

FINSLER MANIFOLDS WITH NON-RIEMANNIAN HOLONOMY

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ABSTRACT. The aim of this paper is to show that holonomy properties of Finsler manifolds can be very different from those of Riemannian manifolds. We prove that the holonomy group of a positive definite non-Riemannian Finsler manifold of non-zero constant curvature with dimension > 2 cannot be a compact Lie group. Hence this holonomy group does not occur as the holonomy group of any Riemannian manifold. In addition, we provide an example of left invariant Finsler metric on the Heisenberg group, so that its holonomy group is not a (finite dimensional) Lie group. These results give a positive answer to the following problem formulated by S. S. Chern and Z. Shen: *Is there a Finsler manifold whose holonomy group is not the holonomy group of any Riemannian manifold?*

1. INTRODUCTION

The notion of the holonomy group of a Riemannian manifold can be generalized very naturally for a Finsler manifold (cf. e.g. S. S. Chern and Z. Shen, [2], Chapter 4): it is the group at a point x generated by the canonical homogeneous (nonlinear) parallel translations along all loops emanated from x . Until now the holonomy groups of non-Riemannian Finsler manifolds have been described only in special cases: for Berwald manifolds there exist Riemannian metrics with the same holonomy group (cf. Z. I. Szabó, [11]), for positive definite Landsberg manifolds the holonomy groups are compact Lie groups consisting of isometries of the

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indicatrix with respect to an induced Riemannian metric (cf. L. Kozma, [4], [5]). A thorough study of the holonomy group of homogeneous (nonlinear) connections was initiated by W. Barthel in his basic work [1] in 1963; he gave a construction for a holonomy algebra of vector fields on the tangent space. A general setting for the study of infinite dimensional holonomy groups and holonomy algebras of nonlinear connections was initiated by P. Michor in [7]. However the introduced holonomy algebras could not be used to estimate the dimension of the holonomy group since their tangential properties to the holonomy group were not clarified.

The aim of this paper is to show that holonomy properties of Finsler manifolds can be very different from those of Riemannian manifolds. We prove that if the holonomy group of a non-Riemannian Finsler manifold of non-zero constant curvature with dimension $n > 2$ is a (finite dimensional) Lie group then its dimension is strictly greater than the dimension of the orthogonal group acting on the tangent space and hence it can not be a compact Lie group. An estimate for the dimension of the holonomy group will be obtained by investigation of a Lie algebra of tangent vector fields on the indicatrix, algebraically generated by curvature vector fields of the Finsler manifold. We call this Lie algebra the *curvature algebra* and prove that its elements are tangent to one-parameter families of diffeomorphisms contained in the holonomy group. For non-Riemannian Finsler manifolds of constant curvature $\neq 0$ with dimension $n > 2$ we construct more than $\frac{n(n-1)}{2}$ linearly independent curvature vector fields.

In addition, we provide an example of a left invariant singular (non y -global) Finsler metric of Berwald-Moór-type on the Heisenberg group which has infinite dimensional curvature algebra and hence its holonomy is not a (finite dimensional) Lie group. These results give a positive answer to the following problem formulated by S. S. Chern and Z. Shen in [2] (p. 85): *Is there a Finsler manifold whose holonomy group is not the holonomy group of any Riemannian manifold?* This question is contained also in the list of open problems in Finsler geometry by Z. Shen [10], (March 8, 2009, Problem 34).

2. PRELIMINARIES

Finsler manifold and its canonical connection. A Minkowski functional on a vector space V is a continuous function \mathcal{F} , positively homogeneous of degree two, i.e. $\mathcal{F}(\lambda y) = \lambda^2 \mathcal{F}(y)$, smooth on $\hat{V} := V \setminus \{0\}$, and for any $y \in \hat{V}$ the symmetric bilinear form $g_y: V \times V \rightarrow \mathbb{R}$ defined by

$$g_y: (u, v) \mapsto g_{ij}(y)u^i v^j = \frac{1}{2} \frac{\partial^2 \mathcal{F}(y + su + tv)}{\partial s \partial t} \Big|_{t=s=0}$$

is non-degenerate. If g_y is positive definite for any $y \in \hat{V}$ then \mathcal{F} is said positive definite and (V, \mathcal{F}) is called *positive definite Minkowski space*. A Minkowski functional \mathcal{F} is called *semi-Euclidean* if there exists a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on V such that $g_y(u, v) = \langle u, v \rangle$ for any $y \in \hat{V}$ and $u, v \in V$. A semi-Euclidean positive definite Minkowski functional is called *Euclidean*.

A *Finsler manifold* is a pair (M, \mathcal{F}) where M is an n -dimensional manifold and $\mathcal{F}: TM \rightarrow \mathbb{R}$ is a function (called *Finsler metric*, cf. [9]) defined on the tangent bundle of M , smooth on $\hat{TM} := TM \setminus \{0\}$ and its restriction $\mathcal{F}_x = \mathcal{F}|_{T_x M}$ is a Minkowski functional on $T_x M$ for all $x \in M$. If the restriction $\mathcal{F}_x = \mathcal{F}|_{T_x M}$ of the Finsler metric $\mathcal{F}: TM \rightarrow \mathbb{R}$ is positive definite on $T_x M$ for all $x \in M$ then (M, \mathcal{F}) is called *positive definite Finsler manifold*. A point $x \in M$ is called *(semi-)Riemannian* if the Minkowski functional \mathcal{F}_x is (semi-)Euclidean.

We remark that in many applications the metric \mathcal{F} is smooth only on an open cone $\mathcal{C}M \subset TM \setminus \{0\}$, where $\mathcal{C}M = \cup_{x \in M} \mathcal{C}_x M$ is a fiber bundle over M such that each $\mathcal{C}_x M$ is an open cone in $T_x M \setminus \{0\}$. In such case (M, \mathcal{F}) is called *singular* (or *non y-global*) Finsler space (cf. [9]).

Geodesics of Finsler manifolds are determined by a system of 2nd order ordinary differential equation:

$$\ddot{x}^i + 2G^i(x, \dot{x}) = 0, \quad i = 1, \dots, n$$

where $G^i(x, \dot{x})$ are locally given by

$$G^i(x, y) := \frac{1}{4} g^{il}(x, y) \left(2 \frac{\partial g_{jl}}{\partial x^k}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y) \right) y^j y^k.$$

The associated homogeneous (nonlinear) parallel translation can be defined as follows: a vector field $X(t) = X^i(t) \frac{\partial}{\partial x^i}$ along a curve $c(t)$ is said to be parallel if it satisfies

$$(1) \quad \nabla_{\dot{c}} X(t) := \left(\frac{dX^i(t)}{dt} + \Gamma_j^i(c(t), X(t)) \dot{c}^j(t) \right) \frac{\partial}{\partial x^i},$$

where $\Gamma_j^i = \frac{\partial G^i}{\partial y^j}$.

