in Definition 4.

Definition 3. A $\mathcal{C}^\infty$-differentiable k-parameter family

$$\{\phi_{(t_1,\dots,t_k)} \in \mathsf{Diff}^\infty(P)\}_{t_i\in(-\varepsilon,\varepsilon)}$$

of diffeomorphisms of P is called a *commutator-like family* if it satisfies the equations

$$\phi_{(t_1,\dots,t_k)} = \mathsf{Id}, \quad \text{whenever} \quad t_j = 0 \quad \text{for some} \quad 1 \le j \le k.$$

We remark, that the commutators of commutator-like families are commutator-like, and the inverse of commutator-like families are commutator-like.

Definition 4. A vector field $X \in \mathfrak{X}^\infty(P)$ is called *strongly tangent* to the subgroup H of $\mathsf{Diff}^\infty(P)$, if there exists a commutator-like family

$$\{\phi_{(t_1,\dots,t_k)} \in \mathsf{Diff}^\infty(P)\}_{t_i\in(-\varepsilon,\varepsilon)}$$

of diffeomorphisms satisfying the conditions

(A) $\phi_{(t_1,\dots,t_k)} \in H \quad \text{for all} \quad t_i \in (-\varepsilon,\varepsilon), \quad 1 \le i \le k,$

(B) $\frac{\partial^k \phi_{(t_1,\dots,t_k)}}{\partial t_1 \cdots \partial t_k}\Big|_{(0,\dots,0)} = X.$

It follows from the commutator-like property that $\frac{\partial^k \phi_{(t_1,\dots,t_k)}}{\partial t_1 \dots \partial t_k}\Big|_{(0,\dots,0)}$ is the first non-necessarily vanishing derivative of the diffeomorphism family $\{\phi_{(t_1,\dots,t_k)}\}$ at any point $x \in P$, and therefore it determines a vector field. On the other hand, by reparametrizing the commutator like family of diffeomorphism, it can be shown that if a vector field is strongly tangent to a group H, then it is also tangent to H. Moreover, we have the following

Theorem 1. *Let $\mathcal{V}$ be a set of vector fields strongly tangent to the group $H \subset \mathsf{Diff}^\infty(P)$. The Lie subalgebra $\mathfrak{v}$ of $\mathfrak{X}^\infty(P)$ generated by $\mathcal{V}$ is tangent to H.*

The proof of the theorem is based on two important observations. The first is a generalization of the well-known relation between the commutator of vector fields and the commutator of their induced flows. Namely, if $\{\phi_{(s_1,\dots,s_k)}\}$ and $\{\psi_{(t_1,\dots,t_l)}\}$ are commutator-like k-parameter, respectively l-parameter families of local diffeomorphisms, then the family of (local) diffeomorphisms $[\phi_{(s_1,\dots,s_k)}, \psi_{(t_1,\dots,t_l)}]$ defined by the commutator of the group $\mathsf{Diff}^\infty(U)$ is a commutator-like $(k+l)$-parameter family and

$$\frac{\partial^{k+l}[\phi_{(s_1,\dots,s_k)}, \psi_{(t_1,\dots,t_l)}]}{\partial s_1 \dots \partial s_k \, \partial t_1 \dots \partial t_l}\Big|_{(0,\dots,0;0,\dots,0)}(x) = -\left[\frac{\partial^k \phi_{(s_1,\dots,s_k)}}{\partial s_1 \dots \partial s_k}\Big|_{(0,\dots,0)}, \frac{\partial^l \psi_{(t_1,\dots,t_l)}}{\partial t_1 \dots \partial t_l}\Big|_{(0,\dots,0)}\right](x)$$

at any point $x \in U$. The second important fact to prove the theorem is that the linear combinations of vector fields tangent to H are also tangent to H. The detailed computations can be found in [6].

4 The curvature algebra at a point

Now, se summarize our results on the tangent Lie algebras of the holonomy group $\mathsf{Hol}(p)$ at a point $p \in M$, their proofs can be found in [6].

Definition 5. A vector field $\xi \in \mathfrak{X}(\mathcal{I}_pM)$ on the indicatrix $\mathcal{I}_pM$ of the Finsler manifold $(M, \mathcal{F})$ is called a *curvature vector field at the point* $p \in M$, if it is in the image of the curvature tensor, i.e. if there exist $X, Y \in T_pM$ such that $\xi = r_p(X, Y)$, where

$$r_p(X,Y)(y) := R_{(p,y)}(X^h, Y^h) \tag{6}$$

The Lie subalgebra $\mathfrak{R}_p := \langle r_p(X,Y); X, Y \in T_pM \rangle$ of $\mathfrak{X}(\mathcal{I}_pM)$ generated by the curvature vector fields at the point $p \in M$ is called the *curvature algebra at the point* $p \in M$.