Horizontal distribution, curvature. The geometric structure associated to ∇ can be given on TM in terms of the horizontal distribution. Let $\mathcal{V}TM \subset TTM$ denote the vertical distribution on TM , $\mathcal{V}_y TM := \text{Ker } \pi_{*,y}$. The horizontal distribution $\mathcal{H}TM \subset TTM$ associated to (1) is locally generated by the vector fields

$$(2) \quad l_{(x,y)} \left(\frac{\partial}{\partial x^i} \right) := \frac{\partial}{\partial x^i} + \Gamma_i^k(x, y) \frac{\partial}{\partial y^k}, \quad i = 1, \dots, n.$$

For any $y \in TM$ we have $T_y TM = \mathcal{H}_y TM \oplus \mathcal{V}_y TM$. The projectors corresponding to this decomposition will be denoted by h_y and v_y . The isomorphism $l_{(x,y)} : T_x M \rightarrow \mathcal{H}_y TM$ defined by the formula (2) is called *horizontal lift*. Then a vector field $X(t)$ along a curve $c(t)$ is parallel if and only if it is a solution of the differential equation

$$(3) \quad \frac{d}{dt} X(t) = l_{X(t)}(\dot{c}(t)).$$

The *curvature tensor* field characterizes the integrability of the horizontal distribution:

$$(4) \quad R_{(x,y)}(\xi, \eta) := v[h\xi, h\eta], \quad \xi, \eta \in T_{(x,y)} TM.$$

Using local coordinate system we have

$$R_{(x,y)} = \left(\frac{\partial \Gamma_i^k}{\partial x^j} - \frac{\partial \Gamma_j^k}{\partial x^i} + \Gamma_i^m \frac{\partial \Gamma_j^k}{\partial y^m} - \Gamma_j^m \frac{\partial \Gamma_i^k}{\partial y^m} \right) dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^k}.$$

The manifold is called of constant curvature $c \in \mathbb{R}$, if for any $x \in M$ the local expression of the curvature is

$$(5) \quad R_{(x,y)} = c (\delta_i^k g_{jm}(y) y^m - \delta_j^k g_{im}(y) y^m) dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^k}.$$

In this case the flag curvature of the Finsler manifold (cf. [2], Section 2.1 pp. 43-46) does not depend either on the point or on the 2-flag.

Indicatrix bundle. Let (M, \mathcal{F}) be an n -dimensional Finsler manifold. The *indicatrix* $\mathfrak{I}_x M$ at $x \in M$ is a hypersurface of $T_x M$ defined by

$$\mathfrak{I}_x M := \{y \in T_x M; \mathcal{F}(y) = \pm 1\}.$$

If the Finsler manifold (M, \mathcal{F}) is positive definite then the indicatrix $\mathfrak{I}_x M$ is a compact hypersurface in the tangent space $T_x M$, diffeomorphic to the standard $(n-1)$ -sphere. In this case the group $\text{Diff}(\mathfrak{I}_x M)$ of all smooth diffeomorphisms of $\mathfrak{I}_x M$ is a regular infinite dimensional Lie group modeled on the vector space $\mathfrak{X}(\mathfrak{I}_x M)$ of smooth vector fields on $\mathfrak{I}_x M$. The Lie algebra of the infinite dimensional Lie group $\text{Diff}(\mathfrak{I}_x M)$ is the vector space $\mathfrak{X}(\mathfrak{I}_x M)$, equipped with the negative of the usual Lie bracket, (c.f. A. Kriegl and P. W. Michor [6], Section 43).

We denote by $(\mathfrak{I}M, \pi, M)$ the *indicatrix bundle* of (M, \mathcal{F}) and by $i : \mathfrak{I}M \hookrightarrow TM$ the natural embedding of the indicatrix bundle into the tangent bundle (TM, π, M) .

Parallel translation and holonomy. Let (M, \mathcal{F}) be a Finsler manifold. The *parallel translation* $\tau_c : T_{c(0)}M \rightarrow T_{c(1)}M$ along a curve $c : [0, 1] \rightarrow \mathbb{R}$ is defined by vector fields $X(t)$ along $c(t)$ which are solutions of the differential equation (1). Since $\tau_c : T_{c(0)}M \rightarrow T_{c(1)}M$ is a differentiable map between $\hat{T}_{c(0)}M$ and $\hat{T}_{c(1)}M$ preserving the value of the Finsler metric, it induces a map

$$(6) \quad \tau_c^{\mathfrak{J}} : \mathfrak{J}_{c(0)}M \longrightarrow \mathfrak{J}_{c(1)}M$$

between the indicatrices.

Definition. The *holonomy group* $\text{Hol}(x)$ of a Finsler space (M, \mathcal{F}) at $x \in M$ is the subgroup of the group of diffeomorphisms $\text{Diff}(\mathfrak{J}_x M)$ of the indicatrix $\mathfrak{J}_x M$ determined by parallel translation of $\mathfrak{J}_x M$ along piece-wise differentiable closed curves initiated at the point $x \in M$.

We note that the holonomy group $\text{Hol}(x)$ is a topological subgroup of the regular infinite dimensional Lie group $\text{Diff}(\mathfrak{J}_x M)$, but its differentiable structure is not known in general.

3. TANGENT LIE ALGEBRAS TO SUBGROUPS OF $\text{Diff}^\infty(M)$

Let H be a subgroup of the diffeomorphism group $\text{Diff}^\infty(M)$ of a differentiable manifold M and let $\mathfrak{X}^\infty(M)$ be the Lie algebra of smooth vector fields on M .

Definition. A vector field $X \in \mathfrak{X}^\infty(M)$ is called *strongly tangent* to H , if there exists a \mathcal{C}^∞ -differentiable k -parameter family $\{\phi_{(t_1, \dots, t_k)} \in H\}_{t_i \in (-\varepsilon, \varepsilon)}$ of diffeomorphisms such that

- (i) $\phi_{(t_1, \dots, t_k)} = \text{Id}$, if $t_j = 0$ for some $1 \leq j \leq k$;
- (ii) $\frac{\partial^k \phi_{(t_1, \dots, t_k)}}{\partial t_1 \cdots \partial t_k} \Big|_{(t_1, \dots, t_k) = (0, \dots, 0)} = X$.