Since the Finsler function is preserved by parallel translations, its derivatives with respect to horizontal vector fields are identically zero. According to [7], eq. (10.9), the derivative of the Finsler metric with respect to $R_{(p,y)}(X^h, Y^h)$ vanishes, i.e.

$$g_{(p,y)}\big(y, R_{(p,y)}(X^h, Y^h)\big) = 0, \quad \text{for any} \quad y, X, Y \in T_xM.$$

This means that the curvature vector fields $\xi = r_p(X, Y)$ are tangent to the indicatrix. In the sequel we investigate the tangential properties of the curvature algebra to the holonomy group of the canonical connection ∇ of a Finsler manifold.

Proposition 1. *Any curvature vector field at a point $p \in M$ is strongly tangent to the holonomy group* $\mathsf{Hol}(p)$.

Proposition 2. *The curvature algebra $\mathfrak{R}_p$ at any point $p \in M$ of a Riemannian manifold (M, g) is isomorphic to the linear Lie algebra on the tangent space T_pM generated by the curvature operators of (M, g) at $p \in M$.*

Remark 1. *The dimension of the curvature algebra at any point $p \in M$ of a Finsler surface is ≤ 1.*

5 Fibred holonomy group and fibred holonomy algebra

Now, we introduce the notion of the fibred holonomy group of a Finsler manifold $(M, \mathcal{F})$ as a subgroup of the diffeomorphism group of the total manifold $\mathcal{I}M$ of the bundle $(\mathcal{I}M, \pi, M)$ and apply our results on tangent vector fields to an abstract subgroup of the diffeomorphism group to the study of tangent Lie algebras to the fibred holonomy group.

Definition 6. The *fibred holonomy group* $\mathsf{Hol_f}(M)$ of $(M, \mathcal{F})$ consists of fibre preserving diffeomorphisms $\Phi \in \mathsf{Diff}^\infty(\mathcal{I}M)$ of the indicatrix bundle $(\mathcal{I}M, \pi, M)$ such that for any $p \in M$ the restriction $\Phi_p = \Phi|_{\mathcal{I}_pM} \in \mathsf{Diff}^\infty(\mathcal{I}_pM)$ belongs to the holonomy group $\mathsf{Hol}(p)$.

We note that the holonomy group $\mathsf{Hol}(p)$ and the fibred holonomy group $\mathsf{Hol_f}(M)$ are topological subgroups of the infinite dimensional Lie groups $\mathsf{Diff}^\infty(\mathcal{I}_pM)$ and $\mathsf{Diff}^\infty(\mathcal{I}M)$ respectively.

The definition of strongly tangent vector fields yields

Remark 2. *A vector field $\xi \in \mathfrak{X}^\infty(\mathcal{I}M)$ is strongly tangent to the fibred holonomy group $\mathsf{Hol_f}(M)$ if and only if there exists a family $\big\{\Phi_{(t_1,\dots,t_k)}\big|_{\mathcal{I}M}\big\}_{t_i\in(-\varepsilon,\varepsilon)}$ of fibre preserving diffeomorphisms of the bundle $(\mathcal{I}M, \pi, M)$ such that for any indicatrix $\mathcal{I}_p$ the induced family $\big\{\Phi_{(t_1,\dots,t_k)}\big|_{\mathcal{I}_pM}\big\}_{t_i\in(-\varepsilon,\varepsilon)}$ of diffeomorphisms is contained in the holonomy group $\mathsf{Hol}(p)$ and $\xi\big|_{\mathcal{I}_pM}$ is strongly tangent to $\mathsf{Hol}(p)$.*

Since $\pi\big(\Phi_{(t_1,\dots,t_k)}(p)\big) \equiv p$ and $\pi_*(\xi) = 0$ for every $p \in U$, we get the

Corollary 1. *Strongly tangent vector fields to the fibred holonomy group $\mathsf{Hol_f}(M)$ are vertical vector fields. If $\xi \in \mathfrak{X}^\infty(\mathcal{I}M)$ is strongly tangent to $\mathsf{Hol_f}(M)$ then its restriction $\xi_p := \xi\big|_{\mathcal{I}_p}$ to any indicatrix $\mathcal{I}_p$ is strongly tangent to the holonomy group $\mathsf{Hol}(p)$.*

The curvature vector fields and the curvature algebra at a point has been defined on an indicatrix of the manifold M. Now we extend the domain of their definition to the total manifold of the indicatrix bundle.