A vector field $X \in \mathfrak{X}^\infty(M)$ is called *tangent* to H , if there exists a \mathcal{C}^1 -differentiable 1-parameter family $\{\phi_t \in H\}_{t \in (-\varepsilon, \varepsilon)}$ of diffeomorphisms of M such that $\phi_0 = \text{Id}$ and $\frac{\partial \phi_t}{\partial t} \Big|_{t=0} = X$.

A Lie subalgebra \mathfrak{g} of $\mathfrak{X}^\infty(M)$ is called *tangent* to H , if all elements of \mathfrak{g} are tangent vector fields to H .

Theorem 1. *Let \mathcal{V} be a set of vector fields strongly tangent to the group $H \subset \text{Diff}^\infty(M)$. The Lie subalgebra \mathfrak{v} of $\mathfrak{X}^\infty(M)$ generated by \mathcal{V} is tangent to H .*

PROOF. First, we investigate some properties of vector fields strongly tangent to the group H .

Lemma 2. Let $\{\psi_{(t_1, \dots, t_h)} \in \text{Diff}^\infty(U)\}_{t_i \in (-\varepsilon, \varepsilon)}$ be a C^∞ -differentiable h -parameter family of (local) diffeomorphisms on a neighbourhood $U \subset \mathbb{R}^n$, satisfying $\psi_{(t_1, \dots, t_h)} = \text{Id}$, if $t_j = 0$ for some $1 \leq j \leq h$. Then

- (i) $\frac{\partial^{i_1 + \dots + i_h} \psi_{(t_1, \dots, t_h)}}{\partial t_1^{i_1} \dots \partial t_h^{i_h}} \Big|_{(0, \dots, 0)}(x) = 0$, if $i_p = 0$ for some $1 \leq p \leq h$;
- (ii) $\frac{\partial^h (\psi_{(t_1, \dots, t_h)})^{-1}}{\partial t_1 \dots \partial t_h} \Big|_{(0, \dots, 0)}(x) = -\frac{\partial^h \psi_{(t_1, \dots, t_h)}}{\partial t_1 \dots \partial t_h} \Big|_{(0, \dots, 0)}(x)$;
- (iii) $\frac{\partial^h \psi_{(t_1, \dots, t_h)}}{\partial t_1 \dots \partial t_h} \Big|_{(0, \dots, 0)}(x) = \frac{\partial \psi_{(\sqrt[h]{t_1}, \dots, \sqrt[h]{t_h})}}{\partial t} \Big|_{t=0}(x)$

at any point $x \in U$.

PROOF. Assertions (i) and (ii) can be obtained by direct computation. It follows from (i) that $\frac{\partial^h \psi_{(t_1, \dots, t_h)}}{\partial t_1 \dots \partial t_h} \Big|_{(0, \dots, 0)}(x)$ is the first non-necessarily vanishing derivative of the diffeomorphism family $\{\psi_{(t_1, \dots, t_h)}\}$ at any point $x \in M$. Using

$$\psi_{(t_1, \dots, t_k)}(x) = x + t_1 \cdots t_k (X(x) + \omega(x, t_1, \dots, t_k)),$$

where $\lim_{t_i \rightarrow 0} \omega(x, t_1, \dots, t_k) = 0$ we obtain, that

$$\frac{\partial}{\partial t} \Big|_{t=0} \psi_{(\sqrt[h]{t}, \dots, \sqrt[h]{t})}(x) = \frac{\partial}{\partial t} \Big|_{t=0} (x + t(X(x) + \omega(x, \sqrt[h]{t}, \dots, \sqrt[h]{t}))) = X(x),$$

which proves (iii). \square

We remark that the assertion (iii) means that any vector field strongly tangent to H is tangent to H .

Now, we generalize a well-known relation between the commutator of vector fields and the commutator of their induced flows.

Lemma 3. Let $\{\phi_{(s_1, \dots, s_k)}\}$ and $\{\psi_{(t_1, \dots, t_l)}\}$ be C^∞ -differentiable k -parameter, respectively l -parameter families of (local) diffeomorphisms defined on a neighbourhood $U \subset \mathbb{R}^n$. Assume that $\phi_{(s_1, \dots, s_k)} = \text{Id}$, respectively $\psi_{(t_1, \dots, t_l)} = \text{Id}$, if some of their variables equals 0. Then the family of (local) diffeomorphisms $[\phi_{(s_1, \dots, s_k)}, \psi_{(t_1, \dots, t_l)}]$ defined by the commutator of the group $\text{Diff}^\infty(U)$ fulfills $[\phi_{(s_1, \dots, s_k)}, \psi_{(t_1, \dots, t_l)}] = \text{Id}$, if some of its variables equals 0. Moreover

$$\frac{\partial^{k+l} [\phi_{(s_1 \dots s_k)}, \psi_{(t_1 \dots t_l)}]}{\partial s_1 \dots \partial s_k \partial t_1 \dots \partial t_l} \Big|_{(0 \dots 0; 0 \dots 0)}(x) = - \left[\frac{\partial^k \phi_{(s_1 \dots s_k)}}{\partial s_1 \dots \partial s_k} \Big|_{(0 \dots 0)}, \frac{\partial^l \psi_{(t_1 \dots t_l)}}{\partial t_1 \dots \partial t_l} \Big|_{(0 \dots 0)} \right](x)$$

at any point $x \in U$.

PROOF. The group theoretical commutator $[\phi_{(s_1, \dots, s_k)}, \psi_{(t_1, \dots, t_l)}]$ of the families of diffeomorphisms satisfies $[\phi_{(s_1, \dots, s_k)}, \psi_{(t_1, \dots, t_l)}] = \text{Id}$, if some of its variables equals 0. Hence

$$\frac{\partial^{i_1 + \dots + i_k + j_1 + \dots + j_l} [\phi_{(s_1, \dots, s_k)}, \psi_{(t_1, \dots, t_l)}]}{\partial s_1^{i_1} \dots \partial s_k^{i_k} \partial t_1^{j_1} \dots \partial t_l^{j_l}} \Big|_{(0, \dots, 0; 0, \dots, 0)} = 0,$$

if $i_p = 0$ or $j_q = 0$ for some index $1 \leq p \leq k$ or $1 \leq q \leq l$. The families of diffeomorphisms $\{\phi_{(s_1, \dots, s_l)}\}$, $\{\psi_{(t_1, \dots, t_l)}\}$, $\{\phi_{(s_1, \dots, s_l)}^{-1}\}$ and $\{\psi_{(t_1, \dots, t_l)}^{-1}\}$ are the constant family Id , if some of their variables equals 0. Hence one has