Definition 7. A vector field $\xi \in \mathfrak{X}^\infty(\mathcal{I}M)$ on the indicatrix bundle $\mathcal{I}M$ is a *curvature vector field* of the Finsler manifold $(M, \mathcal{F})$, if there exist $X, Y \in \mathfrak{X}^\infty(M)$ such that $\xi = r(X, Y)$, where $r(X, Y)(x, y) := R_{(x,y)}(X_x, Y_x)$ for $x \in M$ and $y \in \mathcal{I}_xM$. The Lie algebra $\mathfrak{R}(M)$ generated by the curvature vector fields of $(M, \mathcal{F})$ is called the *curvature algebra* of the Finsler manifold $(M, \mathcal{F})$.

Proposition 3. *If the Finsler manifold $(M, \mathcal{F})$ is diffeomorphic to $\mathbb{R}^n$ then any curvature vector field $\xi \in \mathfrak{X}^\infty(\mathcal{I}M)$ of $(M, \mathcal{F})$ is strongly tangent to the fibred holonomy group $\mathsf{Hol}_\mathsf{f}(M)$.*

Proof. Since M is diffeomorphic to $\mathbb{R}^n$ we can identify the manifold M with the vector space $\mathbb{R}^n$. Let $\xi = r(X,Y) \in \mathfrak{X}^\infty(\mathcal{I}\mathbb{R}^n)$ be a curvature vector field with $X, Y \in \mathfrak{X}^\infty(\mathbb{R}^n)$. According to Proposition 1 its restriction $\xi\big|_{\mathcal{I}_p\mathbb{R}^n}$ to any indicatrix $\mathcal{I}_p\mathbb{R}^n$ is strongly tangent to the holonomy groups $\mathsf{Hol}(p)$. We have to prove that there exists a family $\{\Phi_{(t_1,\dots,t_k)}\big|_{\mathcal{I}\mathbb{R}^n}\}_{t_i\in(-\varepsilon,\varepsilon)}$ of fibre preserving diffeomorphisms of the indicatrix bundle $(\mathcal{I}\mathbb{R}^n, \pi, \mathbb{R}^n)$ such that for any $p \in \mathbb{R}^n$ the family of diffeomorphisms induced on the indicatrix $\mathcal{I}_p$ is contained in $\mathsf{Hol}(p)$ and $\xi\big|_{\mathcal{I}_p\mathbb{R}^n}$ is strongly tangent to $\mathsf{Hol}(p)$.

For any $p \in \mathbb{R}^n$ and $-1 < s,t < 1$ let $\Pi(sX_p, tY_p)$ be the parallelogram in $\mathbb{R}^n$ determined by the vertexes $p, p + sX_p, p + sX_p + tY_p, p + tY_p \in \mathbb{R}^n$ and let $\tau_{\Pi(sX_p,tY_p)} \colon \mathcal{I}_p \to \mathcal{I}_p$ denote the (nonlinear) parallel translation of the indicatrix $\mathcal{I}_p$ along the parallelogram $\Pi(sX_p, tY_p)$ with respect to the associated homogeneous (nonlinear) parallel translation of the Finsler manifold $(\mathbb{R}^n, \mathcal{F})$. Clearly we have $\tau_{\Pi(sX_p,tY_p)} = \mathsf{Id}_{\mathcal{I}\mathbb{R}^n}$, if $s = 0$ or $t = 0$ and

$$\frac{\partial^2 \tau_{\Pi(sX_p,tY_p)}}{\partial s\, \partial t}\bigg|_{(s,t)=(0,0)} = \xi_p \quad \text{for every} \quad p \in \mathbb{R}^n.$$