$$\begin{aligned} (7) \quad & \frac{\partial^{k+l} [\phi_{(s_1, \dots, s_k)}, \psi_{(t_1, \dots, t_l)}]}{\partial s_1 \dots \partial s_k \partial t_1 \dots \partial t_l} \Big|_{(0, \dots, 0; 0, \dots, 0)}(x) = \\ & = \frac{\partial^k}{\partial s_1 \dots \partial s_k} \Big|_{(0, \dots, 0)} \left(\frac{\partial^l \left(\phi_{(s_1 \dots s_k)}^{-1} \circ \psi_{(t_1 \dots t_l)}^{-1} \circ \phi_{(s_1 \dots s_k)} \circ \psi_{(t_1 \dots t_l)}(x) \right)}{\partial t_1 \dots \partial t_l} \Big|_{(0, \dots, 0)} \right) \\ & = \frac{\partial^k}{\partial s_1 \dots \partial s_k} \Big|_{(0, \dots, 0)} \left(d(\phi_{(s_1 \dots s_k)}^{-1})_{\phi_{(s_1 \dots s_k)}(x)} \frac{\partial^l \psi_{(t_1 \dots t_l)}^{-1}}{\partial t_1 \dots \partial t_l} \Big|_{(0, \dots, 0)} (\phi_{(s_1 \dots s_k)}(x)) \right), \end{aligned}$$

where $d(\phi_{(s_1, \dots, s_k)}^{-1})_{\phi_{(s_1, \dots, s_k)}(x)}$ denotes the Jacobi operator of the map $\phi_{(s_1, \dots, s_k)}^{-1}$ at the point $\phi_{(s_1, \dots, s_k)}(x)$. Using the fact, that $\{\phi_{(s_1, \dots, s_k)}\}$ is the constant family Id , if some of its variables equals 0, and the relation $d(\phi_{(0, \dots, 0)}^{-1})_{\phi_{(s_1, \dots, s_k)}(x)} = \text{Id}$, we obtain that (7) can be written as

$$d\left(\frac{\partial^k \phi_{(s_1 \dots s_k)}^{-1}}{\partial s_1 \dots \partial s_k} \Big|_{(0, \dots, 0)}\right)_x \frac{\partial^l \psi_{(t_1 \dots t_l)}^{-1}(x)}{\partial t_1 \dots \partial t_l} \Big|_{(0, \dots, 0)} + d\left(\frac{\partial^l \psi_{(t_1 \dots t_l)}^{-1}}{\partial t_1 \dots \partial t_l} \Big|_{(0, \dots, 0)}\right)_x \frac{\partial^k \phi_{(s_1 \dots s_k)}(x)}{\partial s_1 \dots \partial s_k} \Big|_{(0, \dots, 0)}.$$

According to assertion (ii) of Lemma 2 the last formula gives

$$d\left(\frac{\partial^k \phi_{(s_1 \dots s_k)}}{\partial s_1 \dots \partial s_k} \Big|_{(0, \dots, 0)}\right)_x \frac{\partial^l \psi_{(t_1 \dots t_l)}(x)}{\partial t_1 \dots \partial t_l} \Big|_{(0, \dots, 0)} - d\left(\frac{\partial^l \psi_{(t_1 \dots t_l)}}{\partial t_1 \dots \partial t_l} \Big|_{(0, \dots, 0)}\right)_x \frac{\partial^k \phi_{(s_1 \dots s_k)}(x)}{\partial s_1 \dots \partial s_k} \Big|_{(0, \dots, 0)},$$

which is the Lie bracket of vector fields

$$\left[\frac{\partial^l \psi_{(t_1, \dots, t_l)}}{\partial t_1 \dots \partial t_l} \Big|_{(0, \dots, 0)}, \frac{\partial^k \phi_{(s_1, \dots, s_k)}}{\partial s_1 \dots \partial s_k} \Big|_{(0, \dots, 0)} \right] : U \rightarrow \mathbb{R}^n.$$

□

Lemma 4. *The Lie algebra \mathfrak{v} has a basis consisting of vector fields strongly tangent to the group H .*

PROOF. The iterated Lie brackets of vector fields belonging to \mathcal{V} linearly generate the vector space \mathfrak{v} . It follows from Lemma 3 that iterated Lie brackets of vector

fields belonging to \mathcal{V} are strongly tangent to the group H . Hence \mathfrak{v} is linearly generated by vector fields strongly tangent to H . \square

Lemma 5. *Linear combinations of vector fields tangent to H are tangent to H .*

PROOF. If X and Y are vector fields tangent to H then there exist \mathcal{C}^1 -differentiable 1-parameter families of diffeomorphisms $\{\phi_t \in H\}$ and $\{\psi_t \in H\}$ such that

$$\phi_0 = \psi_0 = \text{Id}, \quad \frac{\partial}{\partial t} \Big|_{t=0} \phi_t = X, \quad \frac{\partial}{\partial t} \Big|_{t=0} \psi_t = Y.$$

Considering the \mathcal{C}^1 -differentiable 1-parameter families of diffeomorphisms $\{\phi_t \circ \psi_t\}$ and $\{\phi_{ct}\}$ one has

$$X + Y = \frac{\partial}{\partial t} \Big|_{t=0} (\phi_t \circ \psi_t), \quad cX = \frac{\partial}{\partial t} \Big|_{t=0} \phi_{(ct)}, \quad \text{for all } c \in \mathbb{R}^n,$$

which proves the assertion. \square

Lemmas 2 – 5 prove Theorem 1. \square

4. CURVATURE ALGEBRA

Definition. A vector field $\xi \in \mathfrak{X}(\mathfrak{J}_x M)$ on the indicatrix $\mathfrak{J}_x M$ is called a *curvature vector field* of the Finsler manifold (M, \mathcal{F}) at $x \in M$, if there exists $X, Y \in T_x M$ such that $\xi = r_x(X, Y)$, where

$$(8) \quad r_x(X, Y)(y) := R_{(x,y)}(l_y X, l_y Y)$$

The Lie subalgebra $\mathfrak{R}_x := \langle r_x(X, Y); X, Y \in T_x M \rangle$ of $\mathfrak{X}(\mathfrak{J}_x M)$ generated by the curvature vector fields is called the *curvature algebra* of the Finsler manifold (M, \mathcal{F}) at the point $x \in M$.

Since the Finsler metric is preserved by parallel translations, its derivatives with respect to horizontal vector fields are identically zero. Using (4) we obtain, that the derivative of the Finsler metric with respect to (8) vanishes, and hence

$$g_{(x,y)}(y, R_{(x,y)}(l_y X, l_y Y)) = 0, \quad \text{for any } y, X, Y \in T_x M$$

(c.f. [9], eq. (10.9)). This means that the curvature vector fields $\xi = r_x(X, Y)$ are tangent to the indicatrix. In the sequel we investigate the tangential properties of the curvature algebra to the holonomy group of the canonical connection ∇ of a Finsler manifold.

Proposition 6. *Any curvature vector field at $x \in M$ is strongly tangent to the holonomy group $\text{Hol}(x)$.*

PROOF. Indeed, let us consider the curvature vector field $r_x(X, Y) \in \mathfrak{X}(\mathfrak{J}_x M)$, $X, Y \in T_x M$ and let $\hat{X}, \hat{Y} \in \mathfrak{X}(M)$ be commuting vector fields i.e. $[\hat{X}, \hat{Y}] = 0$ such that $\hat{X}_x = X$, $\hat{Y}_x = Y$. By the geometric construction, the flows $\{\phi_t\}$ and $\{\psi_s\}$ of the horizontal lifts $l(\hat{X})$ and $l(\hat{Y})$ are fiber preserving diffeomorphisms of the bundle $\mathfrak{J}M$ for any $t \in \mathbb{R}$, corresponding to parallel translations along integral curves of \hat{X} and \hat{Y} respectively. Then the commutator

$$\theta_{t,s} = [\phi_t, \psi_s] = \phi_t^{-1} \circ \psi_s^{-1} \circ \phi_t \circ \psi_s : \mathfrak{J}M \rightarrow \mathfrak{J}M$$

is also a fiber preserving diffeomorphism of the bundle $\mathfrak{J}M$ for any $t, s \in \mathbb{R}$. Therefore for any $x \in M$ the restriction

$$\theta_{t,s}(x) = \theta_{t,s}|_{\mathfrak{J}_x M} : \mathfrak{J}_x M \rightarrow \mathfrak{J}_x M$$

to the fiber $\mathfrak{J}_x M$ is a 2-parameter C^∞ -differentiable family of diffeomorphisms contained in the holonomy group $\text{Hol}(x)$ such that

$$\theta_{0,s}(x) = \text{Id}, \quad \theta_{t,0}(x) = \text{Id}, \quad \text{and} \quad \left. \frac{\partial^2}{\partial t \partial s} \right|_{t=0, s=0} \theta_{t,s}(x) = r_x(X, Y),$$

which proves that the curvature vector field $r_x(X, Y)$ is strongly tangent to the holonomy group $\text{Hol}(x)$ and hence we obtain the assertion. \square

Theorem 7. *The curvature algebra \mathfrak{R}_x of a Finsler manifold (M, \mathcal{F}) is tangent to the holonomy group $\text{Hol}(x)$ for any $x \in M$.*

PROOF. Since by Proposition 6 the curvature vector fields are strongly tangent to $\text{Hol}(x)$ and the curvature algebra \mathfrak{R}_x is algebraically generated by the curvature vector fields, the assertion follows from Theorem 1. \square

Proposition 8. *The curvature algebra \mathfrak{R}_x of a Riemannian manifold (M, g) at any point $x \in M$ is isomorphic to the linear Lie algebra over the vector space $T_x M$ generated by the curvature operators of (M, g) at $x \in M$.*

PROOF. The curvature tensor field of a Riemannian manifold given by the equation (4) is linear with respect to $y \in T_x M$ and hence

$$R_{(x,y)}(\xi, \eta) = (R_x(\xi, \eta))_i^k y^l \frac{\partial}{\partial y^k},$$

where $\{R_x(\xi, \eta)_i^k\}$ is the matrix of the curvature operator $R_x(\xi, \eta) : T_x M \rightarrow T_x M$ with respect to the natural basis $\{\frac{\partial}{\partial x^1}|_x, \dots, \frac{\partial}{\partial x^n}|_x\}$. Hence any curvature vector field $r_x(\xi, \eta)(y)$ with $\xi, \eta \in T_x M$ has the shape $r_x(\xi, \eta)(y) = R_{(x,y)}(\xi, \eta) = (R_x(\xi, \eta))_i^k y^l \frac{\partial}{\partial y^k}$. It follows that the flow of $r_x(\xi, \eta)(y)$ on the indicatrix $\mathfrak{J}_x M$ generated by the vector field $r_x(\xi, \eta)(y)$ is induced by the action of the linear 1-parameter group $\exp t R_x(\xi, \eta)$ on $T_x M$, which implies the assertion. \square

Remark 9. *The curvature algebra of Finsler surfaces is one-dimensional.*

PROOF. For Finsler surfaces the curvature vector fields form a one-dimensional vector space and hence the generated Lie algebra is also one-dimensional. \square

5. CONSTANT CURVATURE

Now, we consider a Finsler manifold (M, \mathcal{F}) of non-zero constant curvature. In this case for any $x \in M$ the curvature vector field $r_x(X, Y)(y)$ has the shape (cf. (5))

$$r(X, Y)(y) = c (\delta_j^i g_{km}(y) y^m - \delta_k^i g_{jm}(y) y^m) X^j Y^k \frac{\partial}{\partial y^i}, \quad 0 \neq c \in \mathbb{R}.$$

Putting $y_j = g_{jm}(y) y^m$ we can write $r(X, Y)(y) = c (\delta_j^i y_k - \delta_k^i y_j) X^j Y^k \frac{\partial}{\partial y^i}$. Any linear combination of curvature vector fields has the form

$$r(A)(y) = A^{jk} (\delta_j^i y_k - \delta_k^i y_j) \frac{\partial}{\partial y^i},$$

where $A = A^{jk} \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k} \in T_x M \wedge T_x M$ is arbitrary bivector at $x \in M$.

Lemma 10. *Let (M, \mathcal{F}) be a Finsler manifold of non-zero constant curvature. The curvature algebra \mathfrak{R}_x at any point $x \in M$ satisfies*

$$(9) \quad \dim \mathfrak{R}_x \geq \frac{n(n-1)}{2},$$

where $n = \dim M$.