Since the mapping $(p,s,t) \mapsto \Pi(sX_p, tY_p)$ is a differentiable field of parallelograms in $\mathbb{R}^n$, the maps $\tau_{\Pi(sX_p,tY_p)}$, $p \in \mathbb{R}^n$, $0 < s,t < 1$, define a 2-parameter family of fibre preserving diffeomorphisms of the indicatrix bundle $\mathcal{I}\mathbb{R}^n$. The diffeomorphisms induced by the family $\{\tau_{\Pi(sX_p,tY_p)}\}_{s,t\in(-1,1)}$ on any indicatrix $\mathcal{I}_p$ are contained in $\mathsf{Hol}(p)$. Hence the vector field $\xi \in \mathfrak{X}^\infty(\mathbb{R}^n)$ is strongly tangent to the fibred holonomy group $\mathsf{Hol}_\mathsf{f}(M)$, hence the assertion is proved. $\square$

Corollary 2. *If M is diffeomorphic to $\mathbb{R}^n$ then the curvature algebra $\mathfrak{R}(M)$ of $(M, \mathcal{F})$ is tangent to the fibred holonomy group $\mathsf{Hol}_\mathsf{f}(M)$.*

The following assertion shows that similarly to the Riemannian case, the curvature algebra can be extended to a larger tangent Lie algebra containing all horizontal covariant derivatives of the curvature algebra vector fields.

Proposition 4. *If $\xi \in \mathfrak{X}^\infty(\mathcal{I}M)$ is strongly tangent to the fibred holonomy group $\mathsf{Hol}_\mathsf{f}(M)$ of $(M, \mathcal{F})$, then its horizontal covariant derivative $\nabla_X \xi$ along any vector field $X \in \mathfrak{X}^\infty(M)$ is also strongly tangent to $\mathsf{Hol}_\mathsf{f}(M)$.*

Proof. Let τ be the (nonlinear) parallel translation along the flow φ of the vector field X, i.e. for every $p \in M$ and $t \in (-\varepsilon_p, \varepsilon_p)$ the map $\tau_t(p) \colon \mathcal{I}_pM \to \mathcal{I}_{\varphi_t(p)}M$ is the (nonlinear) parallel translation along the integral curve of X. If $\{\Phi_{(t_1,\dots,t_k)}\}_{t_i\in(-\varepsilon,\varepsilon)}$ is a $\mathcal{C}^\infty$-differentiable k-parameter family $\{\Phi_{(t_1,\dots,t_k)}\}_{t_i\in(-\varepsilon,\varepsilon)}$ of fibre preserving diffeomorphisms of the indicatrix bundle $(\mathcal{I}M, \pi|_M, M)$ satisfying the conditions of Definition 1 then the commutator

$$[\Phi_{(t_1,\dots,t_k)}, \tau_{t_{k+1}}] := \Phi^{-1}_{(t_1,\dots,t_k)} \circ (\tau_{t_{k+1}})^{-1} \circ \Phi_{(t_1,\dots,t_k)} \circ \tau_{t_{k+1}}$$

of the group $\mathsf{Diff}^\infty(\mathcal{I}M)$ fulfills $[\Phi_{(t_1,\dots,t_k)}, \tau_{t_{k+1}}] = \mathsf{Id}$, if some of its variables equals 0. Moreover

$$\left.\frac{\partial^{k+1}[\Phi_{(t_1,\dots,t_k)}, \tau_{(t_{k+1})}]}{\partial t_1 \dots \partial t_{k+1}}\right|_{(0,\dots,0)} = -\big[\xi, X^h\big] \tag{7}$$

at any point of M, which shows that the vector field $\big[\xi, X^h\big]$ is strongly tangent to $\mathsf{Hol}_{\mathsf{f}}(M)$. Moreover, since the vector field ξ is vertical, we have $h[X^h, \xi] = 0$, and using $\nabla_X \xi := [X^h, \xi]$ we obtain

$$-[\xi, X^h] = [X^h, \xi] = v[X^h, \xi] = \nabla_X \xi$$

which yields the assertion. □

Definition 8. Let $\mathfrak{hol}_{\mathfrak{f}}(M)$ be the smallest Lie algebra of vector fields on the indicatrix bundle $\mathcal{I}M$ satisfying the properties

(i) any curvature vector field ξ belongs to $\mathfrak{hol}_{\mathfrak{f}}(M)$,

(ii) if $\xi \in \mathfrak{hol}_{\mathfrak{f}}(M)$ and $X \in \mathfrak{X}^\infty(M)$, then the covariant derivative $\nabla_X \xi$ also belongs to $\mathfrak{hol}_{\mathfrak{f}}(M)$.

The Lie algebra $\mathfrak{hol}_{\mathfrak{f}}(M) \subset \mathfrak{X}^\infty(\mathcal{I}M)$ is called the *fibred holonomy algebra* of the Finsler manifold $(M, \mathcal{F})$.