PROOF. Let us consider the curvature vector fields $r_{jk} = r_x(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k})(y)$ at a fixed point $x \in M$. If a linear combination

$$A^{jk} r_{jk} = A^{jk} (\delta_j^i y_k - \delta_k^i y_j) \frac{\partial}{\partial y^i} = (A^{ik} y_k - A^{ji} y_j) \frac{\partial}{\partial y^i} = 2A^{ik} y_k \frac{\partial}{\partial y^i}$$

of curvature vector fields r_{jk} with constant coefficients $A^{jk} = -A^{kj}$ satisfies $A^{jk} r_{jk} = 0$ for any $y \in T_x M$ then one has the linear equation $A^{ik} y_k = 0$ for any fixed index i . Since the covector fields y_1, \dots, y_n are linearly independent we obtain $A^{jk} = 0$ for all $j, k \in \{1, \dots, n\}$. It follows that the curvature vector fields r_{jk} are linearly independent for any $j < k$ and hence $\dim \mathfrak{R}_x \geq \frac{n(n-1)}{2}$. \square

Corollary 11. *Let (M, g) be a Riemannian manifold of non-zero constant curvature with $n = \dim M$. The curvature algebra \mathfrak{R}_x at any point $x \in M$ is isomorphic to the orthogonal Lie algebra $\mathfrak{o}(n)$.*

PROOF. The holonomy group of a Riemannian manifold is a subgroup of the orthogonal group $O(n)$ of the tangent space $T_x M$ and hence the curvature algebra \mathfrak{R}_x is a subalgebra of the orthogonal Lie algebra $\mathfrak{o}(n)$. Hence the previous assertion implies the corollary. \square

Theorem 12. *Let (M, \mathcal{F}) be a Finsler manifold of non-zero constant curvature with $n = \dim M > 2$. If the point $x \in M$ is not (semi-)Riemannian then the curvature algebra \mathfrak{R}_x at $x \in M$ satisfies*

$$(10) \quad \dim \mathfrak{R}_x > \frac{n(n-1)}{2}.$$

PROOF. We assume $\dim \mathfrak{R}_x = \frac{n(n-1)}{2}$. For any constant skew-symmetric matrices $\{A^{jk}\}$ and $\{B^{jk}\}$ the Lie bracket of vector fields $A^{ik} y_k \frac{\partial}{\partial y^i}$ and $B^{ik} y_k \frac{\partial}{\partial y^i}$ has the shape $C^{ik} y_k \frac{\partial}{\partial y^i}$, where $\{C^{ik}\}$ is a constant skew-symmetric matrix, too. Using the homogeneity of g_{hl} we obtain

$$(11) \quad \frac{\partial y_h}{\partial y^m} = \frac{\partial g_{hl}}{\partial y^m} y^l + g_{hm} = g_{hm}$$

and hence

$$\begin{aligned} \left[A^{mk} y_k \frac{\partial}{\partial y^m}, B^{ih} y_h \frac{\partial}{\partial y^i} \right] &= \left(A^{mk} B^{ih} \frac{\partial y_h}{\partial y^m} - B^{mk} A^{ih} \frac{\partial y_h}{\partial y^m} \right) y_k \frac{\partial}{\partial y^i} \\ &= (B^{ih} g_{hm} A^{mk} - A^{ih} g_{hm} B^{mk}) y_k \frac{\partial}{\partial y^i} = C^{ik} y_k \frac{\partial}{\partial y^i}. \end{aligned}$$

Particularly, for the skew-symmetric matrices $E_{ab}^{ij} = \delta^i_a \delta^j_b - \delta^i_b \delta^j_a$, $a, b \in \{1, \dots, n\}$, we have

$$\left[E_{ab}^{ij} y_j \frac{\partial}{\partial y^i}, E_{cd}^{kl} y_l \frac{\partial}{\partial y^k} \right] = (E_{cd}^{ih} g_{hm} E_{ab}^{mk} - E_{ab}^{ih} g_{hm} E_{cd}^{mk}) y_k \frac{\partial}{\partial y^i} = \Lambda_{ab,cd}^{im} y_m \frac{\partial}{\partial y^i},$$

where the constants $\Lambda_{ab,cd}^{ij}$ satisfy $\Lambda_{ab,cd}^{ij} = -\Lambda_{ab,cd}^{ji} = -\Lambda_{ba,cd}^{ij} = -\Lambda_{ab,dc}^{ij} = -\Lambda_{cd,ab}^{ij}$. Putting $i = a$ and computing the trace for these indices we obtain

$$(12) \quad (n-2)(g_{bd} y_c - g_{bc} y_d) = \Lambda_{b,cd}^l y_l,$$

where $\Lambda_{b,cd}^l := \Lambda_{ib,cd}^{il}$. The right hand side is a linear form in variables y_1, \dots, y_n . According to the identity (12) this linear form vanishes for $y_c = y_d = 0$, hence $\Lambda_{b,cd}^l = 0$ for $l \neq c, d$. Denoting $\lambda_{bd}^{(c)} := \frac{1}{n-2} \Lambda_{b,cd}^c$ (no summation for the index c) we get the identities

$$g_{bd} y_c - g_{bc} y_d = \lambda_{bd}^{(c)} y_c - \lambda_{bc}^{(d)} y_d \quad (\text{no summation for } c \text{ and } d).$$

Putting $y_d = 0$ we obtain $g_{bd}|_{y_d=0} = \lambda_{bd}^{(c)}$ for any $c \neq d$. It follows $\lambda_{bd}^{(c)}$ is independent of the index c ($\neq d$). Defining $\lambda_{bd} := \lambda_{bd}^{(c)}$ with some c ($\neq d$) we obtain from (12) the identity

$$(13) \quad g_{bd} y_c - g_{bc} y_d = \lambda_{bd} y_c - \lambda_{bc} y_d$$

for any $b, c, d \in \{1, \dots, n\}$. We have

$$\lambda_{cd} y_b - \lambda_{cb} y_d = (g_{bd} y_c - g_{bc} y_d) - (g_{db} y_c - g_{dc} y_b) = (\lambda_{bd} y_c - \lambda_{bc} y_d) - (\lambda_{db} y_c - \lambda_{dc} y_b),$$

which implies the identity

$$(14) \quad \begin{aligned} & (\lambda_{cd} y_b - \lambda_{cb} y_d) + (\lambda_{db} y_c - \lambda_{dc} y_b) + (\lambda_{bc} y_d - \lambda_{bd} y_c) = \\ & = (\lambda_{cd} - \lambda_{dc}) y_b + (\lambda_{db} - \lambda_{bd}) y_c + (\lambda_{bc} - \lambda_{cb}) y_d = 0. \end{aligned}$$

Since $\dim M > 2$, we can consider 3 different indices b, c, d and we obtain from the identity (14) that $\lambda_{bc} = \lambda_{cb}$ for any $b, c \in \{1, \dots, n\}$.