Remark 3. *The fibred holonomy algebra $\mathfrak{hol}_{\mathfrak{f}}(M)$ is invariant with respect to the horizontal covariant derivation with respect to the Berwald connection, i.e.*

$$\xi \in \mathfrak{hol}_{\mathfrak{f}}(M) \quad \textit{and} \quad X \in \mathfrak{X}^\infty(M) \quad \Rightarrow \quad \nabla_X \xi \in \mathfrak{hol}_{\mathfrak{f}}(M). \tag{8}$$

The results of this sections yield the following

Theorem 2. *If M is diffeomorphic to $\mathbb{R}^n$ then the fibred holonomy algebra $\mathfrak{hol}_{\mathfrak{f}}(M)$ is tangent to the fibred holonomy group $\mathsf{Hol}_{\mathsf{f}}(M)$.*

6 Infinitesimal holonomy algebra

Let $\mathfrak{hol}_{\mathfrak{f}}(M) \subset \mathfrak{X}^\infty(\mathcal{I}M)$ be the fibred holonomy algebra of the Finsler manifold $(M, \mathcal{F})$ and let p be a a given point in M.

Definition 9. The Lie algebra $\mathfrak{hol}^*(p) := \big\{\xi_p; \xi \in \mathfrak{hol}_{\mathfrak{f}}(M)\big\} \subset \mathfrak{X}^\infty(\mathcal{I}_pM)$ of vector fields on the indicatrix $\mathcal{I}_pM$ is called the *infinitesimal holonomy algebra at the point $p \in M$.*

Clearly, $\mathfrak{R}_p \subset \mathfrak{hol}^*(p)$ for any $p \in M$.

The following assertion is a direct consequence of the definition. It shows that the infinitesimal holonomy algebra at a point p of $(M, \mathcal{F})$ can be calculated in a neighbourhood of p.

Remark 4. *Let $(U, \mathcal{F}|_U)$ be an open submanifold of $(M, \mathcal{F})$ such that $U \subset M$ is diffeomorphic to $\mathbb{R}^n$ and let $p \in U$. The infinitesimal holonomy algebras at p of the Finsler manifolds $(M, \mathcal{F})$ and $(U, \mathcal{F}|_U)$ coincide.*

Now, we can prove the following

Theorem 3. *The infinitesimal holonomy algebra $\mathfrak{hol}^*(p)$ at any point p of the Finsler manifold $(M,\mathcal{F})$ is tangent to the holonomy group $\mathsf{Hol}(p)$.*

Proof. Let $U \subset M$ be an open submanifold of M, diffeomorphic to $\mathbb{R}^n$ and containing $p \in M$. According to the previous remark we have $\mathfrak{hol}^*(p) := \{\xi_p; \xi \in \mathfrak{hol}_{\mathfrak{f}}(U)\}$. Since the fibred holonomy algebra $\mathfrak{hol}_{\mathfrak{f}}(U)$ is tangent to the fibred holonomy group $\mathsf{Hol}_{\mathfrak{f}}(U)$ we obtain that $\mathfrak{hol}^*(p)$ is tangent to the holonomy group $\mathsf{Hol}(p)$. $\square$

7 Holonomy algebra

Let $x(t)$, $0 \le t \le a$ be a smooth curve joining the points $q = x(0)$ and $p = x(a)$ in the Finsler manifold $(M,\mathcal{F})$. If $y(t) = \tau_t y(0) \in \mathcal{I}_{x(t)}M$ is a parallel vector field along $x(t)$, $0 \le t \le a$, where $\tau_t \colon \mathcal{I}_q M \to \mathcal{I}_{x(t)}M$ denotes the homogeneous (nonlinear) parallel translation, then we have $D_{\dot x} y(t) := \left(\frac{dy^i(t)}{dt} + G^i_j(x(t),y(t))\dot x^j(t)\right)\frac{\partial}{\partial x^i} = 0$. Considering a vector field ξ on the indicatrix $\mathcal{I}_q M$, the map $\tau_{a*}\xi \circ \tau_a^{-1} \colon (p,y) \mapsto \tau_{a*}\xi(y(a))$ gives a vector field on the indicatrix $\mathcal{I}_p M$. Hence we can formulate