By derivation the identity (13) we get

$$\frac{\partial g_{bd}}{\partial y_a} y_c - \frac{\partial g_{bc}}{\partial y_a} y_d + g_{bd} \delta_c^a - g_{bc} \delta_d^a = \lambda_{bd} \delta_c^a - \lambda_{bc} \delta_d^a.$$

Using (11) we obtain

$$\begin{aligned} & \frac{\partial y_a}{\partial y^q} \left(\frac{\partial g_{bd}}{\partial y_a} y_c - \frac{\partial g_{bc}}{\partial y_a} y_d \right) + g_{bd} g_{cq} - g_{bc} g_{dq} = \\ & = \frac{\partial g_{bd}}{\partial y^q} y_c - \frac{\partial g_{bc}}{\partial y^q} y_d + g_{bd} g_{cq} - g_{bc} g_{dq} = \lambda_{bd} g_{cq} - \lambda_{bc} g_{dq}. \end{aligned}$$

Since

$$\left(\frac{\partial g_{bd}}{\partial y^q} y_c - \frac{\partial g_{bc}}{\partial y^q} y_d \right) y^b = 0$$

we get the identity

$$y_d g_{cq} - y_c g_{dq} = \lambda_{bd} y^b g_{cq} - \lambda_{bc} y^b g_{dq}.$$

Multiplying both sides of this identity by the inverse $\{g^{qr}\}$ of the matrix $\{g_{cq}\}$ and taking the trace with respect to the indices c, r we obtain the identity

$$(n-1) y_d = (n-1) \lambda_{bd} y^b.$$

Hence we obtain that $g_{bd} y^b = \lambda_{bd} y^b$ and hence $g_{bd} = \lambda_{bd}$, which means that the point $x \in M$ is (semi-)Riemannian. From this contradiction follows the assertion. \square

Theorem 13. *Let (M, \mathcal{F}) be a positive definite Finsler manifold of non-zero constant curvature with $n = \dim M > 2$. The holonomy group of (M, \mathcal{F}) is a compact Lie group if and only if (M, \mathcal{F}) is a Riemannian manifold.*

PROOF. We assume that the holonomy group of a Finsler manifold (M, \mathcal{F}) of non-zero constant curvature with $\dim M \geq 3$ is a compact Lie transformation group on the indicatrix $\mathfrak{J}_x M$. The curvature algebra \mathfrak{R}_x at a point $x \in M$ is tangent to the holonomy group $\text{Hol}(x)$ and hence $\dim \text{Hol}(x) \geq \dim \mathfrak{R}_x$. If there exists a not (semi-)Riemannian point $x \in M$ then $\dim \mathfrak{R}_x > \frac{n(n-1)}{2}$. The $(n-1)$ -dimensional indicatrix $\mathfrak{J}_x M$ at x can be equipped with a Riemannian metric which is invariant with respect to the compact Lie transformation group $\text{Hol}(x)$. Since the group of isometries of an $n-1$ -dimensional Riemannian manifold is of dimension at most $\frac{n(n-1)}{2}$ (cf. Kobayashi [3], p. 46,) we obtain a contradiction, which proves the assertion. \square

Since the holonomy group of a Landsberg manifold is a subgroup of the isometry group of the indicatrix, we obtain that any Landsberg manifold of non-zero constant curvature with dimension > 2 is Riemannian (c.f. Numata [8]).

We can summarize our results as follows:

Theorem 14. *The holonomy group of any non-Riemannian positive definite Finsler manifold of non-zero constant curvature with dimension > 2 does not occur as the holonomy group of any Riemannian manifold.*

6. APPENDIX: FINSLER METRIC WITH INFINITE DIMENSIONAL CURVATURE ALGEBRA

Let us consider the singular (non y -global) Finsler manifold (H_3, \mathcal{F}) , where H_3 is the 3-dimensional Heisenberg group and \mathcal{F} is a left-invariant Berwald-Moór metric (c.f. [9], Example 1.1.5, p. 8).

The group H_3 can be realized as the Lie group of matrices of the form $\begin{bmatrix} 1 & x^1 & x^2 \\ 0 & 1 & x^3 \\ 0 & 0 & 1 \end{bmatrix}$, where $x = (x^1, x^2, x^3) \in \mathbb{R}^3$ and hence the multiplication can be written as

$$(x^1, x^2, x^3) \cdot (y^1, y^2, y^3) = (x^1 + y^1, x^2 + y^2 + x^1 y^3, x^3 + y^3).$$

The vector $0 = (0, 0, 0) \in \mathbb{R}^3$ gives the unit element of H_3 . The Lie algebra $\mathfrak{h}_3 = T_0 H_3$ consists of matrices of the form $\begin{bmatrix} 0 & a^1 & a^2 \\ 0 & 0 & a^3 \\ 0 & 0 & 0 \end{bmatrix}$, corresponding to the tangent vector $a = a^1 \frac{\partial}{\partial x^1} + a^2 \frac{\partial}{\partial x^2} + a^3 \frac{\partial}{\partial x^3}$ at the unit element $0 \in H_3$. A left-invariant

Berwald-Moór Finsler metric \mathcal{F} is induced by the (singular) Minkowski functional $\mathcal{F}_0: \mathfrak{h}_3 \rightarrow \mathbb{R}$:

$$\mathcal{F}_0(a) := (a^1 a^2 a^3)^{\frac{2}{3}}$$

of the Lie algebra in the following way: if $y = (y^1, y^2, y^3)$ is a tangent vector at $x \in H_3$, then

$$\mathcal{F}(x, y) := \mathcal{F}_0(x^{-1}y).$$

The coordinate expression of the singular (non y -global) Finsler metric \mathcal{F} is

$$\mathcal{F}(x, y) = (y^1 (y^2 - x^1 y^3) y^3)^{\frac{2}{3}}.$$

Since \mathcal{F} is left-invariant, the associated geometric structures (connection, geodesics, curvature) are also left-invariant and the curvature algebras at different points are isomorphic. Using the notation

$$r_x(i, j) = r_x\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \quad i, j = 1, 2, 3,$$

for curvature vector fields, a direct computation yields

$$\begin{aligned} r_x(1, 2) &= \frac{1}{4} \left(\frac{5y^{1^2}y^{3^2}}{(x^1y^3 - y^2)^3} \frac{\partial}{\partial y^1} + \frac{y^1y^{3^2}(3x^1y^3 + y^2)}{(y^2 - x^1y^3)^3} \frac{\partial}{\partial y^2} + \frac{4y^1y^{3^3}}{(y^2 - x^1y^3)^3} \frac{\partial}{\partial y^3} \right), \\ r_x(1, 3) &= \frac{1}{4} \left(\frac{y^{1^2}y^3(6x^1y^3 - 11y^2)}{(x^1y^3 - y^2)^3} \frac{\partial}{\partial y^1} + \frac{4y^1y^{3^2}x^1(2x^1y^3 - 3y^2)}{(y^2 - x^1y^3)^3} \frac{\partial}{\partial y^2} + \right. \\ &\quad \left. + \frac{y^1y^{3^2}(7x^1y^3 - 11y^2)}{(y^2 - x^1y^3)^3} \frac{\partial}{\partial y^3} \right), \\ r_x(2, 3) &= \frac{1}{4} \left(\frac{4y^{1^3}y_3}{(x^1y^3 - y^2)^3} \frac{\partial}{\partial y^1} + \frac{y^{1^2}y^3(6x^1y^3 - y^2)}{(y^2 - x^1y^3)^3} \frac{\partial}{\partial y^2} + \frac{5y^{1^2}y^{3^2}}{(y^2 - x^1y^3)^3} \frac{\partial}{\partial y^3} \right). \end{aligned}$$

The curvature vector fields $r_0(i, j)$, $i, j = 1, 2, 3$, at the unit element $0 \in H_3$ generate the curvature algebra \mathfrak{r}_0 . Let us denote $Y^{k,m} := \frac{y^{1^k}y^{3^m}}{y^{2^{k+m-1}}}$, $k, m \in \mathbb{N}$, and consider the vector fields

$$(15) \quad A^{k,m}(a^1, a^2, a^3) = a^1 Y^{k+1,m} \frac{\partial}{\partial y^1} \Big|_0 + a^2 Y^{k,m} \frac{\partial}{\partial y^2} \Big|_0 + a^3 Y^{k,m+1} \frac{\partial}{\partial y^3} \Big|_0,$$

with $(a^1, a^2, a^3) \in \mathbb{R}^3$ and $k, m \in \mathbb{N}$. Then the curvature vector fields $r_0(i, j)$ at $0 \in H_3$ can be written in the form

$$r_0(1, 2) = \frac{1}{4} A^{1,2}(-5, 1, 4), \quad r_0(1, 3) = \frac{1}{4} A^{1,1}(11, 0, -11), \quad r_0(2, 3) = \frac{1}{4} A^{2,1}(-4, -1, 5).$$

Proposition 15. *The curvature algebra \mathfrak{r}_x at any point $x \in M$ is a Lie algebra of infinite dimension.*

PROOF. Since the Finsler metric is left-invariant, the curvature algebras at different points are isomorphic. Therefore it is enough to prove that the curvature algebra \mathfrak{r}_0 at $0 \in H_3$ has infinite dimension. We prove the statement by contradiction: let us suppose that \mathfrak{r}_0 is finite dimensional.

A direct computation shows that for any $(a^1, a^2, a^3), (b^1, b^2, b^3) \in \mathbb{R}^3$ one has

$$[A^{k,m}(a^1, a^2, a^3), A^{p,q}(b^1, b^2, b^3)] = A^{k+p,m+q}(c^1, c^2, c^3)$$

with some $(c^1, c^2, c^3) \in \mathbb{R}^3$. It follows that any iterated Lie bracket of curvature vector fields $r_0(i, j)$, $i, j = 1, 2, 3$, has the shape (15) and hence there exists a basis of the curvature algebra \mathfrak{r}_0 of the form $\{A^{k_i, m_i}(a_i^1, a_i^2, a_i^3)\}_{i=1}^N$, where $N \in \mathbb{N}$ is the dimension of \mathfrak{r}_0 . We can assume that $\{(k_i, m_i)\}_{i=1}^N$ forms an increasing sequence, i.e. $(k_1, m_1) \leq (k_2, m_2) \leq \dots \leq (k_N, m_N)$ holds with respect to the lexicographical ordering of $\mathbb{N} \times \mathbb{N}$. We can consider the vector fields

$$\frac{4}{11}r_0(1, 3) = A^{1,1}(1, 0, -1), \quad 4r_0(1, 2) = A^{1,2}(-5, 1, 4), \quad 4r_0(2, 3) = A^{2,1}(-4, -1, 5)$$

as the first three members of this sequence. Hence $1 \leq k_N, m_N$ and

$$[A^{1,1}(1, 0, -1), A^{k_N, m_N}(a_N^1, a_N^2, a_N^3)] = A^{1+k_N, 1+m_N}(c^1, c^2, c^3)$$

belongs to \mathfrak{r}_0 , too, where $c^1 = (k_N - m_N - 1)a_N^1 + 2a_N^2 - a_N^3$, $c^2 = (k_N - m_N)a_N^2$ and $c^3 = a_N^1 - 2a_N^2 + (k_N - m_N + 1)a_N^3$. Since $k_N < 1 + k_N$, $m_N < 1 + m_N$ we have $c^1 = c^2 = c^3 = 0$ and hence the homogeneous linear system

$$\begin{aligned} 0 &= (k_N - m_N - 1)a_N^1 + 2a_N^2 - a_N^3, \\ 0 &= (k_N - m_N)a_N^2, \\ 0 &= a_N^1 - 2a_N^2 + (k_N - m_N + 1)a_N^3 \end{aligned}$$

has a solution $(a_N^1, a_N^2, a_N^3) \neq (0, 0, 0)$. It follows that $k_N = m_N$.

Similarly, computing the Lie bracket

$$0 = [A^{1,2}(-5, 1, 4), A^{k_N, k_N}(a_N^1, a_N^2, a_N^3)] = A^{1+k_N, 2+k_N}(d^1, d^2, d^3)$$

Since $k_N < 1 + k_N < 2 + k_N$ we have $d^1 = d^2 = d^3 = 0$ giving the homogeneous linear system

$$\begin{aligned} 0 &= (-3k_N + 5)a_N^1 - 15a_N^2 + 10a_N^3, \\ 0 &= -a_N^1 + (3 - 3k_N)a_N^2 - 2a_N^3, \\ 0 &= -4a_N^1 + 12a_N^2 - (3k_N + 8)a_N^3 \end{aligned}$$

for (a_N^1, a_N^2, a_N^3) . The determinant of this system vanishes only for $k_N = 0$ which is a contradiction. \square

Corollary 16. *The holonomy group of the Finsler manifold (H_3, \mathcal{F}) has an infinite dimensional tangent Lie algebra.*

We remark here, that it remains an interesting open question: *Is there a nonsingular (y-global) Finsler manifold whose holonomy group is infinite dimensional?*

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