Lemma 1. *For any vector field $\xi \in \mathfrak{hol}^*(q) \subset \mathfrak{X}^\infty(\mathcal{I}_q M)$ in the infinitesimal holonomy algebra at q the vector field $\tau_{a*}\xi \circ \tau_a^{-1} \in \mathfrak{X}^\infty(\mathcal{I}_p M)$ is tangent to the holonomy group $\mathsf{Hol}(p)$.*

Proof. Let $\{\phi_t \in \mathsf{Hol}(q)\}_{t\in(-\varepsilon,\varepsilon)}$ be a $\mathcal{C}^1$-differentiable 1-parameter family of diffeomorphisms of $\mathcal{I}_q M$ belonging to the holonomy group $\mathsf{Hol}(q)$ and satisfying the conditions $\phi_0 = \mathsf{Id}$, $\frac{\partial \phi_t}{\partial t}\big|_{t=0} = \xi$. Since the 1-parameter family

$$\tau_a \circ \phi_t \circ \tau_a^{-1} \in \mathsf{Diff}^\infty(\mathcal{I}_p M)\}_{t\in(-\varepsilon,\varepsilon)}$$

of diffeomorphisms consists of elements of the holonomy group $\mathsf{Hol}(p)$ and satisfies the conditions

$$\tau_a \circ \phi_0 \circ \tau_a^{-1} = \mathsf{Id}, \qquad \frac{\partial\big(\tau_a \circ \phi_t \circ \tau_a^{-1}\big)}{\partial t}\bigg|_{t=0} = \tau_{a*}\xi \circ \tau_a^{-1},$$

the assertion follows. $\square$

Definition 10. A vector field $\mathbf{B}_\gamma\xi \in \mathfrak{X}^\infty(\mathcal{I}_p M)$ on the indicatrix $\mathcal{I}_p M$ will be called *the Berwald translate* of the vector field $\xi \in \mathfrak{X}^\infty(\mathcal{I}_q M)$ along the curve $\gamma = x(t)$ if

$$\mathbf{B}_\gamma \xi = \tau_{a*}\xi \circ (\tau_a)^{-1}.$$

Remark 5. *Let $y(t) = \tau_t y(0) \in \mathcal{I}_{x(t)}M$ be a parallel vector field along $\gamma = x(t)$, $0 \le t \le a$, started at $y(0) \in \mathcal{I}_{x(0)}M$. Then, the vertical vector field $\xi_t = \xi(x(t),y(t))$ along $(x(t),y(t))$ is the Berwald translate $\xi_t = \tau_{t*}\xi_0 \circ {\tau_t}^{-1}$ if and only if*

$$\nabla_{\dot x}\xi = \left(\frac{\partial \xi^i(x,y)}{\partial x^j} - G^k_j(x,y)\frac{\partial \xi^i(x,y)}{\partial y^k} + G^i_{jk}(x,y)\xi^k(x,y)\right)\dot x^j \frac{\partial}{\partial y^i} = 0.$$

Now, lemma 1 yields the following

Corollary 3. *If $\xi \in \mathfrak{hol}^*(q)$ then its Berwald translate $\mathbf{B}_\gamma \xi \in \mathfrak{X}^\infty(\mathcal{I}_p M)$ along any curve $\gamma = x(t)$, $0 \leq t \leq a$, joining $q = x(0)$ with $p = x(a)$ is tangent to the holonomy group* $\mathsf{Hol}(p)$.

This last statement motivates the following

Definition 11. The *holonomy algebra* $\mathfrak{hol}_p(M)$ of the Finsler manifold $(M, \mathcal{F})$ at the point $p \in M$ is defined by the smallest Lie algebra of vector fields on the indicatrix $\mathcal{I}_p M$, containing the Berwald translates of all infinitesimal holonomy algebras along arbitrary curves $x(t)$, $0 \leq t \leq a$ joining any points $q = x(0)$ with the point $p = x(a)$.

Clearly, the holonomy algebras at different points of the Finsler manifold $(M, \mathcal{F})$ are isomorphic. The previous lemma and corollary yield the following

Theorem 4. *The holonomy algebra $\mathfrak{hol}_p(M)$ at $p \in M$ of a Finsler manifold $(M, \mathcal{F})$ is tangent to the holonomy group* $\mathsf{Hol}(p)$.

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Authors' address:
Institute of Mathematics, University of Debrecen, H-4010 Debrecen, P.O.B. 12, Hungary

E-mail: `muzsnay@science.unideb.hu`

E-mail: `petert.nagy@science.unideb.hu`

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